Mathematica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

Mathematica
Balkanica

New Series Vol. 2, 1988, Fasc. 4

Operators for Onesided Approximation by Algebraic Polynomials in $L_p([-1, 1]^d)$

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Presented by V. Popov

In this paper we construct a sequence of pair of operators $Q_n^{\pm}$ mapping the set of all real-valued, bounded and measurable in $[-1, 1]^d$ functions on the set of algebraic polynomials of total degree n. The two polynomials $Q_n^+ f$ and $Q_n^- f$ are under and above the function f and the sequence $\{\| Q_n^+ f - Q_n^- f \|_p : n = 1, 2, \ldots,\}$ tends to zero with the same order as the sequence of best onesided algebraic approximations of f.

1. Introduction

This paper is a continuation of our investigations in [4]. We follow in general the program sketched in Section 5.4 in this article. The new difficulties overcomed here in comparison with [4] come from the effect of the edges for the best algebraic approximations and from the lack of translation operators defined on the whole domain.

A characterization of the best onesided algebraic approximations in the univariate case is obtained by M. Stoyanova [7]. Our method of proving the direct statement essentialy differs from this one in [7].

The paper is organized so that the reader can follow it independently from [4]. In section 2 we give the notations and some auxiliary results. An equivalence of weighted τ-moduli and appropriate weighted onesided Peetre K-functionals is established in Section 3. As a consequence we get one important property of the weighted τ-moduli — the possibility a multiplier to be taken out of the modulus. In Section 4 we construct a sequence of pair of operators providing algebraic polynomials of near best onesided approximation and prove a direct theorem. A converse theorem and a characterization rezult for the best onesided algebraic approximations are established in Section 5. Using the equivalence from Section 3 we simplify the proof of the converse theorem and make it transparant as in the classical case of best approximation by trigonometric polynomials.

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* Both authors were supported by contract No 50 with the Committee of Sciences, Bulgaria.

2. Notations and auxiliary results

We work with bounded measurable realvalued functions defined on $\Omega=[-1, 1]^d\subset R^d$. R^d is considered as a normed vector space with elements $x=(x_1, x_2,\dots, x_d)$, y, h and norm $|x|=\max\{|x_s|:s=1, 2,\dots, d\}$.

Let X be a measurable subset of Ω. We shall consider the following spaces

$$L_p(X)=\{f:\|f\|_p=\|f\|_{p(X)}=(\int_X |f(x)|^p dx)^{1/p}<\infty\}$$

$1\leqq p<\infty$, dx — the Lebesgue measure on X and

$$L_\infty(X)=\{f:\|f\|_\infty=\|f\|_{\infty(X)}=\sup\{|f(x)|:x\in X\}<\infty\}.$$

For the restriction of f on X we use the same symbol f.

W_p^r denotes the Sobolev space of all functions $f\in L_p(\Omega)$ possessing weak derivatives $D^\alpha f\in L_p(\Omega)$ for any α, $|\alpha|=r$.

α, β, ε are multi-indices. $|\alpha|=\alpha_1+\alpha_2+\dots+\alpha_d$ is the length of α, where $\alpha=(\alpha_1, \alpha_2,\dots, \alpha_d)$. $\beta\leqq\alpha$ means $\beta_s\leqq\alpha_s$ for any $s=1, 2,\dots, d$ and $\binom{\beta}{\alpha}=\prod_{s=1}^{a}\binom{\beta_s}{\alpha_s}$. D^α as usual denotes a differential operator in Ω.

Let N be a fixed natural number. We set

$$Z=\{0, 1,\dots, N-1\}^d;\quad Z'=\{0, 1,\dots, N\}^d;$$

$$z_\nu=\cos(\pi-\nu\pi/N),\ \nu=0, 1,\dots, N,\quad z_{-1}=z_0=-1,\quad z_{N+1}=z_N=1.$$

For every $j=(j_1, j_2,\dots, j_d)\in Z$ we denote

$$\Omega_j=[z_{j_1}, z_{j_1+1}]\times\dots\times[z_{j_d}, z_{j_d+1}],$$

and for every $j\in Z'$ we denote

$$\Omega'_j=[z_{j_1-1}, z_{j_1+1}]\times\dots\times[z_{j_d-1}, z_{j_d+1}].$$

We set $\mu(v)=\int_0^v \exp\{-1/(u-u^2)\}du/\int_0^1 \exp\{-1/(u-u^2)\}du$ for $0<v<1$, $\mu(v)=0$ for $v\leqq 0$ and $\mu(v)=1$ for $v\geqq 1$. Therefore $\mu\in C^\infty(R)$. We set

$$\mu_\nu(v)=\begin{cases}1-\mu((v-z_0)/(z_1-z_0)) & \text{for } \nu=0;\\ \mu((v-z_{\nu-1})/(z_\nu-z_{\nu-1}))(1-\mu((v-z_\nu)/(z_{\nu+1}-z_\nu))) & \text{for } \nu=1, 2,\dots, N-1;\\ \mu((v-z_{N-1})/(z_N-z_{N-1})) & \text{for } \nu=N.\end{cases}$$

For every $j\in Z$ we set $\mu_j(x)=\prod_{s=1}^{d}\mu_{j_s}(x_s)$. Therefore for any $x\in\Omega$ we have

(2.1) $$0 \leqq \mu_j(x) \leqq 1; \quad \mu_j(x) = 0 \quad \text{if} \quad x \notin \Omega'_j;$$

(2.2) $$\sum_{j \in Z'} \mu_j(x) = 1.$$

For $v \in [-1, 1]$, $t > 0$ we set $\psi(t, v) = t\sqrt{1-v^2} + t^2$. For $x \in \Omega$ we denote $\Psi(t, x) = \prod_{s=1}^{d} \psi(t, x_s)$ and $\Psi^\alpha(t, x) = \prod_{s=1}^{d} \psi(t, x_s)^{\alpha_s}$. A t neighbourhood of the point $x \in \Omega$ we define by

$$U(t, x) = \{y \in \Omega : |x_s - y_s| \leqq \psi(t, x_s), \quad s = 1, 2, \dots, d\}.$$

.... Therefore the neighbourhoods are rectangles in Ω and for $t \leqq 1/2$

(2.3) $$\Psi(t, x) \leqq \operatorname{meas} U(t, x) \leqq 2^d \Psi(t, x).$$

By H_n^d we denote the set of all algebraic polynomials in R^d of total degree not greater than n.

Let $\tilde{E}_n(f)_p$ be the best one-sided algebraic approximation of f in $L_p(\Omega)$, $1 \leqq p < \infty$, i.e.

$$\tilde{E}_n(f)_p = \inf\{\|g^+ - g^-\|_{p(\Omega)} : g^\pm \in H_n^d, \ g^- \leqq f \leqq g^+\}.$$

By $\Delta_h^r f(x)$ we denote the r-th finite difference of f (defined in Ω) with step h in point x, i.e.

$$\Delta_h^r f(x) = \sum_{i=0}^{r} (-1)^{r-i} \binom{r}{i} f(x + ih), \quad \text{if} \quad x, \ x + rh \in \Omega.$$

For a convex subset X of Ω we set

$$\omega_r(f; X) = \sup\{|\Delta_h^r f(y)| : y, \ y + rh \in X\}.$$

The local modulus of f in the point x is given by

$$\omega_r(f; U(t, x))$$

and the average modulus of f is defined by

(2.4) $$\tau_r^*(f; t)_p = \|\omega_r(f; U(t, \cdot))\|_{p(\Omega)}.$$

The usual moduli of smoothness

$$\omega_r(f; t)_p = \sup\{\|\Delta_h^r f(\cdot)\|_{p(\Omega)} : |h| \leqq t\}$$

will be also used.

The properties of ω_r are assumed to be known. The following properties of τ_r^* follows immediately by the definition

(2.5) $$\tau_r^*(f+g;\ t)_p \leqq \tau_r^*(f;\ t)_p + \tau_r^*(g;\ t)_p;$$

(2.6) $\tau_r^*(f;\ t)_p$ is a nondecreasing function of $t \in (0,\ 1]$.

Other properties of τ_r^* will be given in the next section.

In the paper r, d and p are fixed numbers, r, d — naturals standing for the order of the moduli and for the dimension of the space respectively, $1 \leqq p < \infty$. By c we denote positive numbers which may depend only on r, d and p. The c's may differ at each occurence. The number $l = \max\{[d/p]+1, r\}$ ($[.]$ integral part) is also fixed.

We introduce the onesided weighted K-functional as the quantity

(2.7) $$K_r^*(f;\ t)_p = \inf\{\|g^+ - g^-\|_p + \sum_{|\alpha|=r,l} (\|\Psi^\alpha(t) D^\alpha g^+\|_p + \|\Psi^\alpha(t) D^\alpha g^-\|_p)\},$$

where the inf is taken over all $g^\pm \in W_p^l \cap C$ such that $g^- \leqq f \leqq g^+$. Let us mention that in (2.7) we have only the sum for $|\alpha| = r$ when $r > d/p$ and the two sums for $|\alpha| = r$ and $|\alpha| = l$ when $r \leqq d/p$. The use of such complicated expression in the definition (2.7) is forced by two reasons — the treatment of the multivariate case and the necessity to make difference between the behaviour of the function in the interior of the domain and near to the boundary.

In the second half of the section we collect some lemmas which will be used later.

Lemma 1. *Let $f \in W_p^l$. Then f is equivalent to $F \in C(\Omega)$ and*

$$\|F\|_{\infty(\Omega)} \leqq c\{\|f\|_{p(\Omega)} + \sum_{|\alpha|=l} \|D^\alpha f\|_{p(\Omega)}\}.$$

This lemma follows from Theorems 18.10 and 18.11 in [1, p. 302, 303] because of $l \geqq [d/p]+1 > d/p$.

Lemma 2. *Let $g \in W_p^l$. Then g is equivalent to $G \in C(\Omega)$ and for every $x \in \Omega$ and $0 < t \leqq 1/2$ we have*

$$\|G\|_{\infty(U(t,x))} \leqq c\ \Psi(t,\ x)^{-1/p}\{\|g\|_{p(U(t,x))} + \sum_{|\alpha|=l} \Psi^\alpha(t,\ x)\|D^\alpha g\|_{p(U(t,x))}\}.$$

P r o o f. Let $U(t,\ x) = \prod_{s=1}^{d} [a_s,\ b_s]$. We have $\psi(t,\ x_s) \leqq b_s - a_s \leqq 2\psi(t,\ x_s)$. If we set $f(y_1, \dots, y_d) = g(x_1, \dots, x_d)$, $x_s = (a_s + b_s)/2 + (b_s - a_s)y_s/2$, then Lemma 1 and (2.3) will give the statement of the lemma.

Lemma 3. *Let $u,\ v \in [-1,\ 1]$ and $0 < t \leqq 1/2$. Then* i) *If $|u - v| \leqq \psi(t,\ u)$ then $\psi(t,\ u) \leqq 6\psi(t, v)$ and $\psi(t, v) \leqq 4\psi(t, u)$;* ii) *$\operatorname{meas}\{u : [u - \psi(t,\ u),\ u + \psi(t, u)] \ni v\} \leqq 12\,\psi(t, v)$.*

P r o o f. i) follows from inequality (2.5) in K. I v a n o v [5]. If $|v - u| \leqq \psi(t,\ u)$ then $\psi(t,\ u) \leqq 6\psi(t,\ v)$ by i) and hence $\{u : [u - \psi(t,\ u),\ u + \psi(t,\ u)] \ni v\} \subset [v - 6\psi(t,\ v),\ v + 6\psi(t, v)]$, which proves ii).

Lemma 4. *Let $G \in L_p(\Omega)$ and $0 < t \leqq 1/2$. Then*

$$\| \Psi(t, \cdot)^{-1/p} \| G \|_{p(U(t,\cdot))} \|_{p(\Omega)} \leqq c \| G \|_{p(\Omega)}.$$

Proof. Set $G(x)=0$ for $x \in \mathrm{R}^d \setminus \Omega$. Using Lemma 3 i), ii) and Fubini's theorem we have

$$\| \Psi(t,\cdot)^{-1/p} \| G \|_{p(U(t,\cdot))} \|_{p(\Omega)}$$

$$=\Big(\int_{-1}^{1} \cdots \int_{-1}^{1} \Psi(t, x)^{-1} \int_{x_1-\psi(t,x_1)}^{x_1+\psi(t,x_1)} \cdots \int_{x_d-\psi(t,x_d)}^{x_d+\psi(t,x_d)} |G(y_1, \dots, y_d)|^p dy_1 \dots dy_d dx_1 \dots dx_d \Big)^{1/p}$$

$$\leqq \Big(\int_{-1}^{1} \cdots \int_{-1}^{1} 4^d \int_{x_1-\psi(t,x_1)}^{x_1+\psi(t,x_1)} \cdots \int_{x_d-\psi(t,x_d)}^{x_d+\psi(t,x_d)} \Psi(t, y)^{-1} |G(y_1, \dots, y_d)|^p dy_1 \dots dy_d dx_1 \dots dx_d \Big)^{1/p}$$

$$\leqq \Big(\int_{-1}^{1} \cdots \int_{-1}^{1} 4^d \prod_{s=1}^{d} \frac{\operatorname{meas}\{x_s : [x_s - \psi(t, x_s), x_s + \psi(t, x_s)] \ni y_s\}}{\psi(t, y_s)} |G(y_1, \dots, y_d)|^p dy_1 \dots dy_d \Big)^{1/p}$$

$$\leqq c \| G \|_{p(\Omega)}.$$

Lemma 5. *Let $g \in W_p^l$. Then for every $x \in \Omega$ and $0 < t \leqq 1/2$ there exists $R \in H_{r-1}^d$ such that*

$$\| g - R \|_{p(U(t,x))} \leqq c \sum_{|\beta|=r} \Psi^\beta(t, x) \| D^\beta g \|_{p(U(t,x))}.$$

Proof. From a generalization of Whitney's theorem (see e.g. [6]) for any $f \in W_p^l$ we get $Q \in H_{r-1}^d$ such that

$$\| f - Q \|_{p(\Omega)} \leqq c \omega_r(f;\ 1/r)_{p(\Omega)} \leqq c \sum_{|\beta|=r} \| D^\beta f \|_{p(\Omega)}$$

because of $r \leqq l$. Applying in this inequality the same linear change of the variables as in the proof of Lemma 2 we prove the lemma.

Lemma 6. *Let g be bounded and measurable in Ω. Then $E_{r-1}(g)_\infty \leqq c \omega_r(f;\ \Omega)$.*

This is a generalization of Whitney theorem. A proof can be found in [6].

Lemma 7. $\omega_r(f;\ \Omega) \leqq c \tau_r^*(f;\ 1/2)_p$.

Proof. Let y, h be such that $|h| \leqq 1/(4r)$ and $\omega_r(f;\ 1/(4r))_\infty \leqq 2 |\Delta_h^r f(y)|$. Then for every $x \in \Omega$, $|x - y - hr/2| \leqq 1/8$ we have $\omega_r(f;\ 1/(4r))_\infty \leqq 2\omega_r(f;\ U(1/2, x))$. Hence $\omega_r(f;\ 1/(4r))_\infty \leqq 2.8^{d/p} \tau_r^*(f;\ 1/2)_p$ and $\omega_r(f;\ \Omega) = \omega_r(f;\ 2/r)_\infty \leqq 8^r \omega_r(f;\ 1/(4r))_\infty \leqq 2.8^{r+d/p} \tau_r^*(f;\ 1/2)_p$, which proves the lemma.

Lemma 8. *For any $0 < \delta \leqq \Delta$ and $P \in H_{r-1}^1$ we have*

$$\| P \|_{\infty[-\delta, \Delta]} \leqq c (\Delta/\delta)^{r-1} \| P \|_{\infty[-\delta, \delta]}.$$

The proof follows immediately from the extremal property of the Chebyshev polynomials to grow faster than any other polynomial of the same degree out of $[-1, 1]$.

Lemma 9. *For any* $0<\delta\leqq\Delta$, $Q\in H^1_{r-1}$ *and* $f\in L_\infty[-\delta,\Delta]$ *we have*

$$\|f-Q\|_{\infty[-\delta,\Delta]}\leqq c(\Delta/\delta)^{r-1}(\omega_r(f;[-\delta,\Delta])+\|f-Q\|_{\infty[-\delta,\delta]}).$$

Proof. The Whitney theorem (see Lemma 6) gives an $R\in H^1_{r-1}$, such that

$$(2.8)\qquad \|f-R\|_{\infty[-\delta,\Delta]}\leqq c\omega_r(f;[-\delta,\Delta]).$$

Lemma 8 with $P=Q-R$ gives

$$\|f-Q\|_{\infty[-\delta,\Delta]}\leqq\|f-R\|_{\infty[-\delta,\Delta]}+\|R-Q\|_{\infty[-\delta,\Delta]}$$

$$\leqq\|f-R\|_{\infty[-\delta,\Delta]}+c(\Delta/\delta)^{r-1}\|R-Q\|_{\infty[-\delta,\delta]}$$

$$\leqq\|f-R\|_{\infty[-\delta,\Delta]}+c(\Delta/\delta)^{r-1}(\|f-R\|_{\infty[-\delta,\delta]}+\|f-Q\|_{\infty[-\delta,\delta]})$$

which proves the lemma in view of (2.8).

Lemma 10. *Let* $d=1$, $y\in[z_\rho, z_{\rho+1}]$ *and* $x\in[z_\sigma, z_{\sigma+1}]$. *Then* $|x-y|\leqq\psi(\pi(|\sigma-\rho|+1)/N, y)$.

Proof. Let $\sigma\leqq\rho$ (the case $\sigma>\rho$ is similar). Set $\xi=\pi(\rho-\sigma+1)/2N$, $\eta=\pi-(\rho+\sigma+1)\pi/2N\, y'$-arccos$y$. Then $|y'-\eta|\leqq\xi$. We have $|x-y|\leqq z_{\rho+1}-z_\sigma=2\sin\xi\,\sin\eta=2\sin\xi\,\sin(y'+\eta-y')=2\sin\xi(\sin y'\,\cos(\eta-y')+\cos y'.\sin(\eta-y'))\leqq2\sin\xi(\sin y'+\sin\xi)\leqq2\xi\sin y'+2\xi^2\leqq\psi(\pi(\rho-\sigma+1)/N, y)$.
Which proves the lemma.

Lemma 11. *Let f be bounded and measurable in* Ω *and let* $R\in H^d_{r-1}$ *be such that*

$$\|f-R\|_{\infty(\Omega_i)}\leqq c\omega_r(f;\Omega_i)\quad(i\in Z).$$

Then for any $x\in\Omega_j$ ($j\in Z$, $v=\max\{|i_s-j_s|, s=1,2,\ldots,d\}$) *we have*

$$\|f-R\|_{\infty(\Omega_j)}\leqq c(1+v)^{2r-2}\omega_r(f;U(\pi(v+1)/N, x)).$$

Proof. Let y be any point in Ω_j. Denote by X the center of the rectangle Ω_j and by J the line determine by y and X. Let $I=[Y,y]$ be the smallest segment in J containing y and $J\cap\Omega_i$. Therefore for some s, $s=1,2,\ldots,d$ we have $|Y_s-X_s|=(z_{i_s+1}-z_{i_s})/2$. Set $i_s=\sigma$. Then $|y-X|/|Y-X|=|y_s-X_s|/|Y_s-X_s|$ because X, Y and y are colinear. Hence

$$|y-X|/|Y-X|\leqq2(z_{\sigma+1}-z_\sigma)^{-1}\max\{z_{\sigma+1+v}-z_\sigma,\ z_{\sigma+1}-z_{\sigma-v}\}$$

$$=2\max\Big\{\frac{\cos\pi(1-(\sigma+1+v)/N)-\cos\pi(1-\sigma/N)}{\cos\pi(1-(\sigma+1)/N)-\cos\pi(1-\sigma/N)},$$

$$\frac{\cos\pi(1-(\sigma+1)/N)-\cos\pi(1-(\sigma-v)/N)}{\cos\pi(1-(\sigma+1)/N)-\cos\pi(1-\sigma/N)}\Big\}$$

$$\leqq2\frac{1+\cos\pi(1-(1+v)/N)}{1+\cos\pi(1-1/N)}=2\frac{\sin^2(\pi(1+v)/2N)}{\sin^2(\pi/2N)}\leqq2(1+v)^2.$$

Applying Lemma 9 for the restrictions of f and R on the segment $I=[Y, y]$ and Lemma 6 we get

$$|f(y)-R(y)|\leqq c(1+v)^{2r-2}(\omega_r(f;\ I)+\omega_r(f;\ \Omega_i\cap J)).$$

But y was arbitrary point in Ω_j. Hence

$$\|f-R\|_{\infty(\Omega_j)}\leqq c(1+v)^{2r-2}\omega_r(f;\ \Omega_{j,i}),$$

where $\Omega_{j,i}$ is the convex hull of $\Omega_j\cup\Omega_i$. Now by Lemma 10 we get $\Omega_{j,i}\subset U(\pi(v+1)/N,\ x)$ for any $x\in\Omega_j$ and hence $\omega_r(f;\ \Omega_{j,i})\leqq 2\omega_r(f;\ U(\pi(v+1)/N,x))$. This proves the lemma.

3. Equivalence of the onesided weighted K-functional and the average moduli

In this section we prove that the K-functional (2.7) and the average moduli (2.6) are equivalent.

Theorem 1. *Let* $0<t\leqq 1/2$. *Then for every* $f\in L_p(\Omega)$ *we have*

$$\tau_r^*(f;\ t)_p\leqq cK_r^*(f;\ t)_p; \tag{3.1}$$

$$K_r^*(f;\ t)_p\leqq c\tau_r^*(f;\ t)_p. \tag{3.2}$$

Proof. Let us begin with the proof of (3.1). Let $g^{\pm}\in W_p^l\cap C,\ g^-\leqq f\leqq g^+$. By (2.5) we have

$$\tau_r^*(f;\ t)_p\leqq\tau_r^*(f-g^-;\ t)_p+\tau_r^*(g^-;\ t)_p. \tag{3.3}$$

Using Lemma 2 with $G=g=g^+-g^-$ and Lemma 3, i) we get

$$\omega_r(f-g^-;\ U(t,\ x))\leqq 2^r\|f-g^-\|_{\infty(U(t,x))}\leqq 2^r\|g^+-g^-\|_{\infty(U(t,x))}$$

$$\leqq c\Psi(t,\ x)^{-1/p}\{\|g^+-g^-\|_{p(U(t,x))}+\sum_{|\alpha|=l}\Psi^\alpha(t,\ x)\|D^\alpha(g^+-g^-)\|_{p(U(t,x))}\}$$

$$\leqq c\Psi(t,\ x)^{-1/p}\{\|g^+-g^-\|_{p(U(t,x))}+\sum_{|\alpha|=l}\|\Psi^\alpha(t)D^\alpha(g^+-g^-)\|_{p(U(t,x))}\}$$

$$\leqq c\Psi(t,\ x)^{-1/p}\{\|g^+-g^-\|_{p(U(t,x))}+\sum_{|\alpha|=l}(\|\Psi^\alpha(t)D^\alpha g^+\|_{p(U(t,x))}+\|\Psi^\alpha(t)D^\alpha g^-\|_{p(U(t,x))})\}.$$

Taking L_p norm in the above inequality, from Lemma 4 we get

$$\tau_r^*(f-g^-;\ t)_p\leqq c\{\|g^+-g^-\|_{p(\Omega)}+\sum_{|\alpha|=l}(\|\Psi^\alpha(t)D^\alpha g^+\|_{p(\Omega)} \tag{3.4}$$

$$+\|\Psi^\alpha(t)D^\alpha g^-\|_{p(\Omega)})\}.$$

In order to estimate the second term in the right-hand side of (3.3) we apply Lemmas 2, 3, i) and 5. Let R be the polynomial from Lemma 5. We have

$$\omega_r(g^-;\ U(t,\ x))=\omega_r(g^- - R;\ U(t,\ x))\leqq 2^r\|g^- - R\|_{\infty(U(t,x))}$$

$$\leqq c\Psi(t,\ x)^{-1/p}\{\|g^- - R\|_{p(U(t,x))} + \sum_{|\alpha|=l} \Psi^\alpha(t,\ x)\|D^\alpha g^-\|_{p(U(t,x))}\}$$

$$\leqq c\Psi(t,\ x)^{-1/p} \sum_{|\alpha|=r,l} \Psi^\alpha(t,\ x)\|D^\alpha g^-\|_{p(U(t,x))}$$

$$\leqq c\Psi(t,\ x)^{-1/p} \sum_{|\alpha|=r,l} \|\Psi^\alpha(t)D^\alpha g^-\|_{p(U(t,x))}.$$

Taking L_p norm in the above inequality, from Lemma 4 we get

$$\tau_r^*(g^-;\ t)_p\leqq c \sum_{|\alpha|=r,l} \|\Psi^\alpha(t)D^\alpha g^-\|_{p(\Omega)}. \tag{3.5}$$

From (3.3), (3.4) and (3.5) we have

$$\tau_r^*(f;\ t)_p\leqq c\{\|g^+ - g^-\|_{p(\Omega)} + \sum_{|\alpha|=r,l} (\|\Psi^\alpha(t)D^\alpha g^+\|_{p(\Omega)} + \|\Psi^\alpha(t)D^\alpha g^-\|_{p(\Omega)})\}.$$

Taking infinum on $g^\pm\in W_p^l$, $g^-\leqq f\leqq g^+$ in the above inequality we prove (3.1).

Now we turn our attention to (3.2). We set $N=[2\pi/t]+1$ and use the notation for Ω_j, Ω_j' and μ_j from the beginning of Section 2. Let $u\in[z_{v-1},\ z_{v+1}]$ for some $v=1,\ 2\dots,\ N-1$. Because of concavity of $\psi(t,\cdot)$ we have

$$\psi(t,\ u)\geqq\min\{\psi(t,\ z_{v-1}),\ \psi(t,\ z_{v+1})\}.$$

Moreover,

$$0\leqq z_{v+1}-z_{v-1}=2\sin(\pi/N)\sin(\pi-(v+1)\pi/N+\pi/N)$$

$$=2\sin(\pi/N)(\sin(\pi-(v+1)\pi/N)\cos(\pi/N)+\sin(\pi/N)\cos(\pi-(v+1)\pi/N))$$

$$\leqq(2\pi/N)(\sqrt{1-z_{v+1}^2}+\pi/N)\leqq\psi(t,\ z_{v+1})$$

and similarly $z_{v+1}-z_{v-1}\leqq\psi(t,\ z_{v-1})$. Therefore $z_{v+1}-z_{v-1}\leqq\psi(t,\ u)$ for any $u\in[z_{v-1},\ z_{v+1}]$ and $[z_{v+1},\ z_{v-1}]\subset[u-\psi(t,\ u),\ u+\psi(t,\ u)]$. Hence

$$\Omega_j'\subset U(t,\ x) \quad \text{for any} \quad x\in\Omega_j'. \tag{3.6}$$

Also $z_{v+1}-z_{v-1}=2\sin(\pi/N)\sin(\pi-v\pi/N)\geqq\psi(2/N,\ z_{v-1})$ because of $\sin(\pi-v\pi/N)=\sqrt{1-z_v^2}$ and $\sin(\pi-v\pi/N)\geqq\sin(\pi/N)\geqq 2/N$. Hence $c\psi(t,\ z_v)\leqq z_{v+1}-z_{v-1}$, $c\Psi(t,\ z_j)\leqq\operatorname{meas}\Omega_j'$ and $c\Psi(t,\ x)\leqq\operatorname{meas}\Omega_j'$ for every $x\in\Omega_j'$ because of (3.6) and Lemma 3, i). The last inequality together with (3.6) and (2.3) gives

$$c\Psi(t,\ x)\leqq\operatorname{meas}\Omega_j'\leqq c\Psi(t,\ y) \text{ for every } x,\ y\in\Omega_j'. \tag{3.7}$$

For $j\in Z'$ we set

$$e_j(f)=\inf\{\|f-R\|_{\infty(\Omega_j')}:R\in H^d_{r-1}\}=\|f-R_j\|_{\infty(\Omega_j')},$$

where $R_j\in H^d_{r-1}$. From Lemma 6 and (3.6) we have

$$e_j(f)\leqq\omega_r(f;\ \Omega_j')\leqq c\omega_r(f;\ U(t,\ x))\ \text{for every}\ x\in\Omega_j'. \tag{3.8}$$

We set $R^{\pm}_j(x)=R_j(x)\pm e_j(f)$. For any $x\in\Omega_j'$ we have

$$R_j^-(x)\leqq f(x)\leqq R_j^+(f). \tag{3.9}$$

Finally we define

$$g^{\pm}(x)=\sum_{j\in Z'}\mu_j(x)\,R^{\pm}_j(x)\in C^\infty(\Omega). \tag{3.10}$$

From (3.10), (3.9), (2.1) and (2.2) we get

$$g^-(x)\leqq f(x)\leqq g^+(x)\ \text{for any}\ x\in\Omega. \tag{3.11}$$

From (3.10) we have $g^+(x)-g^-(x)=2\sum_{j\in Z'}\mu_j(x)e_j(f)$ and hence

$0\leqq g^+(x)-g^-(x)\leqq c\omega_r(f;\ U(t,\ x))$, because of (3.8), (2.1) and (2.2). Therefore

$$\|g^+-g^-\|_{p(\Omega)}\leqq c\tau^*_r(f;\ t)_p. \tag{3.12}$$

Fix α, $|\alpha|=r$ or $|\alpha|=l$. Let $x\in\Omega_j$, $j\in Z$. From (3.10), (2.1) and (2.2) we have

$$g^+(x)=R_j^+(x)+\sum_{\varepsilon_s=0,1}\mu_{j+\varepsilon}(x)(R^+_{j+\varepsilon}(x)-R_j^+(x))$$

and therefore $(D^\alpha R_j^+\equiv 0)$,

$$D^\alpha g^+(x)=\sum_{\varepsilon_s=0,1}\ \sum_{0\leq\beta\leq\alpha}\binom{\alpha}{\beta}D^{\alpha-\beta}\mu_{j+\varepsilon}(x)\,D^\beta(R^+_{j+\varepsilon}(x)-R_j^+(x)).$$

From (3.6), Lemma 3, i), (3.7), the definition of μ_j and R_j and Markov's inequality we have

$$\|\Psi^\alpha(t)\,D^\alpha g^+\|_{p(\Omega_j)}\leqq c\Psi^\alpha(t,\ z_j)\,\|D^\alpha g^+\|_{p(\Omega_j)}$$

$$\leqq c\Psi^\alpha(t,\ z_j)\sum_{\varepsilon_s=0,1}\ \sum_{0\leq\beta\leq\alpha}\binom{\alpha}{\beta}\|D^{\alpha-\beta}\mu_{j+\varepsilon}\|_{\infty(\Omega_j)}\|D^\beta(R^+_{j+\varepsilon}-R_j^+)\|_{p(\Omega_j)}$$

$$\leqq c\Psi^\alpha(t,\ z_j)\sum_{\varepsilon_s=0,1}\ \sum_{0\leq\beta\leq\alpha}(\Psi^{\alpha-\beta}(t,\ z_j))^{-1}.(\Psi^\beta(t,\ z_j))^{-1}\|R^+_{j+\varepsilon}-R_j^+\|_{p(\Omega_j)}$$

$$\leqq c\sum_{\varepsilon_s=0,1}\|R^+_{j+\varepsilon}-R_j^+\|_{p(\Omega_j)}\leqq c\sum_{\varepsilon_s=0,1}(\|f-R_j^+\|_{p(\Omega_j)}+\|f-R^+_{j+\varepsilon}\|_{p(\Omega_j)})$$

$$\leqq c\,\|\omega_r(f;\ U(t,\cdot))\|_{p(\Omega_j)}.$$

Summating the above inequality on $j\in Z$ we get

$$\text{(3.13)}\qquad \|\Psi^\alpha(t)D^\alpha g^+\|_{p(\Omega)}\leqq c\tau_r^*(f;\,t)_p \quad \text{for every}\quad \alpha,\ |\alpha|=r,\ l.$$

Similarly

$$\text{(3.14)}\qquad \|\Psi^\alpha(t)\,D^\alpha g^-\|_{p(\Omega)}\leqq c\tau_r^*(f;\,t)_p \quad \text{for every}\quad \alpha,\ |\alpha|=r,\ l.$$

From (2.7), (3.11), (3.12), (3.13) and (3.14) we finally get

$$K_r^*(f;\,t)_p\leqq \|g^+-g^-\|_{p(\Omega)}+\sum_{|\alpha|=r,\ l}(\|\Psi^\alpha(t)D^\alpha g^+\|_{p(\Omega)}+\|\Psi^\alpha(t)D^\alpha g^-\|_{p(\Omega)})$$
$$\leqq c\tau_r^*(f;\,t)_p.$$

This proves (3.2) and completes the proof of Theorem 1.

Corollary 1. *The conclusion of Theorem* 1 *holds for every* $t>0$.

Proof. We have to investigate only the case $t>1/2$. From (2.4), Lemma 7, (3.1) with $t=1/2$ and the monotonicity of K^* with respect to t we get

$$\tau_r^*(f;\,t)_p\leqq 2^{d/p}\omega_r(f;\,\Omega)\leqq c\tau_r^*(f;1/2)_p$$
$$\leqq cK_r^*(f;1/2)_p\leqq cK_r^*(f;t)_p,$$

which proves (3.1) for every t.

From Lemma 6 we have $R\in H_{r-1}^d$, such that

$$\|f-R\|_{\infty(\Omega)}\leqq c\omega_r(f;\,\Omega).$$

We set $g^\pm=R\pm\|f-R\|_{\infty(\Omega)}\in H_{r-1}^d$. Then by (2.7), Lemma 7 and (2.6) we obtain

$$K_r^*(f;\,t)_\infty\leqq \|g^+-g^-\|_p=2^{1+d/p}\|f-R\|_{\infty(\Omega)}\leqq c\omega_r(f;\,\Omega)$$
$$\leqq c\tau_r^*(f;\,1/2)\leqq c\tau_r^*(f;\,t)_p.$$

Corollary 2. *For every* $\lambda>1$ *we have*

$$\tau_r^*(f;\,\lambda t)_p\leqq c\lambda^{2l}\tau_r^*(f;\,t)_p.$$

This corollary is immediate consequence from Corollary 1 and the inequality

$$K_r^*(f;\,\lambda t)_p\leqq \lambda^{2l}K_r^*(f;\,t)_p$$

which follows directly from (2.7).

4. Algebraic operators for onesided approximation

Let N be a fixed integer. For $\nu=0,1,2,\dots,N-1$ set $u_\nu=\pi-(2\nu+1)\pi/(2N)$ and

$$\varphi_\nu(u)=\sin^4\frac{\pi}{4N}\cdot\left(\frac{\sin^4 N(u-u_\nu)}{\sin^4(u-u_\nu)/2}+\frac{\sin^4 N(u+u_\nu)}{\sin^4(u+u_\nu)/2}\right).$$

φ_ν are even positive trigonometric polynomials of degree $4N-2$ such that $\varphi_\nu(u)\geqq 1$ for $u\in[\pi-(\nu+1)\pi/N,\ \pi-\nu\pi/N]$.

Making the substitution $u=\arccos v$, $v\in[-1,\ 1]$ we define

$$F_\nu(v)=F_{\nu,N}(v)=\varphi_\nu(\arccos v).$$

Lemma 12. *For $\nu=0,1,\dots,\ N-1$ $F_{\nu,N}$ are positive algebraic polynomials of degree $4N-2$ such that*

$$F_{\nu,N}(v)\geqq 1 \ \textit{for}\ v\in[z_\nu,\ z_{\nu+1}]; \tag{4.1}$$

$$\int_{-1}^{1} F_{\nu,N}(v)dv\leqq c(z_{\nu+1}-z_\nu); \tag{4.2}$$

$$1\leqq \sum_{\nu=0}^{N-1} F_{\nu,N}(v)\leqq c \ \textit{for every}\ v\in[-1,\ 1]. \tag{4.3}$$

Proof. $F_\nu\in H^1_{4N-2}$ because φ_ν is an even trigonometric polynomial. The positivity and (4.1) for F_ν follow from the corresponding properties of φ_ν.

$$\int_{-1}^{1} F_\nu(v)dv=\int_0^\pi \varphi_\nu(u)\sin u\,du=\sin^4\frac{\pi}{4N}\int_{-\pi}^{\pi}\frac{\sin^4 N(u-u_\nu).|\sin u|}{\sin^4(u-u_\nu)/2}du$$

$$=\sin^4\pi/(4N)\int_{-\pi}^{\pi}(\sin^4 Nu/\sin^4 u/2)\,|\sin(u+u_\nu)|\,du$$

$$\leqq cN^{-4}\int_{-\pi}^{\pi}(\sin^4 Nu/\sin^4 u/2)(|\sin u|+|\cos u|\sin u_\nu)du$$

$$\leqq cN^{-4}(N^2+N^3\sin u_\nu)\leqq cN^{-1}\sin u_\nu$$

$$\leqq c.2.\ \sin\pi/(2N).\sin(\pi-(2\nu+1)\pi/(2N))=c(z_{\nu+1}-z_\nu)$$

because $\sin u_\nu\geqq\sin\pi/(2N)\geqq 1/N$ for $\nu=0,1,\dots,\ N-1$. This proves (4.2). The first inequality in (4.3) follows from (4.1). Consider

$$G(\theta)=\sin^4\pi/(4N)\sum_{\nu=0}^{2N-1}\sin^4 N(\theta-\nu\pi/N)/\sin^4(\theta-\nu\pi/N)/2.$$

For $|\theta|\leqq\pi/(2N)$ we have

$$G(\theta)\leqq cN^{-4}\Big(N^4+\sum_{v=1}^{\infty}1/(v/N)^4\Big)\leqq c.$$

But G is π/N periodic and hence $G(\theta)\leqq c$ for any θ. If we set $\theta=u-\pi/(2N)$ in this inequality we get $\sum_{\nu=0}^{N-1}\varphi_\nu(u)\leqq c$ for any u, which proves the second inequality in (4.3) and completes the proof of the lemma.

In the multivariate case we define $(j\in \mathbf{Z})$

$$\Phi_{j,N}(x)=\prod_{s=1}^{d} F_{j_s,N}(x_s). \tag{4.4}$$

From Lemma 12 and (4.4) we immediately get

Lemma 13. *Let* $j\in\mathbf{Z}$. *Then*

$$\Phi_{j,N}\in H^d_{d(4N-2)},\ \Phi_{j,N}\geqq 0; \tag{4.5}$$

$$\Phi_{j,N}(x)\geqq 1 \textit{ for } x\in\Omega_j; \tag{4.6}$$

$$\|\Phi_{j,N}\|_{1(\Omega)}\leqq c\operatorname{meas}\Omega_j; \tag{4.7}$$

$$1\leqq\sum_{j\in\mathbf{Z}}\Phi_{j,N}(x)\leqq c \textit{ for every } x\in\Omega. \tag{4.8}$$

Lemma 14. *Let* $a_j\geqq 0$, $j\in\mathbf{Z}$. *Then*

$$\|\sum_{j\in\mathbf{Z}}a_j\Phi_{j,N}\|_{p(\Omega)}\leqq c(\sum_{j\in\mathbf{Z}}a_j^p\operatorname{meas}\Omega_j)^{1/p}.$$

Proof. Let $\bar c$ be the constant in the right-hand side of (4.8). Therefore for every $x\in\Omega$ by Jensen inequality we have

$$(\sum_{j\in\mathbf{Z}}a_j\Phi_{j,N}(x)/\bar c)^p\leqq\sum_{j\in\mathbf{Z}}a_j^p\Phi_{j,N}(x)/\bar c \text{ and}$$

$$(\sum_{j\in\mathbf{Z}}a_j\Phi_{j,N}(x))^p\leqq\bar c^{p-1}\sum_{j\in\mathbf{Z}}a_j^p\Phi_{j,N}(x),$$

which proves the lemma in view of (4.7).

Now we are going to construct a polynomial $Q_N(f)$ providing a good local approximation to a given function f.

Denote by $T_N(u)=\cos(N\arccos u)$ the Chebyshev polynomial of degree N. The zeros of T_N are the points $u_\nu=\cos(\pi-(2\nu+1)\pi/(2N))$, $\nu=0,1,\dots,N-1$. Let $k=r+l+1$. Set $\gamma_\nu=1/\int_{-1}^{1}(T_N(u)/(u-u_\nu))^{2k}du$.

Denote

$$P_0(v)=1-\gamma_1\int_{-1}^{v}(T_N(u)/(u-u_1))^{2k}du,$$

$$P_\nu(v)=\gamma_\nu\int_{-1}^{v}(T_N(u)/(u-u_\nu))^{2k}du-\gamma_{\nu+1}\int_{-1}^{v}(T_N(u)/(u-u_{\nu+1}))^{2k}du$$

for $\nu=1,2,\dots,N-2$ and

$$P_{N-1}(v)=\gamma_{N-1}\int_{-1}^{v}(T_N(u)/(u-u_{N-1}))^{2k}du.$$

These polynomials are defined in D z j a d y k [2, § VII, 4].

Lemma 15. *For* $\nu=0,1,\ldots,N-1$ *we have* $P_\nu\in H^1_{2k(N-1)+1}$,

(4.9) $$\sum_{\nu=0}^{N-1}P_\nu(v)=1 \text{ for every } v\in\mathbf{R} \text{ and}$$

(4.10) $$|P_\nu(v)|\leqq c(|\sigma-\nu|+1)^{-2k+1} \text{ if } v\in[z_\sigma, z_{\sigma+1}],$$

$$\sigma=0,1,\ldots,N-1.$$

P r o o f. u_ν is a zero of T_N and so $T_N(u)/(u-u_\nu)$ is a polynomial of degree $N-1$. Hence $P_\nu\in H_{2k(N-1)+1}$. (4.9) follows from the definition. From §4, Chapter VII in D z j a d y k [2] we get

$$|P_\nu(v)|\leqq c(1+|v-u_\nu|/\psi(1/N, u_\nu))^{-2k+1},$$

which proves (4.10), because of $1+|v-u_\nu|/\psi(1/N, u_\nu)\geqq(1+|\nu-\sigma|)/2$ whenever $v\in[z_\sigma, z_{\sigma+1}]$.

For $j\in\mathbf{Z}$ we set $I_j(x)=\prod_{s=1}^{d}P_{j_s}(x_s)$. From this definition and Lemma 15 we immediately get

Lemma 16. *For* $j\in\mathbf{Z}$ *we have* $I_j\in H^d_{2k(N-1)d+d}$,

(4.11) $$\sum_{j\in\mathbf{Z}}I_j(x)=1 \text{ for every } x\in\mathbf{R}^d \text{ and}$$

(4.12) $$|I_j(x)|\leqq c\prod_{s=1}^{d}(1+|i_s-j_s|)^{-2k+1} \text{ for any } x\in\Omega_i,\ i\in\mathbf{Z}.$$

Let R_j by the polynomial of degree $r-1$ of the best approximation to f on Ω_j, i. e. $\|f-R_j\|_{\infty(\tilde{\Omega}_j)}=E_{r-1}(f)_{\infty(\Omega_j)}$. We set

(4.13) $$Q_N(f;x)=\sum_{j\in\mathbf{Z}}I_j(x)R_j(x)\in H^d_{2k(N-1)d+d+r-1}.$$

The polynomial $Q_N(f)$ provides a good local approximation to the function f. Finally we define

(4.14) $$Q^{\pm}_N(f;x)=Q_N(f;x)\pm\sum_{j\in\mathbf{Z}}\Phi_{j,N}(x)\|f-Q_N(f)\|_{\infty(\Omega_j)}.$$

Theorem 2. *For any bounded and measurable in* Ω *function* f *we have*

(4.15) $$Q^{\pm}_N(f)\in H^d_{2(r+l+1)(N-1)d+d+r-1};$$

(4.16) $$Q^-_N(f;x)\leqq f(x)\leqq Q^+_N(f;x) \text{ for any } x\in\Omega;$$

(4.17) $$\|Q_N^+(f)-Q_N^-(f)\|_{p(\Omega)}\leqq c\tau_r^*(f;\ 1/N)_{p(\Omega)}.$$

Proof. (4.15) follows from (4.14), (4.13) and (4.5).

Let $x\in\Omega_j$ for some $j\in Z$. Then $\Phi_{j,N}(x)\geqq 1$ because of (4.6), which together with (4.14) and positivity of $\Phi_{j,N}$ – (4.5) – gives (4.16).

From (4.13) and (4.11) we have

(4.18) $$f(y)-Q_N(f;\ y)=\sum_{j\in Z} I_j(y)(f(y)-R_j(y)).$$

Let x be any point in Ω_j. From (4.18), (4.12) and Lemma 11 we get

(4.19) $$\|f-Q_N(f)\|_{\infty(\Omega_j)}\leqq \sum_{j\in Z}\|I_i\|_{\infty(\Omega_j)}\|f-R_i\|_{\infty(\Omega_j)}$$

$$\leqq \sum_{j\in Z}\prod_{s=1}^{d}(1+|i_s-j_s|)^{-2k+1}(1+v_i)^{2r-2}\omega_r(f;\ U(\pi(v_i+1)/N,\ x)),$$

where $v_i=\max\{|i_s-j_s|:s=1,2,\dots,\ d\}$. We have $0\leqq v_i\leqq N-1$. From $-2k+1<-1$ for any v, $0\leqq v\leqq N-1$ we obtain

$$\sum_{j\in Z, v_i=v}\ \prod_{s=1}^{d}(1+|i_s-j_s|)^{-2k+1}(1+v_i)^{2r-2}\leqq c(1+v)^{-2k+2r-1}.$$

This inequality, (4.19) and Jensen inequality give

$$\|f-Q_N(f)\|_{\infty(\Omega_j)}\leqq c\sum_{v=0}^{N-1}(1+v)^{-2k+2r-1}\omega_r(f;\ U(\pi(v+1)/N,\ x))$$

for any $x\in\Omega$; and therefore

(4.20) $$\text{meas}\,\Omega_j\,\|f-Q_n(f)\|^p_{\infty(\Omega_j)}$$

$$\leqq c\int_{\Omega_j}\Big(\sum_{j=0}^{N-1}(1+v)^{-2k+2r-1}\omega_r(f;U(\frac{\pi(v+1)}{N},x))\Big)^p dx.$$

Finally from (4.14), Lemma 14, (4.20), Minkowski's inequality and Corollary 2 we get

$$\|Q_N^+(f)-Q_N^-(f)\|_{p(\Omega)}=2\|\sum_{j\in Z}\Phi_{j,N}(\cdot)\,\|f-Q_N(f)\|_{\infty(\Omega_j)}\|_{p(\Omega)}$$

$$\leqq c\Big(\sum_{j\in Z}\text{meas}\Omega_j\|f-Q_N(f)\|^p_{\infty(\Omega_j)}\Big)^{1/p}$$

$$\leqq c\Big(\int_{\Omega}\Big(\sum_{v=0}^{N-1}(1+v)^{-2k+2r-1}\omega_r(f;\ U(\frac{\pi(v+1)}{N},\ x))\Big)^p dx\Big)^{1/p}$$

$$\leqq c\sum_{v=0}^{N-1}(1+v)^{-2k+2r-1}\Big(\int_{\Omega}\omega_r(f;\ U(\frac{\pi(v+1)}{N},\ x))^p dx\Big)^{1/p}$$

$$=c\sum_{\nu=0}^{N-1}(1+\nu)^{-2k+2r-1}\tau_r^*(f;\ \pi(\nu+1)/N)_{p(\Omega)}$$

$$\leqq c\sum_{\nu=0}^{N-1}(1+\nu)^{-2k+2r-1+2l}\tau_r^*(f;\ 1/N)_{p(\Omega)}\leqq c\tau_r^*(f;\ 1/N)_{p(\Omega)}$$

because $-2k+2r-1+2l=-3$. This proves (4.17) and completes the proof of Theorem 2.

Theorem 3. *For* $n\geqq r-1$

$$\tilde{E}_n(f)_{p(\Omega)}\leqq c\tau_r^*(f;\ 1/n)_{p(\Omega)}.$$

Proof. Let $n\geqq d+r-1$. We set $N=[(n-d-r+1)/(2d(r+l+1))]+1\geqq 1$. From Theorem 2 the polynomials $Q_N^{\pm}f=Q_N^{\pm}(f)\in H_n^d$, $Q_N^-f\leqq f\leqq Q_N^+f$ and

$$\tilde{E}_n(f)_{p(\Omega)}\leqq\|Q_N^+f-Q_N^-f\|_{p(\Omega)}\leqq c\tau_r^*(f;\ N^{-1})_{p(\Omega)}\leqq c\tau_r^*(f;\ n^{-1})_{p(\Omega)},$$

where the last inequality follows from Corollary 2. Let $r-1\leqq n<d+r-1$. From Lemma 6 there is $R\in H_{r-1}^d$ such that

$$E_{r-1}(f)_{\infty(\Omega)}=\|f-R\|_{\infty(\Omega)}\leqq c\omega_r(f;\ \Omega).$$

Set $Q_n^{\pm}f=R\pm E_{r-1}(f)_{\infty(\Omega)}$. Then $Q_n^{\pm}f\in H_n^d$, $Q_n^-f\leqq f\leqq Q_n^+f$ and using Lemma 7 and Corollary 2 we get

$$\|Q_n^+f-Q_n^-f\|_{p(\Omega)}=2^{d/p+1}E_{r-1}(f)_{\infty(\Omega)}$$

$$\leqq c\omega_r(f;\ \Omega)\leqq c\tau_r^*(f;\ 1/2)_p\leqq c\tau_r^*(f;\ 1/n)_p$$

which proves the Theorem 3.

In the case $d=1$ Theorem 3 is proved by M. Stoyanova [7] using different arguments.

5. Converse resuts

In this section we shall prove a statement converse to Theorem 3 and hence to obtain a characterization of the best onesided algebraic approximations.

Theorem 4.

$$\tau_r^*(f;\ 1/n)_{p(\Omega)}\leqq cn^{-r}\sum_{\nu=0}^{n}(\nu+1)^{r-1}\tilde{E}_\nu(f)_{p(\Omega)}.$$

Theorem 4 in the univariate case is proved by M. Stoyanova [7]. In the proof of Theorem 4 we shall use

Lemma 17. *For any* $Q\in H_\nu^d$ *and any* α, $|\alpha|=r$, l *we have*

$$\|\Psi^\alpha(1/\nu)\nu^{|\alpha|}D^\alpha Q\|_{p(\Omega)}\leqq c\nu^{|\alpha|}\|Q\|_{p(\Omega)}. \tag{5.1}$$

Proof. (5.1) follows from the corresponding inequality for the univariate case $(0 \leqq k \leqq \max\{r, l\})$

(5.2) $\|(\sqrt{1-x^2}+1/v)^k R^{(k)}(x)\|_{p[-1,1]} \leqq cv^k \|R\|_{p[-1,1]}$ for any $R \in H_v^1$.

From another side (5.2) follows from

$$\|(\sqrt{1-x^2})^k R^{(k)}(x)\|_{p[-1,1]} \leqq cv^k \|R\|_{p[-1,1]} \text{ and} \tag{5.3}$$

$$\|R^{(k)}\|_{p[-1,1]} \leqq cv^{2k} \|R\|_{p[-1,1]}. \tag{5.4}$$

(5.3) is a generalization of Bernstein inequality. It is proved by Potapov [3]. (5.4) is a generalization of Markov inequality and its proof can be found in S. Szegö, E. Hile, J. Tamarkin [8].

Proof of Theorem 4. Let N be such that $2^{N-1} < n \leqq 2^N$. For $k=0, 1, \ldots, N$ set $\tilde{E}_{2^k}(f)_{p(\Omega)} = \|Q_k^+ - Q_k^-\|_{p(\Omega)}$ where $Q_k^\pm \in H_{2^k}^d$, $Q_k^- \leqq f \leqq Q_k^+$. Also we set $\tilde{E}_{2^{-1}}(f)_{p(\Omega)} := \tilde{E}_0(f)_{p(\Omega)} = \|Q_{-1}^+ - Q_{-1}^-\|_{p(\Omega)}$ where $Q_{-1}^\pm \in H_o^d$, $Q_{-1}^- \leqq f \leqq Q_{-1}^+$. From Theorem 1 and (2.7) we get

$$\tau_r^*(f; 1/n)_{p(\Omega)} \leqq cK_r^*(f; 1/n)_p \tag{5.5}$$

$$\leqq c\{\|Q_N^+ - Q_N^-\|_p + \sum_{|\alpha|=r,l} (\|\Psi^\alpha(1/n)D^\alpha Q_N^+\|_p + \|\Psi^\alpha(1/n)D^\alpha Q_N^-\|_p)\}.$$

By the definition of $Q_N^\pm$ we have

$$\|Q_N^+ - Q_N^-\|_p = \tilde{E}_{2^N}(f)_{p(\Omega)} \leqq \tilde{E}_n(f)_{p(\Omega)} \tag{5.6}$$

$$\leqq cn^{-r} \sum_{v=0}^{n} (v+1)^{r-1} \tilde{E}_v(f)_{p(\Omega)}.$$

For any α, $|\alpha|=r$ or $|\alpha|=l$, from Lemma 17 we get

$$n^{|\alpha|}(\|\Psi^\alpha(1/n)D^\alpha Q_N^+\|_p + \|\Psi^\alpha(1/n)D^\alpha Q_N^-\|_p)$$

$$\leqq \sum_{k=0}^{N} (\|\Psi^\alpha(1/n)n^{|\alpha|}D^\alpha(Q_k^+ - Q_{k-1}^+)\|_p + \|\Psi^\alpha(1/n)n^{|\alpha|}D^\alpha(Q_k^- - Q_{k-1}^-)\|_p)$$

$$\leqq \sum_{k=0}^{N} (\|\Psi^\alpha(2^{-k})2^{k|\alpha|}D^\alpha(Q_k^+ - Q_{k-1}^+)\|_p + \|\Psi^\alpha(2^{-k})2^{k|\alpha|}D^\alpha(Q_k^- - Q_{k-1}^-)\|_p)$$

$$\leqq c\sum_{k=0}^{N} 2^{k|\alpha|}(\|Q_k^+ - Q_{k-1}^+\|_p + \|Q_k^- - Q_{k-1}^-\|_p)$$

$$\leqq c\sum_{k=0}^{N} 2^{k|\alpha|}(\|Q_k^+ - f\|_p + \|f - Q_{k-1}^+\|_p + \|Q_k^- - f\|_p + \|f - Q_{k-1}^-\|_p)$$

$$\leqq c\sum_{k=0}^{N} 2^{k|\alpha|}\tilde{E}_{2^{k-1}}(f)_p.$$

Multiplying this inequality by n^{r-l}, $l \geqq r$, we get

$$(5.7)\qquad n^r(\|\Psi^\alpha(1/n)D^\alpha Q_n^+\|_p+\|\Psi^\alpha(1/n)D^\alpha Q_n^-\|_p)$$

$$\leqq c\sum_{k=0}^{N} 2^{kr}\tilde{E}_{2^k-1}(f)_p \leqq c\sum_{\nu=0}^{n}(1+\nu)^{r-1}\tilde{E}_\nu(f)_p$$

for any α, $|\alpha|=r$ or $|\alpha|=l$.
Applying (5.6) and (5.7) in (5.5) we prove the theorem.

Combining Theorems 3 and 4 we obtain

Theorem 5. *Let* $0<\rho<r$. *Then*

$$\tilde{E}_n(f)_p=O(n^{-\rho})\ (n\to\infty)\ \textit{iff}\ \tau_r^*(f;\ t)_p=O(t^\rho)\ (t\to 0+).$$

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Received 07. 03. 1988