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Control of Unreliable Process with Implicit Breakdowns and Mixed Executive Times

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A problem for minimization of the expected total executive time of a task is considered, when the execution process is unreliable and leads to incorrect final results. It is supposed that the unreliability is caused by implicit breakdowns with constant failure rate, the required executive time is a mixture of several times. But tests and copies help shorten the total executive time. An optimal uniform control schedule is determined.

1. Introduction and model description

The present paper continues the authors research [1,2] concerning the minimization of the expected executive time for an unreliable process. It is supposed that there are several tasks which must be executed on a server. The value X of the needed serving time is the muxture

$$X=p_1X_1+\dots+p_rX_r \tag{1}$$

of different values $X_1, X_2, \dots, X_r$ corresponding to the possible type of the executed task. Here p_i and X_i, $i=1,2,\dots,r$ are given, but the arriving task of type i is never known a priory.

During the execution time X the server is subjected to breakdowns which appear successively in a random way, make some affect on the correct current service and form a Poisson process with intensity $\gamma>0$. This breakdowns are implicit — one can discover the breakdown appearance only after doing a special test. If the test is done after the execution time X and shows a breakdown appearance, the execution of this task must be repeated. These repetitions form the total executive time $\tau(x)$ until the successful correct end of the executions.

Several authors like R. M. Chandy [4], I. Kovalenko and L. Stoykova [5], K. Barosov [3] and others note that it is preferable to introduce controller service discipline with carrying out of intermediate tests and copies of the achieved correct execution statement (those instants are known as check-points). If an implicit breakdown is discovered, the repeated actions start from the last

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successful done check-point. The aim of our paper is to determinate that optimal disposition of the check-points on time interval of duration X which minimizes the expected total executive time of a task.

Let $\{a_k\}$ be a sequence of designed time intervals between the consequtive check-points. We say $\{a_k\}$ is a control schedule for our process of unreliable task execution. We show in [2] that under the given conditions (mixed executive times (1), constant breakdown-rate γ and identically distributed with finite expected copy and test durations) the optimal control schedule which minimize $E\tau(X, \{a_k\})$ is piece-wise-uniform on the intervals $(X_i, X_{i+1}]$, $i=1,2,\dots,n$, $X_{n+1}=\infty$. Also a rule for determination of the terms $\{a_k^*\}$ is given in [1].

The serving with a piece-wise-uniform control schedule has some evident deficiencies. During the execution of tasks with different executive times it is necessary to readjust the server for several time intervals, and it leads to additional expenditures for control. It makes sense to find a common uniform control schedule $\{a_k^*\}$ with $a_k^* = a_o^*$, which is optimal and for the case, when the required executive times of the task are mixed in the form (1).

The next principal suppositions form the model:

(i) The breakdowns are implicid (as it is noted above) and form a stationary Poisson process with intensity $\gamma>0$;

(ii) the required executive of a task X has a form (1), where $p_i>0$, $\sum_{i=1}^{r} p_i=1$, and the values are ordered $X_1<X_2<\dots<X_r$. All the values X_i, p_i, $(i=1,2,\dots,r)$ are known;

(iii) each check-point consists of two times δ and θ where δ is the time necessary for making the control test and θ is the corresponding time for making a copy. (If no breakdown is discovered.) These values as well as all the other test- and copy-interval during the execution of the task are mutually independent random variables (with finite mathematical exceptions);

(iv) all the renewals after breakdowns are instantaneous;

(v) all the check-points are equidistant.

Using the notations of [1] we formulate the next optimization problem: Determine the value a^* for which

$$E\tau(X, a^*, \delta, \theta)=\inf_{a>0} E\tau(X, a, \delta, \theta) \tag{2}$$

and the suppositions (i)-(v) are fulfilled.

2. Main results

2.1. The case $r=2$.

Let us describe the process of execution of a task in the case $X=pX_1+(1-p)X_2$ where $P(X_1=\text{const})=P(X_2=\text{const})$, $X_1<X_2$ and $a_k=a$ i.e. when the control schedule $\{a_k\}$ is uniform.

Let $k_i=[\frac{X_i}{a}]$ be the integer part of the value $\frac{X_i}{a}$ $(i=1, 2)$ and

$$\Delta_{k_1k_2} \overset{\text{def}}{=} [\max(\frac{X_1}{k_1+1}, \frac{X_2}{k_2+1}), \min(\frac{X_1}{k_1}, \frac{X_2}{k_2}).$$

Lemma 1. *For given $a\in\Delta_{k_1k_2}$ the expected total executive duration is determined by the relation*

(3) $$T_{k_1k_2}(X,\ a)=pT_{k_1}(X_1,\ a)+(1-p)T_{k_2}(X_2,\ a),$$

where

(4) $$T_{k_i}(X_i,\ a)=k_i(a+E\delta)e^{\gamma a}+k_iE\theta+(X-k_ia+E\delta)e^{\gamma(X_i-k_ia)},\ (i=1,\ 2).$$

Proof. It is shown in [1] that if the required executive time of the task X is a constant equal to $X_i(i=1,\ 2)$, the breakdowns are implicit with constant failure rate $\gamma>0$ and the control schedule is uniform one, determined by interval $a>0$, then the expected total executive time is determined by the expression (4), and the integers $k_i(i=1,\ 2)$ are the shown ones.

Let X be a mixture $X=pX_1+(1-p)X_2,\ x_1<x_2$ and $\tau(X,\ a)$ be the total duration of the task execution. The equation (3) follows from the evident equality

$$\tau(X,\ a)=p\tau(X_1,\ a)+(1-p)\tau(X_2,\ a)$$

after taking mathematical exception of the both sides. Lemma 1 is proved.

Lemma 2. *For any given $a\in(0,\infty)$ the expected total duration of the task execution is representable in the form*

(5) $$E\tau(X,\ a)=\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}I_{k_1k_2}(a)\,T_{k_1k_2}(X,\ a),$$

where $T_{k_1k_2}(X,\ a)$ are determined for any $k_1,k_2=0,1,\dots$ by Lemma 1 and

$$I_{k_1k_2}(a)=\begin{cases}1 & \text{for } a\in\Delta_{k_1k_2}\\ 0 & \text{otherwise}\end{cases}.$$

Proof. The intervals have one of the form

$$\left[\frac{X_1}{k_1+1},\ \frac{X_2}{k_2}\right),\ \left[\frac{X_1}{k_1+1},\ \frac{X_1}{k_1}\right),\ \left[\frac{X_2}{k_2+1},\ \frac{X_1}{k_1}\right)\ \text{and}\ \left[\frac{X_2}{k_2+1},\ \frac{X_2}{k_2}\right).$$

It is easy to see that for any two pairs is fulfilled

$$\Delta_{k_1k_2}\cap\Delta_{k_1'k_2'}=\emptyset.$$

Moreover, it is true that

$$\bigcup_{k_1=0}^{\infty}\bigcup_{k_2=0}^{\infty}\Delta_{k_1k_2}=[0,\infty).$$

Now it is easy to see that for a given $a>0$ the relation $a\in\Delta_{k_1k_2}$ (i. e. $I_{k_1k_2}(a)=1$) is fulfilled if and only if the inequalities

$$\frac{X_1}{k_1+1}\leqq a<\frac{X_1}{k_1}\ \text{and}\ \frac{X_2}{k_2+1}\leqq a<\frac{X_2}{k_2}$$

are true. According to the result of Theorem 2 from [1] $E\tau(X_i, a)=T_{k_i}(X_i, a)$, $i=1, 2$. Therefore the equations (3), (4) and the equation (5) are equivalent. Lemma 2 is proved.

Now we are able to precise the optimal problem (2) for our case: to find that a^* which satisfies (2) with $E\tau(X, a)$ given by the relation (5).

The solution of this problem is given by the next statement:

Theorem 1. *There exists a pair* (k_1^*, k_2^*) *such that*

(6) $$a^* \in [\min(a_1^*, a_2^*), \max(a_1^*, a_2^*)],$$

where $a_i^* = \frac{X_i}{K_i^*+1}$, $(i=1, 2)$ *is determined by the relation*

(7) $$a_i^* = \arg\min_{a>0} E\tau(X_i, a), \ (i=1, 2)$$

as the optimal solution of our control problem in the case of fixed required executive times, i. e. when

(8) $$P(X=X_i=\text{const})=1, \ (i=1, 2)$$

Proof. Let (8) be true. It is shown in [1] that over the intervals $[\frac{X_i}{k_i+1}, \frac{X_i}{k_i}]$ the functions $E\tau(X_i, a)$ are increasing with interruptions at the points $\frac{X_i}{k_i+1}$, $(i=1, 2)$. In addition the lowest values of $E\tau(X_i, a)$ are

$$t_{k_i} = \lim_{a\downarrow\frac{X_i}{k_i+1}} E\tau(X_i, a)$$

and the sequences $\{t_{k_i}\}_{k_i=0}^{\infty}$ are convex $(i=1, 2)$.

Now let us consider the function $E\tau(X, a)$ with $X=pX_1+(1-p)X_2$. Over the intervals $\Delta_{k_1k_2}$ the speciments $T_{k_1k_2}(X, a)$ of $E\tau(X, a)$ given by (3) are increasing too (as convex combination of two increasing functions $E\tau(X_i, a)$, $i=1, 2$).

Denote

$$t_{k_1k_2} = \lim_{a\downarrow\max(\frac{X_1}{k_1+1}, \frac{X_2}{k_2+1})} E\tau(X, a)$$

and

$$\bar{t}_{k_1k_2} = \lim_{a\downarrow\max(\frac{X_1}{k_1+1}, \frac{X_2}{k_2+1})} E\tau(X, a).$$

At the point $\max(\frac{X_1}{k_1+1}, \frac{X_2}{k_2+1})$ at least one of the functions $E\tau(X_i, a)$, $i=1, 2$ has a jump, but the other one preserves its value (as continuous). Using the above we see that the inequality $t_{k_1k_2}<\bar{t}_{k_1k_2}$ is true. It means that the lowest points of the function $E\tau(X, a)$ are the values $t_{k_1k_2}$ at the points $x_{k_1k_2}=\max(\frac{X_1}{k_1+1}, \frac{X_2}{k_2+1})$; $k_1, k_2=0, 1.$

Further we remind that the optimal solutions (7) of two partial problems in the case are given by Theorem 2 in [1] in the form $a_i^* = \frac{X_i}{k_i^*+1}$, where

$$k_i^* = \arg\min_{k_i \geq 0} t_{k_i}.$$

For specifity let us suppose that $a_1^* < a_2^*$. Then because of the convexity of the sequences $\{t_{k_j}\}$ (they are monotonically decreasing for $k_i < k_i^*$ and monotonically increasing for $k_i > k_i^*$) and the monotonicity of $E\tau(X_i, a)$ with respect to a ($E\tau(X_i, a) \to \infty$ when $a \to 0$ or $a \to \infty$) we have

$$E\tau(X, a) > E\tau(X, a_1^*) \text{ for } a < a_1^*$$

and

$$E\tau(X, a) > E\tau(X, a_2^*) \text{ for } a > a_2^*.$$

Therefore $a^* \in [a_1^*, a_2^*]$. This proves the theorem.

The proof of Theorem 1 gives the next algorithm for determining the optimal solution a^*:

1. Determine the points a_i^* as solutions of partial optimal problems (the corresponding algorithm is given in [1]).

2. Calculate according to (3), (4) the values of $T_{k_1k_2}(X, a)$ for all $a = \frac{X_i}{k_i+1}$ belonging to the interval (6).

3. Select a^* for which under the finite number of values, calculated on step 2, it gets the minimal one.

For example, when $p=0.8$, $E\delta=0.8$, $E\theta=0.2$, $\gamma=0.05$, $X_1=10$, $X_2=79$, we find $a_1^*=3.33$ and $a_2^*=3.95$. All the $a_i = \frac{X_i}{k_i+1}$, $i=1, 2$ belonging to the interval $[a_1^*, a_2^*]$ are 3.33, 3.43, 3.59, 3.76, 3.95 with corresponding values 36.124, 36.052, 36.019, 36.021, 36.065 of $E\tau(X, a)$. In this case $a^*=3.59$.

2.2. The case $r>2$.

In an obvious way all the results for the case $r=2$ can be transferred for arbitrary r. We only formulate the result which can be proved analogously to that of Theorem 1:

Theorem 2. *The optimal uniform control schedule is determined by a value a^*, satisfying the conditions:*

a)
$$a^* \in [\min_{1\leq i\leq r} a_i^*, \max_{1\leq i\leq r} a_i^*]$$

b) *a^* has the form*
$$a^* = \frac{X_i}{k_i+1} \qquad \begin{matrix} k_i = 0, 1, 2, \dots \\ i = 1, 2, \dots, r \end{matrix}$$

c) *$E\tau(X, a^*)$ has minimal value between all the other values of $E\tau(X, a)$ with argument a, satisfying the conditions a) and b).*

Here

$$E\tau(X, a) = \sum_{k_1=0}^{\infty} \dots \sum_{k_r=0}^{\infty} I_{k_1k_2\dots k_r}(a)\, T_{k_1k_2\dots k_r}(X, a)$$

where $I_{k_1 k_2 \cdots k_r}(a)$ is the indicator function of the interval

$$[\max(\frac{X_1}{k_1+1}, \frac{X_2}{k_2+1}, \ldots, \frac{X_2}{k_2+1}), \min(\frac{X_1}{k_1}, \frac{X_2}{k_2}, \ldots, \frac{X_r}{k_r}));$$

$$T_{k_1 k_2 \cdots k_2}(X, a) = \sum_{i=1}^{r} p_i T_{k_i}(X_i, a),$$

where $T_{k_i}(X_i, a)$ are determined by (4); a_i^* are determined by the relations (7), (8).

Theorem 2 gives an essential algorithm for determining the solution of the formulated problem.

For example, when $p_1=0.2$, $p_2=0.3$, $p_3=0.5$, $E\delta=0.8$, $E\theta=0.2$, $X_1=20$, $X_2=33$, $X_3=61$, $\gamma=0.01$ we find $a_1^*=10$; $a_2^*=8{,}25$ and $a_3^*=10.17$. $a=\frac{X_i}{k_i+1}$, $i=1,2,3$ belonging to the interval $[\min_{1 \leq i \leq 3} a_i^*, \max_{1 \leq i \leq 3} a_i^*]=[8.25, 10.17]$ are 8.25, 8.71, 10, 10.17 and corresponding values of $E\tau(X, a)$ are 54.47, 53.78, 55.49, 53.97. In this case $a^*=8.71$.

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Errata

Page	Line	Instead of	Please, read
179	10 from bottom	$m \geqq \backslash\circ$	$m \geqq \mathfrak{N}o$
195	8 from bottom	$\sigma u \exists \sigma v$	$\sigma u \equiv \sigma v$
231	2 from bottom	$\tau_k(\tilde{f}, \delta)\ldots,$	Then $\tau_k(f, \delta)_p \leqq c(k, m, p) \sum_{\substack{\lvert\alpha\rvert \geqq k \\ 0 \geqq \alpha_i = k}} \delta^k \lVert D^\alpha f \rVert_{L_p}$
241	3 from bottom	see,	see [11],