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Overconvergence in Rational Approximation of Meromorphic Functions

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Presented by V. Popov

1. Introduction

Let D_ρ denote the disk $\{z \in \mathbf{C}; |z| < \rho\}$ and let $M_\rho(\nu)$ $(1 < \rho < \infty)$ denote the set of all functions $F(z)$ which are meromorphic in the disk D_ρ with precisely ν poles (counting multiplicity), and which are analytic at $z=0$ and on $|z|=1$, but not on $|z|=\rho$. ($M_\rho(0)$ denotes the set of all functions, which are analytic in D_ρ but not on $|z|=\rho$.)

Let $F(z) \in M_\rho(\nu)$ and $z_1, z_2, \ldots, z_\mu$ be the poles of $F(z)$ in D_ρ with multiplicities $\lambda_1, \lambda_2, \ldots, \lambda_\mu$, respectively ($\sum_{j=1}^{\mu} \lambda_j = \nu$). Then $F(z)$ may be expressed in D_ρ as

$$(1.1)\qquad \begin{cases} F(z)=f(z)/B(z), & \text{where} \\ B(z)=\prod_{j=1}^{\mu}(z-z_j)^{\lambda_j} & =\sum_{k=0}^{\nu}\alpha_k z^k (\alpha_\nu=1) \\ f(z)=\sum_{s=0}^{\infty} a_s z^s; & \limsup_{n\to\infty} |a_n|^{1/n}=\rho^{-1}, \\ f(z_j)\neq 0 \qquad 1\leq j\leq \mu. & \end{cases}$$

Everywhere further we will assume that $F(z) \in M_\rho(\nu)$ and the representation (1.1) is valid for $F(z)$.

If l is a positive integer number or ∞, we define the rational function of type (n, ν):

$$(1.2)\qquad P^l_{n,\nu}(z, F)=\frac{U^l_n(z)}{B^l_n(z)}=\frac{\sum_{s=0}^{n} p^l_{s;n,\nu} z^s}{\sum_{s=0}^{\nu} \gamma^l_{s;n,\nu} z^s} \qquad (\nu^l_{\nu;n,\nu}=1)$$

which is to satisfy

$$(1.3)\qquad (z^\nu B(z) U^l(z))^{(k)}_{z=0}=(z^\nu B^l(z)\sum_{s=-\nu}^{n+\nu} A^l_{s;n,\nu} z^s)^{(k)}_{z=0}$$

for $\nu \leqq k \leqq n+2\nu$, where

$$A^l_{s;n,\nu} = \sum_{m=0}^{l-1} a_{m(n+\nu+1)+s}, \quad -\nu \leqq s \leqq n+\nu$$
$$(a_{-1} = a_{-2} = \dots a_{-\nu} = 0).$$

(If there is no risk of mistakes we will write A^l_s, p^l_s and ν^l_s instead of $A^l_{s;n,\nu}$, $p^l_{s;n,\nu}$ and $\gamma^l_{s;n,\nu}$, respectively.)

Also we define the rational interpolant $L_{n,\nu}(z, F)$ of type (n, ν) with

(1.4) $$L_{n,\nu}(\omega, F) = F(\omega) \quad \text{for each} \quad \omega, \ \omega^{n+\nu+1} = 1.$$

In [6] (Theorem 1) are proved the existence and uniqueness of $P^l_{n,\nu}(z, F)$ (for all n large) and the identity

(1.5) $$L_{n,\nu}(z, F) \equiv P^{\infty}_{n,\nu}(z, F).$$

The equiconvergence of the differencies

$$\Delta^l_{n,\nu}(z, F) = L_{n,\nu}(z, F) - P^l_{n,\nu}(z, F)$$

is investigated in [4] for $\nu = 0$ and [6] for $\nu > 0$, where it is proved that

(1.6) $$S(z, f) \leqq K(z)$$

(F is analytic at z if $z \in D_\rho$), where

$$S(z, F) = \limsup_{n \to \infty} |\Delta^l_{n,\nu}(z, F)|^{1/n}$$

and

$$K(z) = \begin{cases} \rho^{-l} & \text{for} \quad |z| < \rho \\ |z|\rho^{-l-1} & \text{for} \quad |z| \geqq \rho. \end{cases}$$

In special cases $l = 1$ and $\nu = 0$ (1.6) reduces to well-known Walsh's theorem [5]. For more detailed information about various generalizations of Walsh's theorem, see [3].

However, from (1.6) it is not clear whether there are points z for which $S(z, F) < K(z)$.

V. Totik examined the situation in the case $\nu = 0$. He proved (Theorem 3, [1]) that $S(z, F) = K(z)$ for all points without almost l with moduli $> \rho$ and for all points without almost $l-1$ with moduli $< \rho$ and > 0. V. Totik also showed (Theorem 4, [1]) that for any l points $\{\omega_j\}^l_{j=1}$ with moduli $> \rho$ there is an $F(z) \in M_\rho(0)$ such that $S(\omega_j, F) < K(\omega_j)$, $1 \leqq j \leqq l$. On the other hand, for any $l-1$ points $\{\omega_j\}^{l-1}_{j=1}$ with moduli $< \rho$ and > 0, there is an $F(z) \in M_\rho(0)$ such that $S(\omega_j, F) < K(\omega_j)$, $1 \leqq j \leqq l-1$. More precise result is obtained from A. Sharma and K. Ivanov in [2].

The purpose of this paper is to examine the situation in the case $\nu > 0$. We will prove the following statements:

Theorem 1. *Let $F(z)\in M_\rho(\nu)$ and $\{z_j^*\}_{j=1}^{\nu}$ be the poles of $F(z)$ in D_ρ (listed according to multiplicities). Then $S(z, F)=K(z)$ for all points without almost $l-1$ in $D_\rho\setminus\bigcup_{j=1}^{\nu}\{z_j^*\}$.*

Theorem 2. *If $|\omega_j|<\rho$, $1\leqq j\leqq l-1$, then there exists an $F(z)\in M_\rho(\nu)$ which is analytic at ω_j and $S(\omega_j, F)<K(\omega_j)$, $1\leqq j\leqq l-1$.*

Theorem 3. *Let $F(z)\in M_\rho(\nu)$. Then $S(z, F)=K(z)$ for all points without almost $l+\nu$ in $\mathbf{C}\setminus\bar{D}_\rho$.*

Theorem 4. *If $|\omega_j|>\rho$, $1\leqq j\leqq l+1-\nu$, then there exists an $F(z)\in M\rho(\nu)$ such that $S(\omega_j, F)<K(\omega_j)$, $1\leqq j\leqq l+1-\nu$.*

Theorem 1 and Theorem 2 will be proved in section 3, and Theorem 3 and Theorem 4 – in section 4. In section 2 we obtain some technical results.

2. Some auxiliary lemmas

Lemma 1. *Let $F(z)\in M_\rho(\nu)$ is expressed in D_ρ with (1.1). Then if $|z|\neq\rho$ and $B(z)\neq 0$, the conditions*

(i) $$S(z, F)<K(z)$$

and

(ii) $$\bar{\Delta}^l_{n,\nu}(z, F)-z^l\bar{\Delta}^l_{n+1,\nu}(z, F)=0((qK(z))^n)$$

are equivalent, where

$$\bar{\Delta}^l_{n,\nu}(z, F)=B^l_n(z)B^\infty_n(z)B(z)\Delta^l_{n,\nu}(z, F).$$

Everywhere further q will be a positive number with $q<1$. These numbers may be different in each case.

Proof of Lemma 1. Since $\lim_{n\to\infty}|B^l_n(z)|^{1/n}=1$ for $B(z)\neq 0$ ([6], Lemma 1), from (i) directly follows (ii).

Let now (ii) be true. Then

(2.1) $$\bar{\Delta}^l_{n,\nu}(z, F)-z^{sl}\bar{\Delta}^l_{n+s,\nu}(z, F)$$
$$\sum_{k=0}^{s-1}[z^{kl}\bar{\Delta}^l_{n+k,\nu}(z, F)-z^{(k+1)l}\bar{\Delta}^l_{n+k+1,\nu}(z, F)$$
$$=0[\sum_{k=0}^{s-1}|z|^{kl}(qK(z))^{n+k}].$$

(a) Let $|z|<\rho$. Using (2.1) with $s=n$ and $s=n+1$ we get

$$\bar{\Delta}^l_{n,\nu}(z, F)-z^{nl}\bar{\Delta}^l_{2n,\nu}(z, F)=0((q/p)^{ln})$$

and

$$\bar{\Delta}^l_{n,\nu}(z, F)-z^{(n+1)l}\bar{\Delta}^l_{2n+1,\nu}(z, F)=0((q/p)^{ln})\varnothing,$$

i.e.

$$z^{[(n+1)/2]l}\bar{\Delta}^l_{n,\nu}(z,\ F)=\bar{\Delta}^l_{[n/2],\nu}(z,\ F)+0((q/p^l)^{n/2}).$$

Let us suppose that $S(z,\ F)>q/\rho^l$. Then

$$|z|^l S^2(z,\ F)=\limsup_{n\to\infty}|z^{[(n+1)/2]l}\bar{\Delta}^l_{n,\nu}(z,\ F)|^{2/n}$$

$$=\limsup_{n\to\infty}|\bar{\Delta}^l_{[n/2],\nu}(z,\ F)|^{2/n}=S(z,\ F)$$

and therefore $S(z,\ F)=|z|^{-l}>\rho^{-l}=K(z)$, which contradicts (1.6). Therefore $S(z,\ F)\leqq q/\rho^l<K(z)$, i.e. in this case (i) is true.

(b) Let now $|z|>\rho$. Using (2.1) with $s=n$ and $s=n+1$ we have

$$\bar{\Delta}^l_{n,\nu}(z,\ F)-z^{nl}\bar{\Delta}^l_{2n,\nu}(z,\ F)=0((\frac{q|z|}{\rho^{l+1}})^n\sum_{k=0}^{n-1}(\frac{|z|}{\rho})^{(l+1)k})$$

$$=0(q^n|z|^{ln}|z/\rho^{l+1}|^{2n});$$

$$\bar{\Delta}^l_{n,\nu}(z,\ F)-z^{(n+1)l}\bar{\Delta}^l_{2n+1,\nu}(z,\ F)=0(q^n|z|^{ln}|z/\rho^{l+1}|^{2n}),$$

i.e.

$$\bar{\Delta}^l_{[n/2],\nu}(z,\ F)=z^{[(n+1)/2]l}\bar{\Delta}^l_{n,\nu}(z,\ F)+0((q\,|z|^{l/2}\,|z/\rho^{l+1}|)^n).$$

Let us suppose that $S(z,\ F)=K(z)$. Then

$$(K(z))^{1/2}=\limsup_{n\to\infty}|\bar{\Delta}^l_{[n/2],\nu}(z,\ F)|^{1/n}$$

$$=\limsup_{n\to\infty}|z^{[(n+1)/2]l}\bar{\Delta}^l_{n,\nu}(z,\ F)|^{1/n}=|z|^{l/2}K(z),$$

i.e.

$$K(z)=|z|^{-l}=|\rho/z|^{l+1}K(z)<K(z).$$

From this contradiction and (1.6) we obtain that (i) is true for $|z|>\rho$.

Lemma 2. *Let $F(z)\in M_\rho(\nu)$ and $F(z)$ has the representation (1.1) in D_ρ. Then*

$$\bar{\Delta}^l_{n,\nu}(z,\ F)=\sum_{s=0}^{n+2\nu}z^s R_{s,n}+0(qK(z)^n),$$

where

$$\sum_{s=0}^{n+2\nu}z^s R_{s,n}=B(z)\sum_{k=0}^{\nu}\alpha_k\sum_{s=0}^{n+\nu}z^s a_{l(n+\nu+1)+s-k}$$

$$+z^{n+\nu+1}\sum_{s=1}^{\nu}\sum_{k=0}^{\nu-s}\sum_{m=0}^{s-1}z^{m+k}a_{n+s}(\alpha_k r_{m+\nu+1-s}-\alpha_{m+\nu+1-s}r_k),$$

and

$$r_s=r_{s,n}=\gamma^l_{s;n,\nu}-\gamma^{\infty}_{s;n,\nu},\quad 0\leqq s\leqq\nu,\quad (r_\nu=0).$$

P r o o f. The conditions (1.2) and (1.3) are equivalent to the following system of equations (see [6], (2.3)):

$$\sum_{k=0}^{\nu} \gamma_k^l A_{s-k}^l = \sum_{k=0}^{\nu} \alpha p_{s-k}^l \quad 0 \leqq s \leqq n+\nu$$

(p_s^l : 0 for $s<0$ and $s>n$). Therefore

$$B(z)U_n^l(z) = \sum_{k=0}^{\nu} \alpha_k z^k \sum_{s=0}^{n} p_s^l z^s \tag{2.2}$$

$$= \sum_{s=0}^{n+\nu} z^s \sum_{k=0}^{\nu} \alpha_k p_{s-k}^l = \sum_{s=0}^{n+\nu} z^s \sum_{k=0}^{\nu} \gamma_k^l A_{s-k}^l$$

$$= \sum_{s=-\nu}^{-1} \sum_{k=-s}^{\nu} A_s^l \gamma_k^l z^{k+s} + \sum_{s=0}^{n} z^s A_s^l \sum_{k=0}^{\nu} \gamma_k^l z^k + \sum_{s=n+1}^{n+\nu} \sum_{k=0}^{n+\nu-s} A_s^l \gamma_k^l z^{s+k}$$

$$= \sum_{k=1}^{\nu} \sum_{s=-k}^{-1} \gamma_k^l A_s^l z^{k+s} + B_n^l(z) \sum_{s=0}^{n} A_s^l z^s + \sum_{k=0}^{\nu-1} \sum_{s=n+1}^{n+\nu-k} \gamma_k^l A_s^l z^{k+s}.$$

From (2.2), (1.4) and (1.5) we obtain

$$\bar{\Delta}_{n,\nu}^l(z, F) = B_n^l(z) B(z) U_n^\infty(z) - B_n^\infty(z) B(z) U_n^l(z) \tag{2.3}$$

$$= B_n^l(z) B_n^\infty(z) \sum_{s=0}^{n} z^s (A_s^\infty - A_s^l) + [(B_n^l(z) - B_n^\infty(z)] \sum_{k=1}^{\nu} \sum_{s=-k}^{-1} \gamma_k^\infty A_s^\infty z^{k+s}$$

$$+ B_n^\infty(z) \sum_{k=1}^{\nu} \sum_{s=-k}^{-1} [A_s^\infty - A_s^l] \gamma_k^\infty z^{k+s} + B_n^\infty(z) \sum_{k=1}^{\nu} \sum_{s=-k}^{-1} A_s^l [\gamma_k^\infty - \gamma_k^l] z^{k+s}$$

$$+ [B_n^l(z) - B_n^\infty(z)] \sum_{k=0}^{\nu-1} \sum_{s=n+1}^{n+\nu-k} \gamma_k^\infty A_s^\infty z^{k+s} + B_n^\infty(z) \sum_{k=0}^{\nu-1} \sum_{s=n+1}^{n+\nu-k} \gamma_k^\infty [A_s^\infty - A_s^l] z^{k+s}$$

$$+ B_n^\infty(z) \sum_{k=0}^{\nu-1} \sum_{s=n+1}^{n+\nu-k} [\gamma_k^\infty - \gamma_k^l] A_s^l z^{k+s}.$$

Since $\limsup_{n\to\infty} |a_n|^{1/n} = \rho^{-1}$, we have

$$a_n = 0((\rho - \varepsilon)^{-n}) \tag{2.4}$$

for each $\varepsilon > 0$, and therefore

$$A_s^\infty - A_s^l = \sum_{m=0}^{\infty} a_{m(n+\nu+1)+s} = a_{l(n+\nu+1)+s} + 0[(\rho-\varepsilon)^{-(l+1)n-s}]; \tag{2.5}$$

$$A_s^l = \sum_{m=0}^{l-1} a_{m(n+\nu+1)+s} = \begin{cases} 0((\rho-\varepsilon)^{-n}) & \text{for } -\nu \leqq s \leqq -1 \\ a_s + 0((\rho-\varepsilon)^{-n-s}) & \text{for } 0 \leqq s \leqq n+\nu. \end{cases} \tag{2.6}$$

Also in [6]—(3.5), (3.6) it is proved

(2.7) $$r_{s,n}=0((\rho-\varepsilon)^{-ln})$$

and

(2.8) $$\gamma_s^l-\alpha_s=0(q^n)\,,\ (0<q<1).$$

Using (2.4)—(2.8) in (2.3) we get

$$\bar{\Delta}^l_{n,\nu}(z,\ F)=B^2(z)\sum_{s=0}^{n}a_{l(n+\nu+1)+s}z^s+B(z)\sum_{k=1}^{\nu}\sum_{s=-k}^{-1}\alpha_k a_{l(n+\nu+1)+s}z^{k+s}$$

$$+\sum_{m=0}^{\nu}r_m z^m\sum_{k=0}^{\nu-1}\sum_{s=n+1}^{n+\nu-k}\alpha_k a_s z^{k+s}+B(z)\sum_{k=1}^{\nu-1}\sum_{s=n+1}^{n+\nu-k}\alpha_k a_{l(n+\nu+1)+s}z^{k+s}$$

$$-B(z)\sum_{k=1}^{\nu-1}\sum_{s=n+1}^{n+\nu-k}a_s r_k z^{k+s}+0((qK(z))^n)$$

$$=B(z)\{\sum_{k=0}^{\nu}\alpha_k\sum_{s=0}^{n}a_{l(n+\nu+1)+s}z^{k+s}+\sum_{k=1}^{\nu}\alpha_k\sum_{s=-k}^{-1}a_{l(n+\nu+1)+s}z^{k+s}$$

$$+\sum_{k=0}^{\nu-1}\alpha_k\sum_{s=n+1}^{n+\nu-k}a_{l(n+\nu+1)+s}z^{k+s}\}$$

$$+z^n\sum_{s=1}^{\nu}\sum_{k=0}^{\nu-s}\sum_{m=0}^{\nu}a_{n+s}(\alpha_k r_m-\alpha_m r_k)z^{k+s+m}+0((qK(z))^n)$$

$$=B(z)\sum_{k=0}^{\nu}\alpha_k\sum_{s=-k}^{n+\nu-k}a_{l(n+\nu+1)+s}z^{k+s}$$

$$+z^n\sum_{s=1}^{\nu}\sum_{k=0}^{\nu-s}\sum_{m=\nu-s+1}^{\nu}a_{n+s}(\alpha_k r_m-\alpha_m r_k)z^{k+s+m}+0((qK(z))^n)$$

$$=B(z)\sum_{k=0}^{\nu}\alpha_k\sum_{s=0}^{n+\nu}a_{l(n+\nu+1)+s-k}z^s$$

$$+z^{n+\nu+1}\sum_{s=1}^{\nu}\sum_{k=0}^{\nu-s}\sum_{m=0}^{s-1}a_{n+s}(\alpha_k r_{m+\nu+1-s}-\alpha_{m+\nu+1-s}r_k)z^{k+m}$$

$$+0((qK(z))^n),$$

and the proof is completed.

From Lemma 2, (2.4) and (2.7) we get immediately the following

Corollary 1. *The coefficients $R_{s,n}$, defined in Lemma 2 satisfy*

(2.9) $$R_{s,n}=0((\rho-\varepsilon)^{-ln-s}),\ 0\leqq s\leqq n+2\nu\ \textit{for each}\ \ \varepsilon>0$$

and

(2.10) $$R_{s,n}=\sum_{m=0}^{\nu}\sum_{k=0}^{\nu}\alpha_m\alpha_k a_{l(n+\nu+1)+s-m-k},\quad \nu\leqq s\leqq n+\nu.$$

Lemma 3. *Let*

(2.11) $$\sum_{k=0}^{\nu}\alpha_k A_{m-k}=0((q/\rho)^m),\ (0<q<1)$$

for each m large, where

$$\sum_{k=0}^{\nu}\alpha_k z^k=\prod_{j=1}^{\nu}(z-z_j);\quad |z_j|<\rho,\quad 1\leqq j\leqq\nu,\ (\alpha_\nu=1)$$

and

(2.12) $$\limsup_{n\to\infty}|A_n|^{1/n}\leqq\rho^{-1}.$$

Then $\limsup_{n\to\infty}|A_n|^{1/n}<\rho^{-1}$.

Proof. Consider first the case when $z_i\neq z_j$ for $i\neq j$. Since $|z_j|<\rho$ from (2.11) we get

(2.13) $$\sum_{n=m}^{\infty}z_j^n\sum_{k=0}^{\nu}\alpha_k A_{n-k}=0\Big(\sum_{n=m}^{\infty}\Big|\frac{qz_j}{\rho}\Big|^n\Big)=0\Big(\Big|\frac{qz_j}{\rho}\Big|^m\Big),\quad 1\leqq j\leqq\nu.$$

But with (2.12)

(2.14) $$\sum_{n=m}^{\infty}z_j^n\sum_{k=0}^{\nu}\alpha_k A_{n-k}=\sum_{k=0}^{\nu}\alpha_k z_j^k\sum_{n=m-k}^{\infty}A_n z_j^n$$

$$=\sum_{n=m-\nu}^{m-1}A_n z_j^n\sum_{k=m-n}^{\nu}\alpha_k z_j^k+\sum_{n=m}^{\infty}A_n z_j^n\sum_{k=0}^{\nu}\alpha_k z_j^k$$

$$=z_j^m\sum_{s=1}^{\nu}A_{m-s}\sum_{k=s}^{\nu}\alpha_k z_j^{k-s}=z_j^m\sum_{s=1}^{\nu}\alpha_s^1(z_j)A_{m-s},$$

where

$$\alpha_s^1(z)=\sum_{k=s}^{\nu}\alpha_k z^{k-s}.$$

From (2.13) and (2.14) it follows

(2.15) $$\sum_{s=1}^{\nu}\alpha_s^1(z_j)A_{m-s}=0((q/\rho)^m),\quad 1\leqq j\leqq\nu.$$

We may consider (2.15) as a system of equations for A_{m-1}, $A_{m-2},\ldots,$ $A_{m-\nu}$ with determinant

$$\Delta=\begin{vmatrix}\alpha_1^1(z_1)\dots\alpha_\nu^1(z_1)\\ \dots\dots\dots\dots\\ \alpha_1^1(z_\nu)\dots\alpha_\nu^1(z_\nu)\end{vmatrix}$$

$$=\begin{vmatrix}1 & z_1\dots z_1^{\nu-1}\\ \dots & \dots\\ 1 & z_\nu\dots z_\nu^{\nu-1}\end{vmatrix}\begin{vmatrix}\alpha_1\dots\alpha_{\nu-1} & \alpha_\nu\\ \alpha_2\dots\alpha_\nu & 0\\ \dots & \dots\\ \alpha_\nu\dots 0 & 0\end{vmatrix}\neq 0.$$

Now from (2.15) with Cramer's Formula we get

$$A_m=0((q/\rho)^m),$$

i.e.

$$\limsup_{n\to\infty}|A_n|^{1/n}\leqq q/\rho<\rho^{-1}$$

which proves the lemma in this case.

In the general case $\left(\prod_{j=1}^{\nu}(z-z_j)=\prod_{j=1}^{\mu}(z-\bar z_j)^{\lambda_j}\right)$ only some slight technical difficulties arise in the proof. If we set

$$\alpha_k^{t+1}(z)=\sum_{s=k}^{\nu}\alpha_s^t(z)z^{s-k},\quad 1\leqq k\leqq\nu,\quad 1\leqq t\leqq\nu-1$$

then (2.15) is replaced by

$$\sum_{s=t}^{\nu}\alpha_s^t(\bar z_j)A_{m-s}=0((q/\rho)^m),\quad 1\leqq j\leqq\mu,\quad 1\leqq t\leqq\lambda_j,$$

with determinant

$$\Delta=\prod_{j=1}^{\mu}\bar z_j^{\lambda_j(\lambda_j-1)}\prod_{k=2}^{\mu}\prod_{i=1}^{k-1}(\bar z_k-\bar z_i)^{\lambda_k\lambda_i}\neq 0$$

which proves the lemma in the general case.

3. Overconvergence in the disk D_ρ

In this section we will prove Theorem 1 and Theorem 2.

Proof of Theorem 1. Let us suppose that there is an $F(z)\in M_\rho(\nu)$ (with representation (1.1) in D_ρ) and l points $\omega_j\in D_\rho\setminus\bigcup_{j=1}^{\mu}\{z_j\}$, such that $S(\omega_j,F)<K(\omega_j)$, $1\leqq j\leqq l$. Then Lemma 1 gives

(3.1) $$\bar\Delta_{n,\nu}^l(\omega_j,F)-\omega_j^l\bar\Delta_{n,\nu}^l(\omega_j,F)=0((qK(\omega_j))^n),\quad 1\leqq j\leqq l.$$

But using Lemma 2 and Corollary 1 we have

$$\overline{\Delta}^l_{n,\nu}(\omega_j, F)-\omega_j^l\overline{\Delta}^l_{n,\nu}(\omega_j, F) \tag{3.2}$$

$$=B(\omega_j)\sum_{k=0}^{\nu}\alpha_k\sum_{s=0}^{n+\nu}a_{l(n+\nu+1)+s-k}\omega_j^s$$

$$-B(\omega_j)\sum_{k=0}^{\nu}\alpha_k\sum_{s=0}^{n+\nu+1}a_{l(n+\nu+1)+s-k}\omega_j^{s+l}+O((qK(\omega_j))^n)$$

$$=B(\omega_j)\sum_{k=0}^{\nu}\alpha_k\Big\{\sum_{s=0}^{n+\nu}a_{l(n+\nu+1)+s-k}\omega_j^s-\sum_{s=0}^{n+\nu+1+l}a_{l(n+\nu+1)+s-k}\omega_j^s\Big\}$$

$$+O((qK(\omega_j))^n)=B(\omega_j)\sum_{k=0}^{\nu}\sum_{s=0}^{l-1}\alpha_k\omega_j^s a_{l(n+\nu+1)+s-k}+O((qK(\omega_j))^n).$$

Since $B(\omega_j)\neq 0$, (3.1) and (3.2) give

$$\sum_{s=0}^{l-1}\omega_j^s\sum_{k=0}^{\nu}\alpha_k a_{l(n+\nu+1)+s-k}=O((qK(\omega_j))^n)=O((q/\rho)^n),\quad 1\leqq j\leqq l.$$

From the last system of equations with Cramer's formula and Lemma 3 we obtain

$$\limsup_{n\to\infty}|a_n|^{1/n}<\rho^{-1}$$

which contradicts (1.1), and the proof is completed.

P r o o f o f T h e o r e m 2. We define the sequence $\{\varphi_n\}_{n=0}^{\infty}$ in the following way:

$$\begin{cases}\varphi_{nl}=\rho^{-ln}\\ \sum_{s=0}^{l-1}\varphi_{ln+s}\omega_j^s=0, \quad 1\leqq j\leqq l-1\end{cases}\qquad n=0, 1,\dots \tag{3.3}$$

Obviously, $\varphi_{n+l}=\rho^{-l}\varphi_n$ for each $n\geqq 0$, and the functions

$$\varphi_n^1(z):=z^{-n}\sum_{m=n}^{\infty}\varphi_m z^m$$

are defined for each $z\in D_\rho$. Moreover

$$\varphi_n^1(z)=O(\rho^{-n}) \tag{3.4}$$

and

$$\varphi_{n+l}^1(z)=\rho^{-l}\varphi_n^1(z). \tag{3.5}$$

Let us choose the points $\{z_j\}_{j=1}^{\nu}$ such that $z_i\neq z_j$ for $i\neq j$, $z_i\neq\omega_j$, $z_i\neq 0$, $|z_i|\neq 1$, $z_i\in D_\rho$, $1\leqq i\leqq\nu$, $1\leqq j\leqq l-1$ and let

$$B(z) := \prod_{j=1}^{\nu} (z-z_j) = \sum_{k=0}^{\nu} \alpha_k z^k.$$

We define $\{a_{n,s}\}_{1\leqq n\leqq\infty}^{0\leqq s\leqq \nu-1}$ to satisfy

$$\text{(3.6)} \qquad \sum_{i=1}^{\nu} \alpha_i^1(z_j)a_{n,\nu-i} = \varphi_{n+\nu}^1(z_j), \quad 1\leqq j\leqq \nu, \quad (n=1,\ 2,\ldots).$$

Since the determinant Δ of the system (3.6) is different from zero, $\{a_{n,s}\}_{1\leqq n\leqq\infty}^{0\leqq s\leqq \nu-1}$ are unique determined from (3.6). Moreover, Δ is undepending on n and therefore with (3.4), (3.5) and Cramer's formula from (3.6) we have

$$\text{(3.7)} \qquad a_{n,s} = O(\rho^{-n}), \quad 0\leqq s\leqq \nu-1,$$

$$\text{(3.8)} \qquad a_{n+l,s} = \rho^{-l}a_{n,s}, \quad 0\leqq s\leqq \nu-1.$$

Also from (3.6) it follows

$$\text{(3.9)} \qquad \sum_{k=0}^{\nu} \alpha_k \sum_{i=1}^{\nu} \alpha_i^1(z_i)a_{n-k,\nu-i} = \sum_{k=0}^{\nu} \alpha_k \varphi_{n+\nu-k}^1(z_j), \quad 1\leqq j\leqq \nu, \quad n>\nu.$$

But

$$\text{(3.10)} \qquad \sum_{k=0}^{\nu} \alpha_k \sum_{i=1}^{\nu} \alpha_i^1(z_j)a_{n-k,\nu-i}$$

$$= \sum_{i=1}^{\nu} \alpha_i^1(z_j) \sum_{k=0}^{\nu} \alpha_k a_{n-k,\nu-i} = \sum_{i=1}^{\nu} \alpha_i^1(z_j) Y_{n,\nu-i},$$

where

$$Y_{n,s} = \sum_{k=0}^{\nu} \alpha_k a_{n-k,s}, \quad n>\nu$$

and

$$\text{(3.11)} \qquad \sum_{k=0}^{\nu} \alpha_k \varphi_{n+\nu-k}^1(z_j) = \sum_{k=0}^{\nu} \alpha_k z_j^{-n-\nu+k} \sum_{m=n+\nu-k}^{\infty} \varphi_m z_j^m$$

$$= \sum_{m=n+\nu}^{\infty} \varphi_m z_j^m \sum_{k=0}^{\nu} \alpha_k z_j^k + \sum_{m=n}^{n+\nu-1} \varphi_m \sum_{k=n+\nu-m}^{\nu} \alpha_k z_j^{k-n-\nu+m}$$

$$= \sum_{m=n}^{n+\nu-1} \varphi_m \alpha_{n+\nu-m}^1(z_j) = \sum_{i=1}^{\nu} \alpha_i^1(z_j)\varphi_{n+\nu-i}.$$

From (3.9)-(3.11) we have

$$\sum_{i=1}^{\nu} \alpha_i^1(z_j)(Y_{n,\nu-i} - \varphi_{n+\nu-i}) = 0, \quad 1\leqq j\leqq \nu.$$

The last system has the unique solution $(\Delta\neq 0)$ $Y_{n,s}-\varphi_{n,s}=0$, $0\leqq s\leqq v-1$, and in particular

$$\sum_{k=0}^{v} \alpha_k a_{n-k,0}=\varphi_n. \tag{3.12}$$

Let us chose n_0 with $ln_0>v$. From (3.3) and (3.12) we get

$$\sum_{k=0}^{v} \alpha_k \alpha_{ln_0-k,0}=\varphi_{ln_0}=\rho^{-ln_0}.$$

Therefore, some of the numbers $a_{ln_0,0}, a_{ln_0-1,0},\dots a_{ln_0-v,0}$ (for example $a_{n_1,0}$) is different from zero. Then with (3.8) we get

$$a_{ln+n_1,0}=\rho^{-ln}a_{n_1,0} \quad n=1,\ 2,\dots$$

which, with (3.7) gives $\limsup_{n\to\infty} |a_{n,0}|^{1/n}=\rho^{-1}$. At last we choose $a_{0,0}$ such that $f^*(z_j)\neq 0$, $1\leqq j\leqq v$, where $f^*(z)=\sum_{n=0}^{\infty} a_{n,0}z^n$. The function $F^*(z)=f^*(z)/B(z)\in M_\rho(v)$. To complete the proof it remains to show that $S(\omega_j,\ F^*)<K(\omega_j)$, $1\leqq j\leqq l-1$. Indeed, with (3.2), (3.3) and (3.12) we have

$$\bar{\Delta}^l_{n,v}(\omega_j,\ F^*)-\omega_j^l\bar{\Delta}^l_{n,v}(\omega_j,\ F^*)$$

$$=B(\omega_j)\sum_{s=0}^{l-1}\omega_j^s\sum_{k=0}^{v}\alpha_k a_{l(n+v+1)+s-k,0}+0((qK(\omega_j))^n)$$

$$=B(\omega_j)\sum_{s=0}^{l-1}\omega_j^s\varphi_{l(n+v+1)+s}+0((qK(\omega_j))^n)=0((qK(\omega_j))^n)$$

from where with Lemma 1 it follows

$$S(\omega_j,\ F^*)<K(\omega_j),\quad 1\leqq j\leqq l-1.$$

Remark. In fact we proved the statement, which is stronger than Theorem 2. Let $M_\rho(B(z))=\{F\ ;\ F(z)\in M_\rho(v),\ F(z)=f(z)/B(z),\ f(z)\in M_\rho(0)\}$. We proved that if $B(z)$ has only simple zeros and $B(\omega_j)\neq 0$, $\omega_j\in D_\rho$, $1\leqq j\leqq l-1$, then there is an $F^*(z)\in M_\rho(B(z))$ such that $S(\omega_j,\ F^*)<K(\omega_j)$, $1\leqq j\leqq l-1$. With some slight modifications of the proof it is easy to see that the statement is true in the general case, i.e. if some of the zeros of $B(z)$ are multiple.

4. Overconvergence in $\mathbb{C}\backslash\bar{D}_\rho$

In this section we will prove Theorem 3 and Theorem 4.

Proof of Theorem 3. Let us suppose that there is an $F(z)\in M_\rho(v)$ and points $\omega_j\in\mathbb{C}\backslash\bar{D}_\rho$, $1\leqq j\leqq l+v+1$, *such that* $S(\omega_j,\ F)<K(\omega_j)$, $1\leqq j\leqq l+v+1$. Then (with Lemma 1)

(4.1) $$\bar{\Delta}^l_{n,\nu}(\omega_j, F)-\omega_j^l\bar{\Delta}^l_{n+1,\nu}(\omega_j, F)=0((qK(\omega_j))^n).$$

Also, with Lemma 2, (2.9) and (2.10) we have

(4.2) $$\bar{\Delta}^l_{n,\nu}(\omega_j, F)-\omega_j^l\bar{\Delta}^l_{n+1,\nu}(\omega_j, F)$$

$$=\sum_{k=0}^{n+2\nu} R_{k,n}\omega_j^k-\sum_{k=0}^{n+2\nu+1} R_{k,n+1}\omega_j^{k+l}+0((qK(\omega_j))^n)$$

$$=\sum_{k=0}^{l-1} R_{k,n}\omega_j^k+\sum_{k=l}^{n+2\nu}(R_{k,n}-R_{k-l,n+1})\omega_j^k$$

$$-\sum_{k=n+2\nu+1}^{n+2\nu+1+l} R_{k-l,n+1}\omega_j^k+0((qK(\omega_j))^n)\ .$$

From (4.1) and (4.2) it follows

$$\sum_{k=0}^{\nu-1}(R_{n+\nu+1+k,n}-R_{n+\nu+1+k-l,n+1})\omega_j^k$$

$$-\sum_{k=\nu}^{\nu+l} R_{n+\nu+1+k-l,n+1}\omega_j^k=0((qK(\omega_j))^n),\quad 1\leqq j\leqq l+\nu+1.$$

From the last system of equations we get

(4.3) $$R_{m,n}-R_{m-l,n+1}=0((q/\rho)^{(l+1)n}),\ n+\nu+1\leqq m\leqq n+2\nu;$$

(4.4) $$R_{m-l,n+1}=0((q/\rho)^{(l+1)n}),\ n+2\nu+1\leqq m\leqq n+2\nu+l+1.$$

We shall prove

(4.5) $$R_{m,n}=0((q/\rho)^{(l+1)n}),\ n+\nu-l\leqq m\leqq n+\nu.$$

Indeed, (4.4) gives

$$R_{m,n}=0((q/\rho)^{(l+1)(n-1)})\quad \max\{n+\nu-l,\ n+2\nu-l\}\leqq m\leqq n+2\nu.$$

Let us suppose that

(4.6) $$R_{m,n}=0((q/\rho)^{(l+1)(n-k-1)})$$

for $\max\{n+\nu-l,\ n+2\nu-(k+1)l-k\}\leqq m\leqq n+2\nu.$

Then with (4.3) and (4.6):

$$R_{m-l,n+1}=0((q/\rho)^{(l+1)(n-k-1)})$$

for $\max\{n+\nu+1,\ n+\nu-l,\ n+2\nu-(k+1)l-k\}\leqq m\leqq n+2\nu$

which, with (4.4) gives

$$R_{m,n}=O((q/\rho)^{(l+1)(n-k-1)})$$

for $\max\{n+\nu-l,\ n+2\nu-(k+2)l-k-1\}\leqq m\leqq n+2\nu.$

And by induction we obtain that (4.6) is true for any positive integer number k. In particular, if $k=\nu$, from (4.6) follows (4.5). But using (2.10) we may write (4.5) in the form

$$\sum_{s=0}^{\nu}\alpha_s\sum_{k=0}^{\nu}\alpha_k a_{l(n+\nu+1)+m-s-k}=O((q/\rho)^{(l+1)n}),\quad n+\nu-l\leqq m\leqq n+\nu,$$

which is equivalent with

$$\sum_{s=0}^{\nu}\alpha_s\sum_{k=0}^{\nu}\alpha_k a_{n-s-k}=O((q/\rho)^n).$$

Now, using Lemma 3 two times (with $A_n=\sum_{k=0}^{\nu}\alpha_k a_{n-k}$ and then with $A_n=a_n$) we obtain

$$\limsup_{n\to\infty}|a_n|^{1/n}<\rho-1$$

which contradicts (1.1).

To prove Theorem 4 we will need the following

Lemma 4. *Let $F(z)\in M_\rho(\nu)$, and let $\{z_j\}_{j=1}^{\nu}$ ($z_i\neq z_j$ for $i\neq j$) be the poles of $F(z)$ in D_ρ, and $l\geqq\nu$. If $|z|>\rho$, then*

$$z^{-n}\{\overline{\Delta}^l_{n-\nu-1,\nu}(z,\ F)-z^l\overline{\Delta}^l_{n-\nu,\nu}(z,\ F)\}$$

$$=-B(z)\sum_{s=-\nu}^{-1}\sum_{k=-s}^{\nu}\alpha_k a_{(l+1)n+s}z^{k+s}-B^2(z)\sum_{s=0}^{l-\nu}a_{(l+1)n+s}z^s$$

$$-B(z)\sum_{s=l-\nu+1}^{l}\sum_{k=0}^{l-s}\alpha_k a_{(l+1)n+s}z^{k+s}$$

$$+\sum_{s=-\nu}^{0}\sum_{k=-\nu}^{l-1}M_{k,s}a_{n+s}a_{ln+s}+O((q/\rho)^{(l+1)n}),$$

where the numbers $M_{k,s}=M_{k,s}(z\,;\,z_1,\dots,z_\nu,f(z_1),\dots,f(z_\nu))$ depend only on marked parameters and $0<q<1$.

P r o o f. The coefficients $\{\gamma^l_k\}_{k=0}^{l-1}$ satisfy the following system of equations (see [6]$-$(2.7)):

$$(4.7)\qquad \sum_{k=0}^{\nu} \gamma_k^l \sum_{s=0}^{n+\nu} z_j^s A_{s-k}^l = 0, \quad 1 \leqq j \leqq \nu.$$

Using (2.5)-(2.8) we obtain

$$(4.8)\qquad \sum_{k=0}^{\nu} \gamma_k^l \sum_{s=0}^{n+\nu} z_j^s A_{s-k}^l - \sum_{k=0}^{\nu} \gamma_k^{\infty} \sum_{s=0}^{n+\nu} z_j A_{s-k}^{\infty}$$

$$= \sum_{k=0}^{\nu} (\gamma_k^l - \gamma_k^{\infty}) z_j^k \sum_{s=-k}^{n+\nu-k} z_j^s A_s^l + \sum_{k=0}^{\nu} \gamma_k^{\infty} \sum_{s=-k}^{n+\nu-k} z_j^{s+k} (A_s^l - A_s^{\infty})$$

$$= \sum_{k=0}^{\nu-1} r_k z_j^k \Big(\sum_{s=0}^{\infty} a_s z_j^s + 0(q^n) \Big)$$

$$- \sum_{k=0}^{\nu} (\alpha_k + 0(q^n)) \sum_{s=-k}^{n+\nu-k} (a_{l(n+\nu+1)+s} + 0((\rho-\varepsilon)^{-(l+1)n-s})) z_j^{s+k}$$

$$= \sum_{k=0}^{\nu-1} r_k z_j^k f(z_j) - \sum_{k=0}^{\nu} \alpha_k \sum_{s=-k}^{n+\nu-k} a_{l(n+\nu+1)+s} z_j^{s+k} + 0(q^n \rho^{-ln})$$

$$= \sum_{k=0}^{\nu-1} r_k z_j^k f(z_j) - \sum_{s=-\nu}^{-1} a_{l(n+\nu+1)+s} \sum_{k=-s}^{\nu} \alpha_k z_j^{s+k}$$

$$- \sum_{s=0}^{n} a_{l(n+\nu+1)+s} z_j^s \sum_{k=0}^{\nu} \alpha_k z_j^k - \sum_{s=n+1}^{n+\nu} a_{l(n+\nu+1)+s} \sum_{k=0}^{n+\nu-s} \alpha_k z_j^{k+s}$$

$$+ 0((q/\rho)^{ln}) = \sum_{k=0}^{\nu-1} r_k z_j^k f(z_j) - \sum_{s=-\nu}^{-1} a_{l(n+\nu+1)+s} \sum_{k=-s}^{\nu} \alpha_k z_j^{s+k} + 0((q/\rho)^{ln}).$$

Now (4.7) and (4.8) give

$$\sum_{k=0}^{\nu-1} r_k z_j^k f(z_j) = \sum_{s=-\nu}^{-1} a_{l(n+\nu+1)+s} \sum_{k=-s}^{\nu} \alpha_k z_j^{s+k} + 0((q/\rho)^{ln})$$

$$1 \leqq j \leqq \nu.$$

From the last system of equations with Cramer's formula follows:

$$(4.9)\qquad r_{k,n} = \sum_{s=-\nu}^{-1} d_{s,k} a_{l(n+\nu+1)+s} + 0((q/\rho)^{ln}), \quad 0 \leqq k \leqq \nu - 1,$$

where the coefficients $d_{s,k}$ depend only on z_j, $f(z_j)$, $1 \leqq j \leqq \nu$. Moreover,

$$(4.10)\quad z^{n+\nu+1} \sum_{s=1}^{\nu} \sum_{k=0}^{\nu-s} \sum_{m=0}^{s-1} a_{n+s} (\alpha_k r_{m+\nu+1-s,n} - \alpha_{m+\nu+1-s} r_{k,n}) z^{k+m}$$

$$=z^{n+\nu+1}\sum_{s=-\nu}^{-1}\sum_{k=0}^{\nu-1}N_{k,s}a_{n+\nu+1+s}r_{k,n},$$

where $N_{k,s}$ depends only on z_j, $1\leqq j\leqq\nu$ and z. Fom (4.9), (4.10) and Lemma 2, if $|z|>\rho$, we obtain

$$\overline{\Delta}^l_{n-\nu-1,\nu}(z,\ F)-z^l\overline{\Delta}^l_{n-\nu,\nu}(z,\ F) \tag{4.11}$$

$$=B(z)\sum_{k=0}^{\nu}\alpha_k\{\sum_{s=0}^{n-1}a_{ln+s-k}z^s-\sum_{s=0}^{n}a_{l(n+1)+s-k}z^{s+l}\}$$

$$+z^n\sum_{s=-\nu}^{-1}\sum_{k=0}^{\nu-1}N_{k,s}(a_{n+s}r_{k,n-\nu-1}-z^{l+1}a_{n+1+s}r_{k,n-\nu})+0((qK(z))^n)$$

$$=B(z)\sum_{k=0}^{\nu}\alpha_k\{\sum_{s=0}^{l-1}a_{ln+s-k}z^s-\sum_{s=n}^{n+l}a_{ln+s-k}z^s\}$$

$$+z^n\sum_{s=-\nu}^{-1}\sum_{k=0}^{\nu-1}N_{k,s}\sum_{m=-\nu}^{-1}d_{m,k}(a_{n+s}a_{ln+m}-z^{l+1}a_{n+1+s}a_{l(n+1)+m})$$

$$+0((qK(z))^n)$$

$$=-z^nB(z)\sum_{k=0}^{\nu}\alpha_k\sum_{s=0}^{l}a_{(l+1)n+s-k}z^s+z^n\sum_{s=-\nu}^{0}\sum_{k=-\nu}^{l-1}M_{k,s}a_{n+s}a_{ln+k}$$

$$+0((qK(z))^n),$$

where

$$M_{k,s}=M_{k,s}(z;\ z_1,\dots,\ z_\nu,\ f(z_1),\dots,\ f(z_\nu)).$$

But

$$\sum_{k=0}^{\nu}\alpha_k\sum_{s=0}^{l}a_{(l+1)n+s-k}z^s=\sum_{k=0}^{\nu}\alpha_kz^k\sum_{s=-k}^{l-k}a_{(l+1)n+s}z^s$$

$$=\sum_{s=-\nu}^{-1}\sum_{k=-s}^{\nu}\alpha_ka_{(l+1)n+s}z^{s+k}+\sum_{s=0}^{l-\nu}a_{(l+1)n+s}z^s\sum_{k=0}^{\nu}\alpha_kz^k$$

$$+\sum_{s=l-\nu+1}^{l}\sum_{k=0}^{l-s}\alpha_ka_{(l+1)n+s}z^{k+s}.$$

Replacing the last expression in (4.11) we obtain the desired representation.

P r o o f o f T h e o r e m 4. We shall prove that if $|\omega_j|>\rho$, $1\leqq j\leqq l-\nu+1$ and $B(z)$ has only simple zeros, then there is an $F^*(z)\in M_\rho(B(z))$, such that $S(\omega_j,\ F^*)<K(\omega_j)$, $1\leqq j\leqq l-\nu+1$.

We define $\{a_n\}_{n=l+1}^{\infty}$ as follows:

$$a_{(l+1)n+s}=A\rho^{-(l+1)n-s}\quad l+1-\nu\leqq s\leqq l,\quad n\geqq 1. \tag{4.12}$$

(A will be determined further.)

(4.13) $$a_{(l+1)n+s}=0,\quad 0\leqq s\leqq l-\nu,\quad 1\leqq n\leqq 2\nu-1.$$

The numbers $a_{(l+1)n+s}$, $0\leqq s\leqq l-\nu$, $n\geqq 2\nu$ we define (by induction in n) as a solution of the system

(4.14) $$\sum_{s=0}^{l-\nu} \omega_j a_{(l+1)n+s}$$

$$=-B^{-1}(\omega_j)\{\sum_{s=-\nu}^{-1}\sum_{k=-s}^{\nu}\alpha_k a_{(l+1)n+s}\omega_j^{k+s}+\sum_{s=l-\nu+1}^{l}\sum_{k=0}^{l-s}\alpha_k a_{(l+1)n+s}\omega_j^{s+k}\}$$

$$+B^{-2}(\omega_j)\sum_{s=-\nu}^{0}\sum_{k=-\nu}^{l-1} M_{k,s}(\omega_j,\ z_1,\dots,\ z_\nu,\ 1,\dots,\ 1)a_{n+s}a_{ln+s}$$

$$1\leqq j\leqq l-\nu+1,\quad (n\geqq 2\nu)$$

($M_{k,s}(\omega_j;\ z_1,\dots,\ z_\nu)$ are the numbers, defined in Lemma 4.) Obviously, the sequence $\{a_n\}_{n=l+1}^{\infty}$ is defined in full with (4.12)-(4.14).

Now we shall choose A, such that

(4.15) $$\limsup_{n\to\infty}|a_n|^{1/n}=\rho^{-1}.$$

From (4.12) and (4.14) we have

(4.16) $$a_{(l+1)n+m}=\sum_{s=-\nu}^{-1} c_{m,s}a_{(l+1)n+s}+\sum_{s=l-\nu+1}^{-l}\bar{c}_{m,s}a_{(l+1)n+s}$$

$$+\sum_{s=-\nu}^{0}\sum_{k=-\nu}^{l-1} N_{k,s}^{m}a_{n+s}a_{ln+k}$$

$$=\rho^{-(l+1)n-m}(c_mA+\sum_{s=-\nu}^{0}\sum_{k=-\nu}^{l-1} N_{k,s}^{m}\rho^{m-k-s}\rho^{n+s}a_{n+s}\rho^{ln+k}a_{ln+k})$$

$$0\leqq m\leqq l-\nu,\quad n\geqq 2\nu.$$

Let

$$M:=\max\{1,\ |c_m|;\quad 0\leqq m\leqq l-\nu\}$$

and

$$N:=\max\{1,\ \sum_{s=-\nu}^{0}\sum_{k=-\nu}^{l-1}|N_{k,s}^{m}|\rho^{m-k-s};\quad 0\leqq m\leqq l-\nu\}.$$

We take $A:=(4MN)^{-1}<(2N)^{-1}$.

From (4.12) and (4.13) follows

$$|a_k|\leqq A\rho^{-k}\leqq(2N)^{-1}\rho^{-k},\quad l+1\leqq k<2\nu(l+1).$$

If for $l+1 \leqq k < n(l+1)$ $(n \geqq 2\nu)$

$$|a_k| \leqq (2N)^{-1}\rho^{-k} \tag{4.17}$$

then with (4.16) we have

$$|a_{(l+1)n+m}| \leqq \rho^{-(l+1)n-m}(MA+N(2N)^{-2}) = \rho^{-(l+1)n-m}(2N)^{-1},$$
$$0 \leqq m \leqq l-1.$$

The last inequalities with (4.12) and (4.17) show that

$$|a_k| \leqq \rho^{-k}(2N)^{-1}, \quad (l+1) \leqq k \leqq (n+1)(l+1)$$

and therefore (4.17) is true for $l+1 \leqq k < \infty$. From (4.12) and (4.17) follows (4.15).

At last we choose $a_0, a_1, \ldots, a_l$ such that

$$f^*(z_j)=1, \quad 1 \leqq j \leqq \nu, \quad (f^*(z) = \sum_{n=0}^{\infty} a_n z^n).$$

(This choice is possible so that $l+1>\nu$.)

Let $F^*(z)=f^*(z)/B(z)$. $F^*(z) \in M_\rho(B(z))$ and with Lemma 4 and (4.14) we get

$$\bar{\Delta}^l_{n-\nu-1,\nu}(\omega_j, F^*) - \omega_j^l \bar{\Delta}^l_{n-\nu,\nu}(\omega_j, F^*) = 0\,((q/\rho)^{(l+1)n}\omega_j^n)$$
$$= 0\,((qK(\omega_j))^n), \quad 1 \leqq j \leqq l-\nu+1$$

which with Lemma 1 shows that $S(\omega_j, F^*) < K(\omega_j)$.

Remark. Theorem 4 concerns only the case $l \geqq \nu$. We do not know what happens if $l<\nu$. Whether in this case there is a function $F^*(z) \in M_\rho(\nu)$ with $S(z, F^*) < K(z)$ for some points $z \in \mathrm{C} \backslash \bar{D}_\rho$? And more generally, whether there is some $F^*(z) \in M_\rho(\nu)$ with $S(z, F^*) < K(z)$ for $1 \leqq j \leqq k$, $(\omega_j \in \mathrm{C} \backslash \bar{D}_\rho$, $1 \leqq j \leqq k)$ if $l-\nu+1<k<l+\nu+1$.?

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