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Some Properties of Proximate Orders for Analytic Functions

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0. To obtain a more refined measure of growth of an analytic function that is given by the growth parameters viz., order and type, O. P. Juneja and G. P. Kapoor [1] considered a real function $\rho(r)$ in $0<r<1$ having the following properties:

(i) $\rho(r)\to\rho$ as $r\to 1$, $0<\rho<\infty$,
(ii) $\lim\limits_{r\to 1} -\rho'(r)\ (1-r)\ \log\ (1-r)=0$,

where $\rho'(r)$ denotes the derivative of $\rho(r)$. Such a function $\rho(r)$ is called a proximate order.

The outlines of this note are: We start with proving an approximation theorem for arbitrary proximate orders and then define a slowly growing function in reference to $\rho(r)$ for giving the result based on it. Ultimately, proximate order is constructed for a function, analytic in $U=\{z : z\in \mathrm{C},\ |z|<1\}$ having finite positive order under certain conditions.

1. Now we prove

Theorem 1. *For every continuous differentiable proximate order $\rho(r)$ of a function, analytic in U, there exists a twice continuously differentiable proximate order $\rho_1(r)$ satisfying*

(1.1) $$|\rho_1(r)-\rho(r)|=o[(\log\ (1-r)^{-1})^{-1}]$$

and

(1.2) $$\lim_{r\to 1}\ (1-r)^2\ \log(1-r)\quad \rho''(r)=0.$$

Proof. Initially, we assume that $\rho_1(r)$ is a proximate order and moreover, it coincides with $\rho(r)$ on the sequence $\{r_n\}$ of positive numbers such as

(1.3) $$\rho_1(r_n)=\rho(r_n),\quad r_n=1-\frac{1}{4^n},\quad n=0,\ 1,\ 2,\ldots$$

For all values of r in the semi-closed interval $[r_n, r_{n+1})$, we observe that

$$|\rho_1(r)-\rho(r)|=|\int_{r_n}^{r}(\rho_1'(t)-\rho'(t))dt|$$
$$=o(\log\frac{\log(1-r)}{\log(1-r_n)})$$
$$=o\{(-\log(1-r))^{-1}\} \quad \text{as } r\to 1.$$

Thus it is sufficient to construct a twice continuously differentiable proximate order $\rho_1(r)$ satisfying the conditions (1.3) and (1.2). On the interval [0, 3/4], we define the functions

$$\Phi(t)=\begin{cases} t & , \quad 0\leqq t\leqq\frac{1}{4}, \\ -2t+3/4, & \frac{1}{4}\leqq t\leqq\frac{1}{2}, \\ t-3/4 & , \quad \frac{1}{2}\leqq t\leqq\frac{3}{4}, \end{cases}$$

and

$$\psi(\sigma)=\int_0^{\sigma}\Phi(t)dt, \quad 0\leqq\sigma\leqq 3/4.$$

Since $\Phi(t)$ is continuous on [0, 3/4], it follows that $\psi(\sigma)$ is continuously differentiable on the same interval. We also note:

(a) $0=\psi(0)=\psi(\frac{3}{4})=\psi'(0)=\psi'(\frac{3}{4})$,

(b) $0\leqq\psi(\sigma)<\frac{3}{16}$,

(c) $|\psi'(\sigma)|\leqq\frac{3}{4}$,

(d) $\int_0^{3/4}\psi(\sigma)d\sigma=\alpha>0.$

Consider a sequence $\{\varepsilon_n\}$ of positive real numbers such that

$$\varepsilon_n=\frac{\rho(r_{n+1})-\rho(r_n)}{\alpha}\log(1-r_n).$$

By the properties (iii), it may be shown that $\varepsilon_n\to 0$ as $n\to\infty$.

Finally, we define the function

$$(1.4) \qquad \rho_1(r)=\rho(r_n)-\frac{\varepsilon_n}{(1-r_n)\log(1-r_n)}\int_{r_n}^{r}\psi(\frac{t-r_n}{1-r_n})dt$$

on the interval $[r_n, r_{n+1}]$, $n=0, 1, 2,\ldots$.

The verification of properties (i) and (ii) and derivation of the result (1.2) for $\rho_1(r)$ defined by (1.4) is left to the reader.

For the view point of studying the properties of a proximate order of an analytic function we need the concept of slowly growing function. A real-valued function $L(r)$, $0<r<1$ is said to be a slowly growing at 1, if for every k, $1<k<\infty$,

$$\lim_{r\to 1} \frac{L(r+\frac{1}{k}(1-r))}{L(r)}=1. \tag{1.5}$$

Theorem 2. *Let $\rho(r)$ be a proximate order of a function, analytic in U and having nonzero finite order ρ. Then, $(1-r)^{-\rho(r)+\rho}$ is a slowly growing function of r in $0<r<1$.*

P r o o f. Let

$$L(r)=(1-r)^{-\rho(r)+\rho}.$$

Then by logarithmic differentiation, we have

$$\frac{L'(r)}{L(r)}=-\rho'(r)\log(1-r)+\frac{\rho(r)-\rho}{(1-r)}.$$

Hence

$$\frac{(1-r)L'(r)}{L(r)}=-\rho'(r)(1-r)\log(1-r)+\rho(r)-\rho. \tag{1.6}$$

From the definition of a proximate order we have asymptotically

$$|\rho(r)-\rho|<\frac{\varepsilon}{2} \quad\text{and}\quad |\rho'(r)(1-r)\log(1-r)|<\frac{\varepsilon}{2}. \tag{1.7}$$

Using (1.7) in (1.6), it follows that

$$\int_r^{r+\frac{1}{k}(1-r)} \frac{L'(t)}{L(t)}dt< \int_r^{r+\frac{1}{k}(1-r)} \frac{\varepsilon}{1-t}dt.$$

Thus, for all values of r sufficiently close to 1,

$$\varepsilon\log(1-\frac{1}{k})<\log\frac{L(r+\frac{1}{k}(1-r))}{L(r)}<-\varepsilon\log(1-\frac{1}{k}).$$

Since $\varepsilon>0$ is arbitrary, the above inequality leads to

$$\lim_{r\to 1}\log\frac{L(r+\frac{1}{k}(1-r))}{L(r)}=0,$$

which implies the required assertion (1.5).

Remark. The proof of the above theorem is extremely simple and markedly different from the proof based on the Lagrange mean value theorem given by O.P. Juneja and G. P. Kapoor [1, Theorem 1.6.2, pp 58-59]. Also, B. Ja. Levin [2, Lemma 5, pp 32-33] used mean value theorem to prove a similar result on slowly growing functions in reference to a proximate order of entire functions.

2. Let $f(z)$ be a function analytic in U and having the order $\rho(0<\rho<\infty)$. If, in addition to (i) and (ii) $\rho(r)$ satisfies also (iii) $\rho(i)$ is continuous and piecewise differentiable for $r>r_0$,

(iv) $$\limsup_{r\to 1} \frac{\log M(r,f)}{(1-r)^{-\rho}} = T^*.$$

If T^* is different from zero and infinity then the function $\rho(r)$ satisfying (i) to (iv) is called a proximate order of the given function f. Next we construct proximate orders for a class of analytic functions in U. To establish the last result we need the following

Lemma. *For a function f, analytic in U and having order ρ and lower order λ, we find*

(2.1) $$\liminf_{r\to 1} \frac{(1-r)M'(r,f)}{M(r,f)\log M(r,f)} \leqq \lambda \leqq \rho \leqq \limsup_{r\to 1} \frac{(1-r)M'(r,f)}{M(r,f)\log M(r,f)},$$

where $M'(r,f)$ is the derivative of $M(r,f)$.

Proof. Let

$$\limsup_{r\to 1} \frac{(1-r)M'(r,f)}{M(r,f)\log M(r,f)} = B.$$

Then, for given $\varepsilon>0$ and r sufficiently close to 1,

$$\frac{(1-r)M'(r,f)}{M(r,f)\log M(r,f)} < B+\varepsilon.$$

Integrating this, we get

$$\log\log M(r,f) < O(1) + (B+\varepsilon)\log(1-r)^{-1}.$$

Passing to limits and using the result [1, p. 43], we obtain

$$\rho \leqq \limsup_{r\to 1} \frac{(1-r)M'(r,f)}{M(r,f)\log M(r,f)}.$$

Similarly, it can be shown that

$$\lambda \geqq \liminf_{r\to 1} \frac{(1-r)M'(r,f)}{M(r,f)\log M(r,f)}.$$

Combining the above two inequalities (2.1) is immediate.

Definition. *An analytic function is said to be of regular growth if* $0<\lambda=\rho<\infty$.

Theorem 3. *Let* $f(z)$ *be analytic in* U *and with an order* $\rho(0<\rho<\infty)$ *such that limits in* (2.1) *exist. Then* $\log(\alpha^{-1}.\log M(r, f))/\log(1-r)^{-1}$ *is a proximate order of* $f(z)$.

Proof. For a given positive number $\alpha(0<\alpha<\infty)$, let

$$\xi(r)=\frac{\log(\alpha^{-1}\log M(r,f))}{\log(1-r)^{-1}}. \tag{2.2}$$

$\log M(r,f)$ is a real, continuous and increasing function of r for $r>r_0>0$, which is differentiable in adjacent open intervals, it follows that $\xi(r)$ satisfies (iii). Since existence of limits in (2.1) implies $f(z)$ is of regular growth and moreover, $\xi(r)\to\rho$ as $r\to 1$.

Differentiating (2.2), we have

$$\xi'(r)=-\frac{M'(r, f)}{\log(1-r)M(r, f)\log M(r, f)}+\frac{\log(\alpha^{-1}\log M(r, f))}{(\log(1-r))^2(1-r)},$$

so that

$$-(1-r)\log(1-r)\xi'(r)=\frac{(1-r)M'(r,f)}{M(r, f)\log M(r,f)}-\frac{\log\log M(r,f)}{\log(1-r)^{-1}}+o(1).$$

On using the hypothesis of Theorem 3 together with the Lemma we conclude that $-\xi'(r)(1-r)\log(1-r)\to 0$ as $r\to 1$. Thus $\xi(r)$ satisfies (ii) also. Finally, (iv) is the direct consequence of (2.2). All the assertions for $\xi(r)$ to be proximate orders are satisfied and hence the theorem.

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