Mathematica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

---

Mathematica Balkanica

New Series Vol. 3, 1989, Fasc. 1

Direct Estimation for Approximation by Bernstein Polynomials in $L_p[0, 1].(0<p<1)$

*Gancho T. Tachev**

Presented by V. Popov

The purpose of the present paper is to evaluate the error of the approximation of the function $f(x)$, defined in [0, 1] by Bernstein polynomials in L_p-metric $(0<p<1)$.

1. Main Result

In this paper we consider the approximation of a function $f(x)$, defined in [0, 1] by Bernstein polynomials:

$$B_n(f, x)=\sum_{k=0}^{n} f(k/n)p_{n,k}(x);\; p_{n,k}(x)=\binom{n}{k}x^k(1-x)^{n-k}.$$

Let $M[0, 1]$ be the set of all measurable and bounded in [0, 1] functions. We set

$$\Delta_h^k f(x)=\begin{cases}\sum_{m=0}^{k}(-1)^{k-m}\binom{k}{m}.f(x+mh), & \text{if } x,\ x+k.h\in[0,1]\\ 0, & \text{otherwise}\end{cases}$$

$$\omega_k(f,\ x;\ \delta(x))=\sup\{|\Delta_h^k f(x)|:t,\ t+k.h\in[x-\frac{k.\delta(x)}{2},\ x+\frac{k.\delta(x)}{2}]\}$$

$$\tau_k(f;\delta)_p=\|\omega_k(f,.,\ (\cdot))\|_p.$$

In this paper we use for δ the following function:

$$\delta(x)=\Delta_{\sqrt{n}}(x)=\sqrt{x(1-x)}/\sqrt{n}+1/(2n).$$

Now let us state our Main Result.

Theorem: *Let* $f(x)\in M[0,\ 1]$, $0<p\leqq\infty$.
Then $\|f(x)-B_n(f,\ x)\|_p\leqq c(p).\tau_2(f,\ \Delta_{\sqrt{n}}(x))_p$,
where $c(p)$ *is a constant, depending only on the parameter* p.

---

* Supported by contract No 50-88 with the Department of Culture, Sciences and Education, Bulgaria.

Remark : The case $1 \leqq p \leqq \infty$ is proved in [3]. So we shall consider only the case $0<p<1$.

Corollary 1. *Taking into consideration the improvement of the approximation near the end-points of the interval, we get as an immediate consequence of the Theorem, that*

$$\|B_n f - f\|_p \leqq c(p) \cdot \tau_2(f, 1/\sqrt{n})_p.$$

This inequality is an analogue of Theorem 4.1 in [1] for the case $0<p<1$.

Corollary 2. *If $g(x)$ is a piecewise linear function, defined on* [0, 1], *then*

$$\|B_n g - g\|_p \leqq c(p) \cdot (1/\sqrt{n})^{1+1/p} = o(1/n), \quad 0<p<1.$$

So we obtain that the order of approximation of the function $g(x)$ by Bernstein operator in L_p-metric $(0<p<1)$ is better than the saturation order of the same operator for the case $1<p \leqq \infty$ (see [3]).

In sections 2., 3., and 4. we shall evaluate by $\tau_2(f, \Delta_{\sqrt{n}}(x))_p$, respectively $\|f-S\|_p$, $\|S-B_n S\|_p$ and $\|B_n S - B_n f\|_p$. (S is an appropriate auxiliary function.) The proof of the Theorem follows from the above estimations.

2. Estimation for $\|f-S\|_p$

We denote:
$$\begin{cases} b_0 = 0, \\ b_1 = b_0 + \Delta_{\sqrt{n}}(b_0), \\ b_2 = b_1 + \Delta_{\sqrt{n}}(b_1), \\ \dots\dots\dots\dots\dots\dots \\ b_{i+1} = b_i + \Delta_{\sqrt{n}}(b_i), \\ \dots\dots\dots\dots\dots\dots \\ b_N \leqq 1, \; b_{N+1} > 1, \end{cases}$$

and let $S(x)$ be the piecewise linear function, which interpolates $f(x)$ at the points b_i, $i=0, \dots, N$.
For $x \in [b_i, b_{i+1}]$, it follows (see Lemma 2.3 in [1])

$$(2.1) \qquad |f(x) - S(x)| \leqq \omega_2(f, [b_i, b_{i+1}]) \leqq \omega_2(f, x; \Delta_{\sqrt{n}}(b_i)).$$

From (2.1) in [2], for $\lambda = 1$ we obtain

$$(2.2) \qquad \frac{1}{6} \cdot \Delta_{\sqrt{n}}(b_i) \leqq \Delta_{\sqrt{n}}(x) \leqq \frac{7}{2} \cdot \Delta_{\sqrt{n}}(b_i), \text{ for } x : |x - b_i| \leqq \Delta_{\sqrt{n}}(b_i).$$

Then for $x \in [b_i, b_{i+1}]$, the following inequality

$$|f(x) - S(x)| \leqq \omega_2(f, x; 6 \cdot \Delta_{\sqrt{n}}(x))$$

holds true. Taking the p-powers of both sides and integrating we obtain

$$\|f-S\|_p^p \leqq \tau_2(f;\ 6\,.\,\Delta_{\sqrt{n}}(x))^p.$$

We can take the const 6 out of the modulus as it is shown in [3] (see p. 426). So we obtain

$$\|f-S\|_p \leqq c(p)\,.\,\tau_2(f,\ \Delta_{\sqrt{n}})_p. \tag{2.3}$$

3. Estimation for $\|S-B_nS\|_p$

Let us suppose that $f(0)=f(1)=0$.
Then $S(x)$ may be written as follows

$$S(x)=\sum_{i=0}^{N} a_i(x-b_i)_+.$$

Later we prove that

$$\|B_n(x-b_i)_+ -(x-b_i)_+\|_p^p \leqq c(p)\,.\,\Delta^{p+1}_{\sqrt{n}}\ (b_i). \text{ Then} \tag{3.1}$$

$$\|B_nS-S\|_p^p \leqq c(p)\,.\,\sum_{i=0}^{N} |a_i|^p\,\Delta^{p+1}_{\sqrt{n}}\ (b_i). \tag{3.2}$$

Let $f_i=f(b_i)=S(b_i)$,
$f_0=0$,
$f_1=S(b_1)=a_0\Delta_{\sqrt{n}}(b_0)$, that is $a_0=f_1/\Delta_{\sqrt{n}}(b_0)$,
....
$f_{i+1}=S(b_{i+1})=a_0\,.\,(b_{i+1}-b_0)+a_1\,.\,(b_{i+1}-b_1)+\ldots+a_i(b_{i+1}-b_i)$,

$$a_i\Delta_{\sqrt{n}}(b_i)=f_{i+1}-D_i, \text{ where we set} \tag{3.3}$$

$$\begin{aligned} D_i=\ & a_0[\Delta_{\sqrt{n}}(b_0)+\Delta_{\sqrt{n}}(b_1)+\ldots+\Delta_{\sqrt{n}}(b_i)] \\ & +a_1[\Delta_{\sqrt{n}}(b_1)+\ldots+\Delta_{\sqrt{n}}(b_i)] \\ & +\ldots.. \\ & +a_{i-1}\,.\,[\Delta_{\sqrt{n}}(b_{i-1})+\Delta_{\sqrt{n}}(b_i)]. \end{aligned} \tag{3.4}$$

$$\begin{aligned} S(b_i)=\ & a_0\,.\,[\Delta_{\sqrt{n}}(b_0)+\ldots+\Delta_{\sqrt{n}}(b_{i-1})] \\ & +a_1\,.\,[\Delta_{\sqrt{n}}(b_1)+\ldots+\Delta_{\sqrt{n}}(b_{i-1})] \\ & +\ldots\ldots. \\ & +a_{i-1}\,.\,\Delta_{\sqrt{n}}(b_{i-1}). \end{aligned} \tag{3.5}$$

Let us rewrite (3.3) in the form

$$a_i \,.\, \Delta_{\sqrt{n}}(b_i) = f(b_{i+1}) - 2\,.\,f((b_{i-1}+b_{i+1})/2) + f(b_{i-1}) \tag{3.6}$$
$$+2\,.\,f((b_{i-1}+b_{i+1})/2) - 2\,.\,S((b_{i-1}+b_{i+1})/2)$$
$$+2\,.\,S((b_{i-1}+b_{i+1})/2) - f(b_{i-1}) - D_i.$$

Using the definition of $S(x)$, after elementary calculations we obtain

$$2\,.\,S((b_{i-1}+b_{i+1})/2) - f(b_{i-1}) - D_i = 0, \text{ for } b_i > \frac{1}{2} \text{ and} \tag{3.7}$$

$$2\,.\,S((b_{i-1}+b_{i+1})/2) - f(b_{i-1}) - D_i = a_i \Delta_{\sqrt{n}}(b_i) - a_i \,.\, \Delta_{\sqrt{n}}(b_{i-1}), \text{ for } b_i \leqq 1/2.$$

In both cases, from (3.7), (3.6), (2.1) and (2.2) it follows:

$$|a_i|\,.\,\Delta_{\sqrt{n}}(b_i) \leqq c_1 \,.\, \omega_2(f,\ x;\ c_2 \,.\, \Delta_{\sqrt{n}}(x)), \text{ for } x \in [b_i,\ b_{i+1}],\ i \geqq 1, \tag{3.8}$$

where c_1, c_2 are absolute constants.

$$\int_{b_i}^{b_{i+1}} |a_i|^p \,.\, \Delta^p_{\sqrt{n}}(b_i)\,dx \leqq c_1^p \,.\, \int_{b_i}^{b_{i+1}} \omega_2^p(f,\ x;\ c_2 \Delta_{\sqrt{n}}(x))\,dx,$$

$$\sum_{i=1}^{N} |a_i|^p \,.\, \Delta^{p+1}_{\sqrt{n}}(b_i) \leqq c_1^p \int_{b_1}^{1} \omega_2^p(f,\ x;\ c_2 \Delta_{\sqrt{n}}(x))\,dx$$

$$\leqq c_1^p \cdot \tau_2^p(f,\ c_2 \,.\, \Delta_{\sqrt{n}})_p \leqq c(p) \,.\, \tau_2^p(f,\ \Delta_{\sqrt{n}})_p.$$

In the last inequality we use the property of τ_2, that we can take the constant c_2 out of the modulus (see [3]). To represent the coefficient a_0, we use that $f(1)=0$.

$$f(1) = a_0 + a_1\,.\,(1-b_1) + \ldots + a_N(1-b_N) = 0.$$

$$|a_0| \leqq \sum_{i=1}^{N} |a_i|,$$

$$|a_0|^p \,.\, \Delta^{p+1}_{\sqrt{n}}(b_0) \leqq \sum_{i=1}^{N} |a_i|^p \,.\, \Delta^{p+1}_{\sqrt{n}}(b_i).$$

From (3.2) we have

$$\|B_n S - S\|_p^p \leqq c(p)\,.\,\tau_2^p(f,\ \Delta_{\sqrt{n}})_p. \tag{3.9}$$

We only need the verification of (3.1). It follows from the following Lemma.

Lemma 3.1. *For every* $a \in [0,\ 1]$ *and* $g(x) = |x-a|$, *the following inequality*

(3.10) $$\| B_n g - g \|_p^p \leqq c(p) . \Delta^{p+1}_{\sqrt{n}} \quad (a),$$

holds.

Proof: It is enough to consider $a \leqq 1/2$.

I. Let $a \in [1/n, 1/2]$. We choose α, so that $0 < \alpha < 1/2$.

(3.11) $$\int_0^1 | B_n |x-a| - |x-a| |^p \, dx = \int_0^{a - n^{\alpha}/\sqrt{n}} + \int_{a-n^{\alpha}/\sqrt{n}}^{a - \Delta_{\sqrt{n}}(a)} + \int_{a - \Delta_{\sqrt{n}}(a)}^{a + \Delta_{\sqrt{n}}(a)} + \int_{a + \Delta_{\sqrt{n}}(a)}^{a + n^{\alpha}/\sqrt{n}}$$

$$+ \int_{a + n^{\alpha}/\sqrt{n}}^{1} |B_n|x-a| - |x-a||^p \, dx = I_1 + I_2 + I_3 + I_4 + I_5 .$$

From Popoviciu's Theorem (see [4]), we have

$$|g(x) - B_n g(x)| \leqq 3 . \omega(g, \sqrt{x . (1-x)/n}) \leqq 3 . \omega(g, \Delta_{\sqrt{n}}(x)) \leqq c . \Delta_{\sqrt{n}}(a),$$

for $x \in [a - \Delta_{\sqrt{n}}(a), a + \Delta_{\sqrt{n}}(a)]$. The last inequality follows from (2.2).

$$I_3 = \int_{a - \Delta_{\sqrt{n}}(a)}^{a + \Delta_{\sqrt{n}}(a)} |B_n g - g|^p \, dx \leqq \int_{a - \Delta_{\sqrt{n}}(a)}^{a + \Delta_{\sqrt{n}}(a)} c^p . \Delta^p_{\sqrt{n}}(a) . dx \leqq c(p) . \Delta^{p+1}_{\sqrt{n}}(a).$$

Now we evaluate I_1 and I_5 integrals. Let $n^{\alpha}/\sqrt{n} = \delta$. At every point $x \in [0, a - \delta]$, $g(x)$ has finite derivative $g''(x) = 0$. From Taylor's formula we get

(3.12) $$g(k/n) = g(x) + g'(x) . (k/n - x) + [g''(x)/2 + \lambda(k/n)] . (k/n - x)^2,$$

where $|\lambda(t) . (t-x)^2|$ is a bounded function.

(3.13) $$B_n(g, x) = B_n g(x) = g(x) + \sum_{k=0}^{n} \lambda(k/n) . (k/n - x)^2 . p_{n,k}(x),$$

(3.14) $$|B_n g - g| \leqq \sum_{k=0}^{n} |\lambda(k/n)| (k/n - x)^2 . p_{n,k}(x) = \sum_{1\, |k/n - x| < \delta} + \sum_{2\, |k/n - x| \geqq \delta}$$

$\Sigma_1 = 0$, because in (3.12) $g(k/n) = g(x) + g'(x) . (k/n - x)$, that is $\lambda(k/n) = 0$. Let M be an upper bound of bounded function $|\lambda(t) . (t-x)^2|$. We use the following statement [see Lemma 3. in [4], p. 248]

$$\sum_{k : |k/n - x| \geqq \left(\frac{1}{n}\right)^{\beta}} p_{n,k}(x) \leqq K(2m) . \left(\frac{1}{n}\right)^{2m . (1/2 - \beta)},$$

where $0 < \beta < \frac{1}{2}$, $m \in N$, K is a constant, depending only on m.

It follows that $\Sigma_2 \leqq M . K(2m) . (\frac{1}{n})^{2m\alpha}$, for $\beta = \frac{1}{2} - \alpha$. The parameter m is so chosen, that the following inequality

$$(\frac{1}{n})^{2m\alpha} \leqq \Delta_{\sqrt{n}}^{1+1/p}(\frac{1}{n}) \leqq \Delta_{\sqrt{n}}^{1+1/p}(a), \quad a \in [1/n, 1/2]$$

holds. Then from (3.14) we get

$$|g(x) - B_n g(x)| \leqq c(p) . \Delta_{\sqrt{n}}^{p+1}(a), \text{ for } x \in [0, a - n^\alpha/\sqrt{n}] \text{ and}$$

$|I_1| \leqq c(p) . \Delta_{\sqrt{n}}^{p+1}(a)$. Analogous is the estimation for the I_5 integral. Now let us consider the I_4 integral.

$$B_n(g, x) - g(x) = \rho_{n,a}(x)$$

$$\int_{a+\Delta_{\sqrt{n}}(a)}^{a+n^\alpha/\sqrt{n}} \rho_{n,a}^p(x)\,dx \leqq \sum_{1 \leqq i \leqq n^\gamma} \rho_{n,a}^p(a + i . \Delta_{\sqrt{n}}(a)) . \Delta_{\sqrt{n}}(a), \text{ where}$$

$$a + ([n^\gamma] + 1) . \Delta_{\sqrt{n}}(a) > a + n^\alpha/\sqrt{n} \text{ and } a + [n^\gamma] . \Delta_{\sqrt{n}}(a) \leqq a + n^\alpha/\sqrt{n}.$$

It is enough to prove that

$$\text{(3.15)} \qquad \sum_{1 \leqq i \leqq n^\gamma} \rho_{n,a}^p(a + i . \Delta_{\sqrt{n}}(a)) \leqq c(p) . \Delta_{\sqrt{n}}^p(a), \text{ for } a \in [1/n, 1/2],$$

$$\rho_{n,a}(x) = \sum_{k=0}^{[na]} (2a - 2 . \frac{k}{n}) . p_{n,k}(x). \text{ Let } k = [na],$$

$$\text{(3.16)} \qquad \frac{p_{n,k}(a + 2 . \Delta_{\sqrt{n}}(a))}{p_{n,k}(a + \Delta_{\sqrt{n}}(a))} \leqq \left(\frac{a + \Delta_{\sqrt{n}}(a)}{a}\right)^{[na]} . \left(\frac{1 - a - \Delta_{\sqrt{n}}(a)}{1 - a}\right)^{n - [na]}$$

We set in (3.16) $a = 1/2$,

$$\frac{p_{n,[n/2]}(1/2 + \Delta_{\sqrt{n}}(1/2))}{p_{n,[n/2]}(1/2)} \approx (1 - \frac{1}{n})^{\frac{n}{2}} \leqq \exp(-1/2) = q < 1,$$

$$\text{(3.17)} \qquad \frac{p_{n,[n/2]}(1/2 + (i+1)\Delta_{\sqrt{n}}(1/2))}{p_{n,[n/2]}(1/2 + i . \Delta_{\sqrt{n}}(1/2))} < \frac{p_{n,[n/2]}(1/2 + \Delta_{\sqrt{n}}(1/2))}{p_{n,[n/2]}(1/2)} \leqq q,$$

for every $i \in [0, n^\gamma]$; $k \in [0, [na]]$, because the degree of the first multiplicand in (3.16) decreases, but of the second increases. So

$$\frac{\rho_{n,1/2}(1/2 + (i+1) . \Delta_{\sqrt{n}}(1/2))}{\rho_{n,1/2}(1/2 + i . \Delta_{\sqrt{n}}(1/2))} \leqq q, \text{ for } 1 \leqq i \leqq n^\gamma.$$

From (3.15) we have

$$\sum_{1\leq i\leq n^{\gamma}} \rho^{p}_{n,1/2}(1/2+i.\Delta_{\sqrt{n}}(1/2))\leq \frac{1}{1-q^{p}}.\ \rho^{p}_{n,1/2}(1/2+\Delta_{\sqrt{n}}(1/2)).$$

The Popoviciu's theorem and (2.2) give

$$\rho^{p}_{n,1/2}(1/2+\Delta_{\sqrt{n}}(1/2))\leq c(p).\Delta^{p}{}_{\sqrt{n}}(1/2).$$

(3.15) *is proved for* $a=1/2$. Following the considerations from (3.16) downward, to prove (3.15) for $a\in[1/n,\ 1/2)$ it is enough to prove that

$$\text{(3.18)}\qquad \varphi(a)=\frac{p_{n,[na]}(a+\Delta_{\sqrt{n}}(a))}{p_{n,[na]}(a)}\leq q<1.$$

The last assertion follows from Lemma 4.1 in the section 4. The estimation for the I_2 integral in (3.11) is similar to that for the I_4

II. Let $a<1/n$. We introduce the function $h(x)=\begin{cases}\Delta_{\sqrt{n}}(a), & 0\leq x\leq a,\\ 0, & a<x\leq 1.\end{cases}$

$$\int_0^1 |B_n|x-a|-|x-a||^p\,dx=2.\int_0^1 |B_n(a-x)_+-(a-x)_+|^p\,dx$$

$$\leq 2.\int_0^1 B_n^p(a-x)_+\,dx\leq 2.\int_0^1 B_n^p h(x)\,dx\leq 2.c(p).\Delta^{p+1}_{\sqrt{n}}\quad(a).$$

The last inequality follows from Lemma 4.2 in section 4. So we proved Lemma 3.1. This completes the proof of (3.1) and (3.9).

4. Estimation for $\|B_nf-B_nS\|_p$

We start with the following Lemma:

Lemma 4.1. *For* $a\in[0,\ 1/2]$, *the following inequality*

$$\text{(4.1)}\qquad \varphi(a)=\frac{p_{n,[na]}(a+\Delta_{\sqrt{n}}(a))}{p_{n,[na]}(a)}\leq q<1$$

holds true.

Proof:

$$\varphi(a)=[1+\sqrt{(1-a)/(na)}+1/(2na)]^{[na]}.[1-\sqrt{a/(n.(1-a))}-1/(2n(1-a))]^{n-[na]}.$$

If $a<2/n$, then $\varphi(a)\leq\exp(-1/2)=q<1$.

Let $\lim_{n\to\infty} na=c\geq 2$, $c=\text{const}$. We have

(4.2) $$\varphi(a) \approx \psi(c)\,.\exp(-1/2), \text{ where } \psi(c) \text{ denotes}$$

(4.3) $$\psi(c)=[1+\sqrt{(1-c/n)/c}+1/(2\,.\,c)]^c\,.\exp(-\sqrt{c\,.\,(1-c/n)}).$$

We use that

(4.4) $$\ln(1+x)\leqq x-x^2/2+x^3/3, \text{ for } x\in(0,\ 1).$$

From (4.4) and (4.3) we obtain

$$\ln \psi(c)\leqq c/(2\,.\,n)+(1/\sqrt{c}+3/(2\,.\,c))/3\leqq c/(2n)+(1/\sqrt{2}+3/4)/3.$$

Then the following inequality

(4.5) $$\psi(c)\,.\exp(-1/2)< \exp(-d),$$

holds, where d is an absolute positive constant and $d<1/2-(1/\sqrt{2}+3/4)/3$.
Now let $\lim_{n\to\infty} na=\infty$. A simple calculation and (4.4) give

$$\ln\varphi(a) \;=-\frac{1}{2}+O\,(\frac{1}{\sqrt{na}}),$$ that is $\varphi(a)\approx\exp(-1/2)$. So (4.1) is proved.

Lemma 4.2. *By denotations in section 2. we have*

(4.6) $$\int_0^1(\sum_{b_i\leqq k/n\leqq b_{i+1}} p_{n,k}(x))^p\,dx\leqq c(p)\,.\,\Delta_{\sqrt{n}}(b_i),\quad i=0,\ldots,\ N.$$

Proof: We set $R_{i,n}(x)=\sum_{b_i\leqq k/n\leqq b_{i+1}} p_{n,k}(x)$,

$$\int_0^1 R^p_{i,n}(x)\,dx=\int_0^{b_i}+\int_{b_i}^{b_{i+1}}+\int_{b_{i+1}}^1 R^p_{i,n}(x)\,dx=I_1+I_2+I_3.$$

The estimation for I_2 is obvious. We consider I_3.

$$I_3\leqq \sum_{0\leqq j\leqq m_i} R^p_{i,n}(b_{i+1}+j\,.\,\Delta_{\sqrt{n}}(b_{i+1}))\,.\,\Delta_{\sqrt{n}}(b_{i+1}), \text{ where}$$

$$b_{i+1}+([m_i]+1)\,.\,\Delta_{\sqrt{n}}(b_{i+1})>1 \text{ and } b_{i+1}+[m_i]\,.\,\Delta_{\sqrt{n}}(b_{i+1})\leqq 1.$$

It is easy to notice that for $k_0=[n\,.\,b_{i+1}]$ we have

(4.7) $$\frac{p_{n,k_0}(b_{i+1}+\Delta_{\sqrt{n}}(b_{i+1}))}{P_{n,k_0}(b_{i+1})}\geqq\frac{P_{n,k_0}(b_{i+1}+j\,.\,\Delta_{\sqrt{n}}(b_{i+1}))}{P_{n,k_0}(b_{i+1}+(j-1)\,.\,\Delta_{\sqrt{n}}(b_{i+1}))},\quad j\geqq 2.$$

Using (4.1), (4.7) and the considerations in 3. from (3.15) to (3.18) we get

$$I_3\leqq\frac{1}{1-q^p}\,.\,\Delta_{\sqrt{n}}(b_{i+1})\leqq c(p)\,.\,\Delta_{\sqrt{n}}(b_i).$$

The estimation for I_1 is similar to the above with a slightly modification of the function $\varphi(a)$. The proof of Lemma 4.2 is completed. Let us formulate and prove the following basic inequality

Lemma 4.3. *For every sequence* $\{a_k\}$ *with the property*

$$\begin{cases} a_k \geqq 0, \ k=0, 1, \dots, n \\ a_k = 0, \ \text{otherwise, we have} \end{cases}$$

$$(4.8) \qquad \int_0^1 \Big| \sum_{k=0}^n a_k \cdot p_{n,k}(x) \Big|^p dx \leqq c(p) \cdot \frac{1}{n} \sum_{k=0}^n \max_{|\frac{j}{n}-\frac{k}{n}| \leqq 6 \cdot \Delta_{\sqrt{n}}(\frac{k}{n})} |a_j|^p.$$

Proof: Using Lemma 4.2 we obtain

$$(4.9) \qquad \int_0^1 \Big| \sum_{k=0}^n a_k \cdot p_{n,k}(x) \Big|^p dx \leqq \sum_{i=0}^N \int_0^1 \Big| \sum_{b_i \leqq k/n \leqq b_{i+1}} a_k p_{n,k}(x) \Big|^p dx$$

$\leqq \sum_{i=0}^N c(p) \cdot \Delta_{\sqrt{n}}(b_i) \cdot \max_{b_i \leqq k/n \leqq b_{i+1}} |a_k|^p$. It is obviously, that

$$(4.10) \qquad \Delta_{\sqrt{n}}(b_i) \cdot \max_{nb_i \leqq k \leqq nb_{i+1}} |a_k|^p \leqq \frac{2}{n} \cdot \sum_{nb_i \leqq k \leqq nb_{i+1}} \max_{|\frac{j}{n}-\frac{k}{n}| \leqq \Delta_{\sqrt{n}}(b_i)} |a_j|^p.$$

Now (4.9), (4.10) and (2.2) give

$$\int_0^1 \Big| \sum_{k=0}^n a_k p_{n,k}(x) \Big|^p dx \leqq c(p) \cdot \frac{1}{n} \cdot \sum_{i=0}^N \sum_{nb_i \leqq k \leqq nb_{i+1}} \max_{|\frac{j}{n}-\frac{k}{n}| \leqq \Delta_{\sqrt{n}}(b_i)} |a_j|^p$$

$$\leqq c(p) \cdot \frac{1}{n} \cdot \sum_{k=0}^n \max_{|\frac{j}{n}-\frac{k}{n}| \leqq 6 \cdot \Delta_{\sqrt{n}}(\frac{k}{n})} |a_j|^p.$$

We are ready to estimate $\|B_n f - B_n S\|_p^p$. Using (2.1) in [2] and Lemma 4.3 we obtain

$$\|B_n f - B_n S\|_p^p = \int_0^1 \Big| \sum_{k=0}^n \Big(f\Big(\frac{k}{n}\Big) - S\Big(\frac{k}{n}\Big)\Big) \cdot p_{n,k}(x) \Big|^p dx$$

$$\leqq \int_0^1 \Big| \sum_{k=0}^n \omega_2 \Big(f, \frac{k}{n}; 6 \cdot \Delta_{\sqrt{n}}\Big(\frac{k}{n}\Big)\Big) \cdot p_{n,k}(x) \Big|^p dx$$

$$\leqq c(p) \cdot \frac{1}{n} \cdot \sum_{k=0}^n \max_{|\frac{j}{n}-\frac{k}{n}| \leqq 6 \cdot \Delta_{\sqrt{n}}(\frac{k}{n})} \omega_2^p \Big(f, \frac{j}{n}; 6 \cdot \Delta_{\sqrt{n}}\Big(\frac{j}{n}\Big)\Big)$$

$$\leqq c(p) \cdot \frac{1}{n} \cdot \sum_{k=0}^n \omega_2^p \Big(f, \frac{k}{n}; c \cdot \Delta_{\sqrt{n}}\Big(\frac{k}{n}\Big)\Big), \quad c = \text{const}$$

$$\frac{1}{n} \cdot \omega_2^p \Big(f, \frac{k}{n}; c \cdot \Delta_{\sqrt{n}}\Big(\frac{k}{n}\Big)\Big) \leqq \int_{k/n}^{(k+1)/n} \omega_2^p (f, x; c' \cdot \Delta_{\sqrt{n}}(x))\, dx.$$

After summing over k in the last inequality we get

$$(4.11)\quad \|B_n f-B_n S\|_p^p \leqq c_1(p).\tau_2^p(f;\ c'.\Delta_{\sqrt{n}}(x))_p \leqq c_2(p).\tau_2^p(f,\ \Delta_{\sqrt{n}}(x))_p.$$

Finally the proof of the Main Theorem follows immediately from (2.3), (3.9) and (4.11).

References

1. Бл. Сендов, В. А. Попов. Усреднени модули на гладкост. С., 1983.
2. K. G. Ivanov. A Constructive Characteristic of the Best Algebraic Approximations in $L_p[-1, 1]$, $(1 \leqq p \leqq \infty)$. Конструктивная теория функций, 81, С., 1983, 357-367.
3. K. G. Ivanov. Approximation by Bernstein Polynomials in L_p-Metric. Constructive Theory of Functions'84. S., 1984, 421-428.
4. И. П. Натансон. Конструктивная теория функций. М., 1948.

Institute of Mathematics
Bulgarian Academy of Sciences
1090 Sofia, P. O. Box 373
BULGARIA

Received 29. 03. 88