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Approximate Determination of the Generalized Polynomial of the Best One-sided Hausdorff Approximation

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In this paper we modify the first Remez' algorithm [1] for finding the polynomial of best uniform approximation to apply it for numerical finding of the polynomial of best Hausdorff approximation. For this purpose we offer an iterative method and the convergence is proved. The efficiency of the proposed algorithm is illustrated by numerical experiments.

1.0. Definitions and denotations

(We keep within the denotations, used in [2])

1.1. Let us denote

$R=\{x : -\infty<x<+\infty\}$, $S^M(R)=\{[a, b] : a, b\in R, -M\leqq a\leqq b\leqq +M, M>0\}$ and $A_X^M=\{f : X\to S^M(R), X=[a, b]\}$ be the set of all segment-valued bounded functions in the interval X.

1.2. If $f\in A_X^M$, we define $F(f)\in A_X^M$ as $F(f;x)=[I(f;x), S(f;x)]$, where

$$I(f;x)=\lim_{\delta\to 0}\inf\{y\in f(t) : t\in[x-\delta, x+\delta]\cap X\}$$

$$S(f;x)=\lim_{\delta\to 0}\sup\{y\in f(t) : t\in[x-\delta, x+\delta]\cap X\}.$$

1.3. Let $H_n=\{p : p(x)=\sum_{i=1}^{n} c_i\cdot g_i(x)\}$ be the set of all generalized polynomials of degree at most n, and the set of base functions $\{g_1, g_2, \ldots, g_n\}$ satisfy the Haar condition, i.e. each g_i is continuous and each determinant

$$\begin{vmatrix} g_1(x_1)\ldots g_n(x_1) \\ \ldots\ldots\ldots\ldots \\ g_1(x_n)\ldots g_n(x_n) \end{vmatrix}$$

composed from n distinct points, $x_1, x_2, \ldots, x_n$ is nonzero [1].

1.4. If the functions $f, g\in A_X^M$ the one-sided Hausdorff distance with parameters $\alpha(x)$ and $\beta(x)$ from g to f is defined as

$$h(X, \alpha, \beta; g, f)=\max_{(x,y)\in F(g)}\ \min_{(u,v)\in F(f)}\ \max\,[|x-u|/\alpha(x), |y-v|/\beta(x)]$$

and the Hausdorff distance with parameters $\alpha(x)$ and $\beta(x)$ between g and f as

$$H(X,\ \alpha,\ \beta;\ g,\ f)=\max\ [h(X,\ \alpha,\ \beta;\ g,\ f),\ h(X,\ \alpha,\ \beta;\ f,\ g)],$$

where $\alpha(x)>0$ and $\beta(x)>0$ for $x\in X$.

1.5. For a given function $f\in A_X^M$ we denote

$$E_h(H_n,\ X,\ \alpha,\ \beta;\ f)=\inf\ \{h(X,\ \alpha,\ \beta;\ p,\ f),\ p\in H_n\}$$

the best approximation of the function f according to the one-sided Hausdorff distance by elements of H_n (the best one-sided Hausdorff approximation). Respectively the best Hausdorff approximation (the best approximation of the function f according to Hausdorff distance in the interval X, with parameters $\alpha(x)$, $\beta(x)$, by elements of H_n) is denote by

$$E_H(H_n,\ X,\ \alpha,\ \beta;\ f)=\inf\ \{H(X,\ \alpha,\ \beta;\ p,\ f),\ p\in H_n\}.$$

The generalized polynomials $p,\ q\in H_n$ are called polynomials of the best one-sided and Hausdorff approximation of $f\in A_X^M$ if they satisfy $E_h(H_n,\ X,\ \alpha,\ \beta;f)=h(X,\ \alpha,\ \beta;\ p,\ f)$ and $E_H(H_n,\ X,\ \alpha,\ \beta;f)=H(X,\ \alpha,\ \beta;\ q,\ f)$ (such $p(x)$ and $q(x)$ exist for every $f\in A_X^M$).

1.6. The function $f\in A_X^M$ is called λ-monotonous if f is monotonous in every subinterval $X'=[a',\ b']$ with $a',\ b'\in X$ such that the length of X' is less than λ and f is a constant in $[a,\ a+\lambda]$ and $[b-\lambda,\ b]$. Let us note that the set of all λ-monotonous functions is enough wide. For example, it contains all step functions.

2.0. Basic theorems and algorithms

2.1 Theorem 1. *If the function* $f\in A_X^M$ *is* λ*-monotonous and* $E_H(H_n,\ X,\alpha,\beta;\ f)\leqq\lambda/B$ $(B=\max_{x\in X}\alpha(x)/\min_{x\in X}\beta(x))$ *then there exists only one function* $p\in H_n$, *such that* $H(X,\ \alpha,\ \beta;\ p,\ f)=E_H(H_n,\ X,\ \alpha,\ \beta;\ f)=E_h(H_n,\ X,\ \alpha,\ \beta;\ f)=h(X,\ \alpha,\ \beta;\ p,f)$.

The proof of this theorem is in some sense similar to the theorem 8.4 in [2].

2.2 Theorem 2. *Let the function* $f\in A_X^M$ *be* λ*-monotonous and* $E_H(H_n,X,\alpha,\beta;f)\leqq\lambda/B$ $(B=\max_{x\in X}\alpha(x)/\min_{x\in X}\beta(x))$. *A necessary and sufficient condition for the generalized polynomial* $p\in H_n$ *to satisfy* $H(X,\ \alpha,\ \beta;\ p,\ f)=E_H(H_n,\ X,\ \alpha,\ \beta;f)$ *is the existance of* $n+1$ *points* $\{x_i\}_{i=0}^n$, $x_i\in X$, $p(x)=p(C,\ x)=\sum_{j=1}^{n} c_j.g_j(x)$, *such that* $d(C,\ x_i)=(-1)^i\sigma.E_H(H_n,\ X,\ \alpha,\ \beta;\ f)$, $\sigma=\pm1$, $i=0,\ 1,\ 2,\dots,\ n$ *where* $d(C,\ x)=\text{sign}\ (f(x)-p(C,\ x)),\ \min_{(u,v)\in F(f)}\ \max\ [|x-u|/\alpha(x),\ |p(C,\ X)-v|/\beta(x)]$.

The above theorem looks like the classical Chebishev's theorem [1, 2] and its proof does not differ much from the proof of theorem 8.3 in [2] in the sufficient condition and from the classical case in the necessary condition.

2.3. It gives a possibility to modify the first Remez' algorithm [1] and we suggest the following numerical method for finding the generalized polynomial of the best Hausdorff approximation of a given λ-monotonous function $f\in A_X^M$.

Algorithm A

step 0. select: $\varepsilon>0$ (accuracy), interval X, function $f\in A_X^M$ $f(x)$ λ-monotonous, degree n, base functions $\{g_i\}_{i=0}^m$, parameter functions $\alpha(x)>0$,

$\beta(x)>0$, ITER upper bound of the number of iteration, $m \geqq n$ and $X^1=\{x_i^1\}_{i=0}^m$, $x_i \in X$, set $k=1$;

step 1. define $D^k(C)=\max\{|d(C,x)|, x \in X^k\}$, where $d(C,x)=\text{sing}(f(x)-p(C,x)) \cdot \min_{(u,v)\in F(f)} \max[|x-u|/\alpha(x), |p(C,x)-v|/\beta(x)]$ and find $C^k=\{c_i^k\}_{i=1}^n$ such that $D^k(C^k)=\min_C D^k(C)$ (see Algorithm B);

step 2. find $x^k \in X: |d(C^k, x^k)|=D(C^k)$, where $D(C)=\max\{|d(C,x)|, x \in X\}$;

step 3. if $x^k \in X^k$ then go to step 5, if $D(C^k)-D^k(C^k)<\varepsilon$ then go to step 5, if $k>$ ITER then go to step 6, else go to step 4;

step 4. $X^{k+1}=X^k \cup \{x^k\}$, set $k=k+1$ and go to step 1;

step 5. we accept that $p(C^k,x)=\sum_{j=0}^{n} c_j \cdot g_j(x)$ is the required generalized polynomial;

step 6. stop the computation.

Theorem 3. *The sequence* $\{C^k\}$ $k=1,2,\ldots$ *is bounded, and its cluster points minimize* $D(C)$ *and* $D^k(C^k) \uparrow E_h(H_n, X, \alpha, \beta; f)=\inf_C D(C)$, i.e. $D^k(C^k)<D^{k+1}(C^{k+1})$ *and* $\lim_{k\to\infty} D^k(C^k)=E_h(H_n, X, \alpha, \beta; f)$.

Lemma 1. [3] *Let* $f \in A_X^M$, $\alpha(x)>0$, $\beta(x)>0$. *For each* $\varepsilon>0$ *there exists a constant* W *and* $f_1 \in \text{Lip}_W 1$ *such that* $|E_h(H_n, X, \alpha, \beta; f)-E_h(H_n, X, \alpha, \beta; f_1)|<\varepsilon$.

Lemma 2. *Let* $f \in \text{Lip}_W 1$, $x_0 \in X$, $f_0=f(x_0)$, $\alpha_0=\alpha(x_0)$, $\beta_0=\beta(x_0)$, $p_1=p(C^1,x_0)$, $p_2=p(C^2,x_0)$, $\text{sign}(f_0-p_1)=\text{sign}(f_0-p_2)=\text{sign}(p_1-p_2)$, $d_1=d(C^1,x_0)$, $d_2=d(C^2,x_0)$ *then*:

a) $|f_0-p_1|/(\beta_0+\alpha_0 W) \leqq |d_i| \leqq |f_0-p_1|/\beta_0$ for $i=1,2$;

b) $|d_1| \leqq |d_2|$ and $|p_2-p_1|/(\beta_0+\alpha_0 W) \leqq |d_2|-|d_1| \leqq |p_2-p_1|/\beta_0$.

Proof of Lemma 2: $|d_1|=\min_{(u,v)\in F(f)} \max[|x_0-u|/d_0, |p_1-v|/\beta_0] = |x_0-u_1|/\alpha_0=|p_1-v_1|/\beta_0$, from $|p_1-v_1| \leqq |p_1-f_0|$ follows $|p_1-v_1|/\beta_0=|d_1| \leqq |p_1-f_0|/\beta_0$; from $|p_1-f_0|=|p_1-v_1+v_1-f_0| \leqq |p_1-v_1|+|v_1-f_0|$ and $f \in \text{Lip}_W 1$ follows $|v_1-f_0| \leqq W.|u_1-x_0|$ and $|p_1-f_0| \leqq |p_1-v_1|+W|u_1-x_0|=\beta_0.(|p_1-v_1|/\beta_0)+W\alpha_0.(|u_1-x_0|/\alpha_0)=|d_1|.(\beta_0+\alpha_0 W)$, i.e. $|f_0-p_1|/(\beta_0+\alpha_0 W) \leqq |d_1|$. By analogy $|f_0-p_2|/(\beta_0+\alpha_0 W) \leqq |d_2| \leqq |f_0-p_2|/\beta_0$

b) $|d_2|-|d_1|=|p_2-v_2|/\beta_0-|p_1-v_1|/\beta_0=|(p_2-v_2)-(p_1-v_1)|/\beta_0 \geqq 0 \to |d_2| \geqq |d_1|$; $\text{sign}(p_1-p_2)=\text{sign}(f_0-p_1)=\text{sign}(f_0-p_2) \leftrightarrow |p_2-f_0|>|p_1-f_0|$ therefore: $|p_2-v_2|>|p_1-v_1|$ and $\text{sign}(v_1-v_2)=\text{sign}(p_1-p_2)=\text{sign}(v_2-p_2)=\text{sign}(v_1-p_1)$

$$|d_2|-|d_1|=|(p_2-p_1)-(v_2-v_1)|/\beta_0=||p_2-p_1|-|v_2-v_1||/\beta_0$$

$$|p_2-p_1|-W.|u_2-u_1| \leqq ||p_2-p_1|-|v_2-v_1|| \leqq |p_2-p_1| \text{ as}$$

$0 \leqq |v_2-v_1| \leqq W.|u_2-u_1|$.

If $\text{sign}(x_0-u_1)=\text{sign}(x_0-u_2)$ then $|u_2-u_1|=|u_2-x_0+x_0-u_1|=|u_2-x_0|-|u_1-x_0|=\alpha_0(|d_2|-|d_1|)$ and $|d_2|-|d_1| \geqq |p_2-p_1|/\beta_0-W.|u_2-u_1|/\beta_0=|p_2-p_1|/\beta_0-W.\alpha_0(|d_2|-|d_1|)$ therefore $|p_2-p_1|/(\beta_0+\alpha_0 W) \leqq |d_2|-|d_1|$.

If $\text{sign}(x_0-u_1)=-\text{sign}(x_0-u_2)$ then p_0 exists between p_1 and p_2 such that $|d_0|=|p_0-v_0|/\beta_0=|x_0-u_{01}|/\alpha_0=|x_0-u_{02}|/\alpha_0$ and $\text{sign}(x_0-u_{01})=\text{sign}(x_0-u_1)=-\text{sign}(x_0-u_2)=-\text{sign}(x_0-u_{02})$. From $|d_2|-|d_0| \geqq |p_2-p_0|/(\beta_0+\alpha_0 W)$ and $|d_1|-|d_0| \geqq |p_1-p_0|/(\beta_0+\alpha_0 W)$ it follows $|d_2|-|d_1| \geqq (|p_2-p_0|+|p_1-p_0|)/(\beta_0+\alpha_0 W)=|p_2-p_0+p_0-p_1|)/(\beta_0+\alpha_0 W)=|p_2-p_1|)/(\beta_0+\alpha_0 W)$ □

Proof of Theorem 3. Define $|C|=\sum_{i=1}^{n}|c_i|$. We consider X^1 arbitrary except that the set of n-tuples $[g_1(x), g_2(x), \dots, g_n(x)]$ corresponding to $x\in X^1$ should be of rank n. Then it follows that the number $W_1=\min_{|C|=1}\max_{x\in X^1}|p(C, x)|$ is positive. Let $W_2=\max_{x\in X}[\beta(x)+\alpha(x)W]$, $W_3=\max_{1\leqq i\leqq n}\max_{x\in X}|g_i(x)|$, $W_4=\min_{x\in X}\beta(x)$, $W_5=\max_{x\in X}|f(x)|/\beta(x)$. Consequently $D^1(C)=\max_{x\in X^1}|d(C, x)|\geqq\max_{x\in X^1}|\Sigma c_i g_i(x)-f(x)|/W_2 \geqq\max_{x\in X^1}(|\Sigma c_i g_i(x)|-|f(x)|)/W_2\geqq(1/W_2).\min_C\max_{x\in X^1}|\Sigma c_i g_i(x)|-(1/W_2).\max_{x\in X}|f(x)| \geqq(1/W_2).\min_C|C|\max_{x\in X^1}|\Sigma(c_i/|C|).g_i(x)|-(1/W_2).\|f\|=(1/W_2).(|C|.W_1-\|f\|)$.
If $|C|>(\|f\|+W_2.W_5)/W_1$ then $D^k(C)\geqq D^1(C)>(\|f\|+W_2.W_5)/W_1.(W_1/W_2)-(1/W_2).\|f\|=W_5\geqq D^k(0)$, $0=\{0, 0, \dots, 0\}$, so that the vector C does not enter the competition to minimize any of the functions D^k. Thus the sequence $\{C^k\}$ generated by the algorithm is bounded.

Now it follows from the inclusions $X^k\subset X^{k+1}\subset X$ that $D^k(C)\leqq D^{k+1}(C)\leqq D(C)$. Hence $D^k(C^k)\leqq D^k(C^{k+1})\leqq D^{k+1}(C^{k+1})\leqq D^{k+1}(C^*)\leqq D(C^*)=E_h(H_n, X, \alpha, \beta; f)$, i.e. $D^k(C^k)\leqq D^{k+1}(C^{k+1})\leqq E_h(H_n, X, \alpha, \beta; f)$.
Thus for some $\varepsilon\geqq 0$, $D^k(C^k)\uparrow E_h(H_n, X, \alpha, \beta; f)-\varepsilon$. We must prove that $\varepsilon=0$. Since

$$|d(C, x)-d(B, x)|\leqq|\Sigma(c_i-b_i)g_i(x)|/\beta(x)\leqq|B-C|.(W_3/W_4)$$

it follows that $|d(B, x)|\leqq|d(C, x)|+|B-C|.(W_3/W_4)$ and $D(B)\leqq D(C)+|B-C|.(W_3/W_4)$ as $D(B)=|d(B, x_B)|\leqq|d(C, x_B)|+|B-C|.(W_3/W_4)\leqq|d(C, x_C)|+|B-C|.(W_3/W_4)=D(C)+|B-C|.(W_3/W_4)$. Suppose now that $\varepsilon>0$. Let B denote any cluster point of sequence $\{C^k\}$. For any $\delta>0$ we may find an index k such that $|B-C^k|<\delta$ and an index i such that $|B-C^i|<\delta$. Then

$$|C^i-C^k|\leqq|C^i-B|+|B-C^k|<2\delta, \text{ and } E_h(H_n, X, \alpha, \beta; f)\leqq D(B)\leqq D(C^k)$$

$$+\delta.(W_3/W_4)=|d(C^k, X^k)|+\delta.(W_3/W_4)\leqq|d(C^i, x^k)+|C^i-C^k|.(W_3/W_4)+\delta.$$

$$(W_3/W_4)\leqq|d(C^i, x^k)|+3\delta.(W_3/W_4)\leqq|d(C^i, x^i)|+3\delta.(W_3/W_4)$$

$$\leqq E_h(H_n, X, \alpha, \beta; f)-\varepsilon+3\delta.(W_3/W_4).$$

If $\delta<\varepsilon W_4/(3W_3)$, this is a contradiction.
The same inequalities show that $D(B)=E_h(H_n, X, \alpha, \beta; f)$. □

It should be especially noted that this theorem does not guarantee that the coefficient vectors $\{C^k\}$ converge. The sequence of generalized polynomials from H_n, which are generated by the algorithm, converges uniformly on the generalized polynomial of the best one-sided Hausdorff approximation, but it does not give the rate of convergence.

2.4. In step 2 on algorithm A we solve the problem for finding the best approximation in discrete set X^k. We suggest the following numerical method:

Algorithm B

step 0. set $Z=X^k$, i.e. $z_i=x_i^k$ for $i=0,\dots,m$, constant $\delta\in(0;1)$ (k is a number of iteration on step 2 in algorithm A) set $Z^1=\{z_1\}_{i=0}^n$ and $J=1$;

step 1. set $Y=Z^J$, i.e. $y_i=z_i^J$ for $i=0,\dots,n$ set $w_i^1=1/\beta(y_i)$, $i=0,\dots,n$ and $L=1$;

step 2. solve the linear system:

$$(-1)^i/w_i^L.c_0+\sum_{j=1}^n c_j.g_j(y_i)=f(y_i),\ i=0,\dots,n \text{ and find } \{c_0,c_1,\dots,c_n\},$$

set $E^L=|c_0|$ and $C^L=\{c_j\}_{j=1}^n$;

step 3. compute $E_i^L=|d(C^L,y_i)|$, $\hat{w}_i^L=E_i^L/|f(y_i)-p(C^L,y_i)|$, $U^L=\max\{E_j^L : j=0,\dots,n\}$ and $D^L=\min\{E_j^L : j=0,\dots,n\}$;

step 4. if $(U^L-D^L)<\varepsilon$ or $L>$ITER then go to step 6;

step 5. compute $w_i^{L+1}=\hat{w}_i^L.\delta+(1-\delta).w_i^L$ for $i=0,\dots,n$ set $L=L+1$ and go to step 2;

step 6. we accept that the generalized polynomial $p(C^L,x)$ is required for the problem $\min_C\max\{|d(C,y)| : y\in Z^J\}$ and $C^J=C^L$;

step 7. find $z^J\in Z : |d(C^J,z^J)|=\max_{z\in Z}|d(C^J,z)|$;

step 8. if $z^J\in Z^J$ then go to step 10, if $\max_{z\in Z}|d(C^J,z)|-\max_{z\in Z^J}|d(C^J,z)|<\varepsilon$ go to step 10, if $J>$ITER then go to step 10;

step 9. find index j such that the new set Z^{J+1} of $n+1$ points where we add z^J (from step 7) and remove $z_j^J\in Z^J$ satisfies sign $d(C^J,z_i^{J+1})=(-1)^i$. sigh $d(C^J,z_0^{J+1})$, $i=1,\dots,n$ set $J=J+1$ and go to step 1;

step 10. we accept that the $p(C^J,x)$ is required for the problem $\min_C\max\{|d(C,y)| : y\in Z\}$, set $C^k=C^J$ and stop the computation.

Theorem 4. *Let the sequence $\{C^L\}$, generated by the algorithm B from step 1 to step 6 for fixed discrete set $Y=\{y_i\}_{i=0}^n$ and a function $f(x)$ is λ-monotonous and $f\in \mathrm{Lip}_w 1$. Then there exists $q\in(0;1)$ such that $U^{L+1}-D^{L+1}\leq q.(U^L-D^L)$, where $U^L=\max_{y\in Y}|d(C^L,y)|$ and $D^L=\min_{y\in Y}|d(C^L,y)|$.*

Proof: At the L-th iteration for given w_j^L and $y_i\in Z$ $(i=0,1,2,\dots,n)$ we solve the linear system

$$(-1)^i/w_i^L.c_0+\sum_{j=1}^n c_j.g_j(y_i)=f(y_i),\ i=0,\dots,n$$

and find $\{c_0,c_1,\dots,c_n\}$; set $E^L=|c_0|$, $C^L=\{c_j\}_{j=1}^n$;

compute $E_i^L=|d(C^L,y_i)|$, $\hat{w}_i^L=E_i^L/|f(y_i)-p(C^L,y_i)|$,

$$z_i^L=|f(y_i)-p(C^L,y_1)|=E^L/w_i^L,$$

$U^L=\max\{E_j^L : j=0,\dots,n\}$ and $D^L=\min\{E_j^L : j=0,\dots,n\}$;

compute $w_1^{L+i}=\hat{w}_i^L.\sigma+(1-\sigma).w_i^L$ for $i=0,\dots,n$.

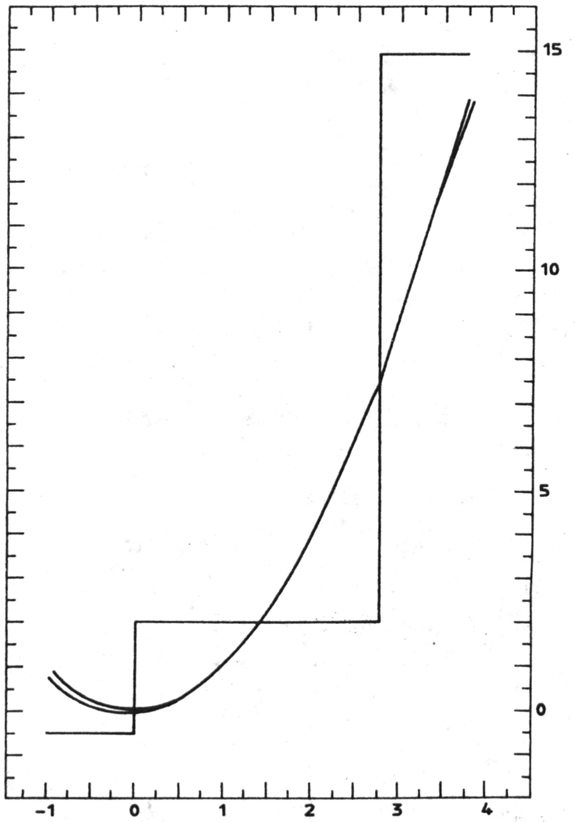

Fig. 1

At $(L+1)$-th iteration we find $C^{L+1}=\{c_j\}_{j=1}^{n}$, E^{L+1}, D^{L+1}, U^{L+1}, E_i^{L+1}, z_i^{L+1}, $i=1, 2, \ldots, n$.

Let $W_2=\max_{x\in X}[\beta(x)+\alpha(x)W]$, $W_4=\min_{x\in X}\beta(x)$. Consequently

$$z_i^{L+1}=E^{L+1}/\,w_1^{L+1}=[E^{L+1}/(\sigma E_i^L+(1-\sigma)E^L)]\,.\,z_i^L.$$

For i, such that $z_i^{L+1}\leqq z_i^L$, we have $E_i^L-E_i^{L+1}\geqq[z_i^L-z_i^{L+1}]/[\beta_i+W\alpha_i]$

$$\geqq[z_i^L-z_i^{L+1}]/W_2=[(\sigma E_i^L+(1-\sigma)E^L)-E^{L+1}]/[w_i^{L+1}\,.\,W_2]\geqq[W_4/W_2]\,.$$

$$[\sigma E_i^L+(1-\sigma)E^L-E^{L+1}].$$

From $E_i^{L+1}\leqq E_i^L\,.\,[1-\sigma W_4/W_2]+[E^{L+1}-E^L(1-\sigma)]\,.\,[W_4/W_2]$ there follows $U^{L+1}=\max E_i^{L+1}\leqq U^L\,.\,[1-\sigma W_4/W_2]+[E^{L+1}-E^L(1-\sigma)]$.

For i, such that $z_i^{L+1}\geqq z_i^L$, we have $E_i^{L+1}-E_i^L\geqq[z_i^{L+1}-z_i^L]/[\beta_i+W\alpha_i]$

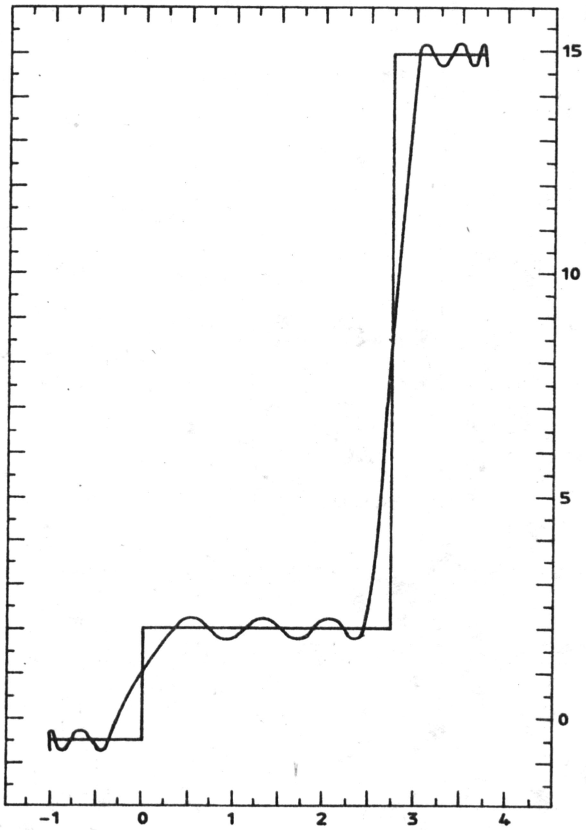

Fig. 2

$$\geqq [z_i^{L+1} - z_i^L]/W_2 = [-(\sigma E_i^L + (1-\sigma)E^L) + E^{L+1}]/[w_i^{L+1} \,.\, W_2] \geqq [W_4/W_2] \,.$$

$$[-\sigma E_i^L - (1-\sigma)E^L + E^{L+1}].$$

From $E_i^{L+1} \geqq E_i^L . [1 - \sigma W_4/W_2] + [E^{L+1} - E^L(1-\sigma)] . W_4/W_2]$ it follows $D^{L+1} = \min\ E_i^{L+1} \geqq D^L . [1 - \sigma W_4/W_2] + [E^{L+1} - E^L(1-\sigma)]$.

Finally, for $\sigma > 0$ and $\sigma < \min\ \{\sigma_0,\ W_2/W_4\}$ we have $(U^{L+1} - D^{L+1}) \leqq (U^L - D^L) . q$, where $q = (1 - \sigma W_4/W_2)$ and $0 < q < 1$. □

3.0. Numerical experiments

The suggested algorithm was used for numerical finding of the generalized polynomials of the best one-sided approximation of the following functions:

Table 1

j	x	Coeff. 1	$(f-p_1)(x)$	$d(C^1, x)$	Coeff. 2	$(f-p_2)(x)$	$d(C^2, x)$
0	-1	—	-1.2497	-1.0000	—	-1.5000	-1.00
1	$+1$	-0.0793345	1.0000	1.0000	0.00	1.0000	1.00
2	$\sqrt{3}$	0.1251261	-1.0000	-1.0000	0.00	-1.0000	-1.00
3	$2+\sqrt{3}$	0.9542059	1.2502	1.0000	1.00	1.0000	1.00

Table 2

j	Coeff.	j	Coeff.
1	5.16007057866	11	-0.045662988383
2	7.95080326808	12	-0.549363843097
3	3.10933405488	13	-0.357958893290
4	0.94967493786	14	-0.115239694038
5	-0.98982748545	15	0.187964620702
6	-1.84385394537	16	0.367514937129
7	-0.55266466424	17	0.095345944210
8	0.41931351139	18	-0.065893599941
9	0.69610076166	19	-0.343084729348
10	0.69558573894	20	-0.094441310959

Table 3

n	E
3	1.000
5	0.798
7	0.568
10	0.430
15	0.326
20	0.254
25	0.214
30	0.185
35	0.164

Table 4

n	E
3	0.0806
5	0.0494
7	0.0363
10	0.0245
15	0.0188
20	0.0142
25	0.0123
30	0.0103
35	0.0093

Table 5

j	Coeff.	j	Coeff.
1	0.499998106280	12	-0.041780732664
2	0.634984971606	13	-0.000002563607
3	0.000003747338	14	0.030776593653
4	-0.207338355096	15	0.000002200345
5	-0.000003633649	16	-0.022507603355
6	0.119329431968	17	-0.000001824441
7	0.000003448958	18	0.016155585912
8	-0.079999008866	19	0.000002004327
9	-0.000003201294	20	-0.020921178116
10	0.057071649074	21	-0.000001185285
11	0.000002901243		

3.1. $$f_1(x)=\begin{cases} -0.50 & -1\leqq x<0 \\ +2 & 0\leqq x<1+\sqrt{3} \\ 8+4.\sqrt{3} & 1+\sqrt{3}\leqq x\leqq 2+\sqrt{3} \end{cases}$$

in interval $X=[a, b]=[-1, 2+\sqrt{3}]$, with parameters $\alpha(x)=1$ and $\beta(x)=1$

a) $\{g_1(x), g_2(x), g_3(x), \ldots\}=\{1, x, x^2, \ldots\}$, $n=3$.

The best one-sided approximation and the element of the best approximation are:

$$E_h(H_3, X, \alpha, \beta; f_1)=1.00, \quad p(C, x)=\sum_{j=1}^{3} c_j . g_j(x), \text{ where}$$

$$c_1=-0.0793345, \quad c_2=0.125126, \quad c_3=0.9542059.$$

Note that the polynomial is not unique ($c_1 = c_2 = 0$ and $c_3 = 1$ is other polynomial of the best approximation [1]), also f_1 is 1-monotonous (λ-monotonous with $\lambda = 1$), and the best one-sided approximation is not equal to the best Hausdorff approximation. (Fig. 1 and Table 1);

$$\text{b)}\ \{g_1(x),\ g_2(x),\ g_3(x),\dots\} = \{T_0(x),\ T_1(x),\ T_2(x),\dots\},$$

where $T_k(x) = \cos\left(k \arc\cos\dfrac{2x-a-b}{b-a}\right)$ is the k-th Chebishev's polynomial in interval [a, b].

Table 3 shows the order of the approximation of the function f_1 at different n. For $n = 20$ the polynomial and the function are given graphically on Fig. 2 and its coefficients are given in Table 2. Note that for $n \geqq 5$ $E_h(H_n,\ X,\ \alpha\ \beta;\ f_1) = E_H(H_n,\ X,\ \alpha,\ \beta; f_1)$ and the polynomial of the best one-sided approximation is unique and is equal to the polynomial of the best Hausdorff approximation.

3.2.
$$f_2(x) = \begin{cases} 1 & 0 \leqq x < \pi/2 \\ 0 & \pi/2 \leqq x \leqq \pi \end{cases}$$

in interval $X = [a,\ b] = [0,\ \pi]$, with parameters $\alpha(x) = 10$ and $\beta(x) = 1$

$$\{g_1(x),\ g_2(x),\ g_3(x),\dots\} = \{1,\ \cos(x),\ \cos(2x),\dots\}.$$

Table 4 shows the order of the approximation of the function f_2 at different n. In this case the best one-sided approximation is equal to the best Hausdorff approximation, i. e. $E_h(H_n,\ X,\ \alpha,\ \beta; f_2) = E_H(H_n,\ X,\ \alpha,\ \beta; f_2)$ and the polynomial of the best one-sided approximation is at the same time polynomial of the best Hausdorff approximation. In Table 5 the coefficients of the polynomial of the best approximation for $n = 21$ are given.

References

1. E. W. Cheney. Introduction to Approximation Theory, N. Y., 1966.
2. Бл. Сендов. Хаусдорфовые приближения, С., 1979.
3. А. С. Андреев. Численный метод построения полинома най-лучшего хаусдорфова приближения, *Доклады БАН*, **29**, 1976, 2, 163

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