Mathematica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

Mathematica Balkanica

New Series Vol. 3, 1989, Fasc. 1

Approximation Splines

Blagovest Sendov

In this paper we construct explicitly local splines with uniform knots having good approximation properties. These splines, called approximation splines or A-splines for conveniency, belong to the so-called quasi-interpolants or to other constructions used in the works of G. Birkhof [1], C. De Boor [2, 3], C. De Boor and G. J. Fix [4], T. Lyche and L. L. Shumaker [5] and others.

The advantage of A-splines is the simplicity of their representation, that proposes an opportunity for estimation with concrete constants.

In this paper we will consider only the definition and some approximation properties of A-splines with uniform knots in the one dimensional case. It is possible to transfer the definition of A-splines with uniform knots to function of more than one variables as well. The idea of defining A-splines could be applied in the case of ununiform knots, but devided differences should be used.

1. Definition of an A-spline

Let $\{x_i\}$ be an uniform net of knots with a stepsize $h(x_{i+1}=x_i+h_i$; $i=0, \pm 1, \pm 2, \ldots)$ and f be a function, defined on the real axis. Let use the denotations $f(x_i)=f_1$ and

$$\Delta_h^k f(x_i)=\Delta^k f_i=\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}f_{i+j}, \tag{1}$$

for k — a natural number.

Denote $Q_{k-1,i}(f;\ x)=Q_i(f;\ x)$ the interpolation algebraic polynomial of degree $k-1$, uniquely defined by

$$Q_i(f;\ x_j)=f_j \text{ for } j=i,\ i+1,\ldots,\ i+k-1. \tag{2}$$

Define the algebraic polynomial for every integer i

$$F_i(f;\ x)=Q_i(f;\ x)+\sum_{j=-q}^{q}P_j\left(\frac{x-x_i}{h}-\frac{k-1}{2}\right)\Delta^k f_{i+j}, \tag{3}$$

where the algebraic polynomials $P_j(x)$; $j=0, \pm 1, \pm 2, \ldots \pm q$, of degree $\leq k$ are chosen in such a way, that the identity

$$(4)\qquad F_{i+1}(f;\ x)-F_i(f;\ x)=\left(\frac{x-x_i}{h}-\frac{k+1}{2}\right)^k \sum_{j=-q}^{q+1} \beta_j \Delta^k f_{i+j},$$

holds, where β_j; $j=-q, -q+1, \ldots, q+1$ are constants appropriately chosen, and q is a given natural number.

Definition 1. *The function $A_{k,q}(f)$ is called A-spline of order (k, q) of the function f for the net $\{x_i\}$ if*

$$(5)\qquad A_{k,q}(f;\ x)=F_i(f;\ x) \text{ for } x\in\left[x_i-\frac{k-1}{2}h,\ x_i+\frac{k+1}{2}h\right).$$

It is obvious from (4) that $A_{k,q}(f)$, defined by (5), is a spline indeed.

So we further need to prove that it is possible to choose the polynomials P_j and the constants β_j in such a way, that the identity (4) is valid. But this will be done later.

Now from the definition (2) of $Q_i(f)$ it follows that

$$(6)\qquad Q_{i+1}(f;\ x)-Q_i(f;\ x)=\varphi\left(\frac{x-x_i}{h}\right)\Delta^k f_i,$$

where

$$(7)\qquad \varphi(x)=\frac{1}{(k-1)!}(x-1)(x-2)\ldots(x-k+1).$$

Lemma 1. *If $2q\geqq k-1$, then there exist constants $\beta_j^k=\beta_j$; $j=-q, -q+1, \ldots, q+1$ for which the identity* (8) *holds.*

$$(8)\qquad \sum_{j=-q}^{q+1} \beta_j(x+j)^k=\varphi\left(x+\frac{k+1}{2}\right).$$

Proof. Comparing the coefficients preceding the degrees of x, we obtain a system of $k+1$ linear equation for the constants β_j. The system has a solution for $2q+2\geqq k+1$ or $2q\geqq k-1$.□

After determining the constants β_j to satisfy the identity (8) we define the polynomials P_j as follows:

$$(9)\qquad P_j(x)=\sum_{p=j+1}^{q+1} \beta_p(x+p-j-1)^k \text{ for } j=0,\ 1,\ 2, \ldots,\ q,$$

$$(10)\qquad P_j(x)=\sum_{p=-q}^{j} \beta_p(x+p-j-1)^k \text{ for } j=-q,\ -q+1, \ldots,\ -1.$$

From the equalities (9) and (10) we directly obtain the following dependences

$$(11)\qquad P_q(x-1)=\beta_{q+1}(x-1)^k,\ P_{-q}(x)=-\beta_{-q}(x-1)^k,$$

$$(12)\qquad P_j(x-1)-P_{j+1}(x)=\beta_{j+1}(x-1)^k \text{ for } j=-q,\ -q+1, \ldots, -2, 0, 1, \ldots, q-1,$$

$$(13)\qquad P_{-1}(x-1)-P_0(x)=\beta_0(x-1)^k-\sum_{p=-q}^{q+1} \beta_p(x+p-1)^k.$$

Lemma 2. *If the polynomials P_j are defined by the equalities* (9) *and* (10) *and the constants β_j satisfy the identity* (8), *then the identity* (4) *is valid.*

Proof. From (3) using (8) and (13) we obtain

$$F_{i+1}(f;\ x)-F_i(f;\ x)=Q_{i+1}(f;\ x)-Q_i(f;\ x)$$

$$+\sum_{j=-q}^{q} P_j\left(\frac{x-x_{i+1}}{h}-\frac{k-1}{2}\right)\Delta^k f_{i+j+1}-\sum_{j=-q}^{q} P_j\left(\frac{x-x_i}{h}-\frac{k-1}{2}\right)\Delta^k f_{i+j}$$

$$=\varphi\left(\frac{x-x_i}{h}\right)\Delta^k f_i+P_q\left(\frac{x-x_i}{h}-\frac{k+1}{2}\right)\Delta^k f_{i+q+1}-P_{-q}\left(\frac{x-x_i}{h}-\frac{k-1}{2}\right)\Delta^k f_{i-q}$$

$$+\sum_{j=-q}^{q-1}\left\{P_j\left(\frac{x-x_i}{h}-\frac{k+1}{2}\right)-P_{j+1}\left(\frac{x-x_i}{h}-\frac{k-1}{2}\right)\right\}\Delta^k f_{i+j+1}$$

$$=\sum_{j=-q}^{q+1}\beta_j\left(\frac{x-x_i}{h}-\frac{k+1}{2}+j\right)^k\Delta^k f_i+\beta_{q+1}\left(\frac{x-x_i}{h}-\frac{k+1}{2}\right)\Delta^k f_{i+q+1}$$

$$+\beta_{-q}\left(\frac{x-x_i}{h}-\frac{k+1}{2}\right)^k\Delta^k f_{i-q}+\sum_{\substack{j=-q\\ j\neq -1}}^{q-1}\beta_{j+1}\left(\frac{x-x_i}{h}-\frac{k+1}{2}\right)^k\Delta^k f_{i+j+1}$$

$$+\beta_0\left(\frac{x-x_i}{h}-\frac{k+1}{2}\right)^k\Delta^k f_i-\sum_{p=-q}^{q+1}\beta_p\left(\frac{x-x_i}{h}-\frac{k+1}{2}+p\right)^k\Delta^k f_i$$

$$=\left(\frac{x-x_i}{h}-\frac{k+1}{2}\right)^k\sum_{j=-q}^{q+1}\beta_j\Delta^k f_{i+j}.\ \square$$

2. Approximation properties of an A-spline

To estimate the distance between a given function and its A-spline, we need the following theorem [6], which represents a precision of the well-known theorem of H. Whitney [7].

Theorem 1. *Let the function f be integrable on a finite segment $[x_{i-1}, x_{i+k}]$ and possess a bounded k-th modulus of smoothness*

$$\omega_k(f:\delta)=\sup\ \{|\Delta_t^k f(x)|:|t|\leqq\delta\} \tag{14}$$

on this segment, where $\{x_i\}$ is a uniform net with a stepsize h. Then for the interpolation polynomial $Q_i(f)$ the estimate (15) *holds*

$$\|f-Q_i(f)\|\leqq W_k\omega_k(f;\ h), \tag{15}$$

where $\|\cdot\|$ is the uniform norm on the segment $[x_{i-1}, x_{i+k}]$ and W_k are constants, depending on k, for which (15) *is valid and*

$$W_1=W_2=1,\ W_3\leqq\frac{14}{9},\ W_4\leqq 3{,}25,\ W_k\leqq 6\ \textit{for}\ k=5,\ 6,\ 7,\ldots \tag{15'}$$

From the definition of A-spline and Theorem 1, we obtain

Lemma 3. *Let the function f be defined on the real axis and has a bounded k-th* modulus of smoothness. Then

(16) $$\|f-A_{k,q}(f)\| \leqq (W_k + \sum_{i=-q}^{q} \max_{0\leqq t\leqq 1} |P_i(t)|) W_k(f;\ h),$$

where $\|\cdot\|$ *is the uniform norm on the whole real axis and* $W_k(f;\ h)$ *is the bounded k-th modulus of smoothness on the real axis.*

Proof. Taking into account that $A_{k,q}(f;\ x)=F_i(f;\ x)$ for $x\in\left[x_i+\frac{k-1}{2}h,\ x_i+\frac{k+1}{2}h\right)$, from (3) and (15), (16) follows directly.

From (5) and (3) we obtain also the estimate for the k-th derivative of $A_{k,q}(f)$ beyond the knots $\{x_i\}$:

(17) $$\|A_{k,q}^{(k)}(f)\| \leqq h^{-k} \sum_{j=-q}^{q} |P_j^{(k)}|\omega_k(f;\ h).$$

To compute the values in the right sides of the estimates (16) and (17), we will express the constants β_j using certain new constants α_j in the following way

(18) $$\alpha_j^k=\alpha_j=k!\sum_{p=j+1}^{q+1}\beta_p \qquad \text{for} \qquad j=0,\ 1,\ 2,\ \dots,\ q$$

(19) $$\alpha_j^k=\alpha_j=k!\sum_{p=-q}^{j}\beta_p \qquad \text{for} \qquad j=-q,\ -q+1,\dots,\ -1.$$

From the identity (8), it follows that

(20) $$\sum_{j=-q}^{q+1}\beta_j=0,$$

since $\varphi(x)$ is a polynomial of degree $k-1$. Taking into consideration (18)-(20), we can express the constants β_j by the constants α_j

(21) $$\beta_{q+1}=\frac{1}{k!}\alpha_q,\ \beta_{-q}=-\frac{1}{k!}\alpha_{-q},\ \beta_j=\frac{1}{k!}(\alpha_{j-1}-\alpha_j)$$

$$\text{for } j=-q+1,\ -q+2,\dots,\ q.$$

Then the identity (8) could be written in the following way

(22) $$\frac{1}{k!}\sum_{j=-q}^{q}\alpha_j[(x+j+1)^k-(x+j)^k]=\varphi\left(x+\frac{k+1}{2}\right).$$

From (22) comparing the coefficients preceding x^{k-1} we obtain

(23) $$\sum_{j=-q}^{q}\alpha_j=1.$$

Based on certain considerations for symmetry, we could take such constants α_j for which

$$(24) \qquad \alpha_{-j}=\alpha_j \qquad \text{holds for } j=1, 2, 3, \ldots, q.$$

So, to define the constants α_j; $j=1, 2, 3, \ldots, q$, we obtain the identity

$$(25) \quad (x+1)^k-x^k+\sum_{j=1}^{q}\alpha_j[(x+j+1)^k-(x+j)^k+(x-j+1)^k-(x-j)^k$$
$$-2(x+1)^k+2x^k]=\varphi\left(x+\frac{k+1}{2}\right).$$

If we denote

$$(26) \qquad R(x)=x^k+\sum_{j=1}^{q}\alpha_j[(x+j)^k+(x-j)^k-2x^k],$$

then the identity (25) could be expressed as follows

$$(27) \qquad R(x+1)-R(x)=k!\ \varphi\left(x+\frac{k+1}{2}\right).$$

It is obvious from (27) that the polynomial $R(x)$ of degree k assumes same values in k different points $x=\frac{k-1}{2}, \frac{k-3}{2}, \ldots, -\frac{k-1}{2}$ since $\varphi(x)=0$ for $x=1, 2, 3, \ldots, k-1$. Therefore

$$(28) \qquad R(x)=A+\left(x-\frac{k-1}{2}\right)\left(x-\frac{k-3}{2}\right)\ldots\left(x+\frac{k-1}{2}\right),$$

as it is evident from (26) that the coefficient of x^k in the polynomial $R(x)$ is equal to 1. In (28) A is an arbitrary constant.

And finally, to determine the constants α_j; $j=1, 2, 3, \ldots, q$ we obtain the following identity

$$(29) \qquad \sum_{j=1}^{q}\alpha_j[(x+j)^k+(x-j)^k-2x^k]=\varphi_k(x),$$

where

$$(30) \qquad \varphi_k(x)=\left(x-\frac{k-1}{2}\right)\left(x-\frac{k-3}{2}\right)\ldots\left(x+\frac{k-1}{2}\right)-x^k+A.$$

Using (21), the polynomials P_j, defined by (9) and (10) can be written using the constants α_j, namely

$$(31) \qquad P_j(x)=\frac{1}{k!}\alpha_j x^k+\frac{1}{k!}\sum_{p=j+1}^{q}\alpha_p[(x+p-j)^k-(x+p-j-1)^k]$$
$$\text{for } j=0, 1, 2, \ldots, q$$

$$(32)\qquad P_j(x)=\frac{1}{k!}\alpha_j(x-1)^k+\frac{1}{k!}\sum_{p=-q}^{j-1}\alpha_p[(x+p-j-1)^k-(x+p-j)^k]$$

$$\text{for } j=-q,\ -q+1,\dots,\ -1.$$

From (31) and (32) we directly obtain

$$(33)\qquad \|P_j^{(k)}\|=|\alpha_j|\qquad \text{for } j=0,\ \pm1,\ \pm2,\dots,\ \pm q\ ;$$

$$(34)\qquad \max_{0\le x\le 1}|P_j(x)|\le\frac{1}{k!}\{|\alpha_j|+\sum_{p=j+1}^{q}|\alpha_p|[(p-j+1)^k-(p-j)^k]\}$$

$$\text{for } j=0,\ 1,\ 2,\dots,q\ ;$$

$$(35)\qquad \max_{0\le x\le 1}|P_j(x)|\le\frac{1}{k!}\{|\alpha_j|+\sum_{p=-q}^{j-1}|\alpha_p|[(j-p+1)^k-(j-p)^k]\}$$

$$\text{for } j=-q,\ -q+1,\dots,\ -1.$$

Theorem 2. *Let f be defined on the real axis, integrable on every finite segment and have a bounded k-th modulus of smoothness. Then for A-spline of the function f on the uniform net $\{x_i\}$ with a stepsize h the following estimates are valid:*

$$(36)\qquad \|f-A_{k,q}(f)\|\le(W_k+1+\frac{2}{k!}\sum_{j=1}^{q}|\alpha_j|(j+1)^k)\omega_k(f;\ h),$$

$$(37)\qquad \|A_{k,q}^{(k)}(f)\|\le h^{-k}(1+4\sum_{j=1}^{q}|\alpha_j|)w_k(f;\ h),$$

where the constants α_j satisfy the identity (29) and $q\ge\left[\frac{k-1}{2}\right]$ is a natural number.

Proof. From (33), according to (17), (23) and (24) we obtain (36). From (16), taking into account (34) and (35), we get the estimate

$$\sum_{j=-q}^{q}\max_{0\le x\le 1}|P_j(x)|\le\frac{1}{k!}[1+\sum_{j=1}^{q}|\alpha_j|((j+1)^k+j^k+2)]\le 1+\frac{2}{k!}\sum_{j=1}^{q}|\alpha_j|(j+1)^k.\ \square$$

The theorem proved is a precized version of the well-known theorem of Ju. Brudnij [8], considered in [9, 10].

3. A-spline of minimal parameter

If the parameter q of A-spline $A_{k,q}$ is chosen possibly least, namely $q=\left[\frac{k-1}{2}\right]$ the respective spline is uniquely defined. We will further study this type of splines. For briefness we use the denotation $\left[\frac{k-1}{2}\right]=s$.

3. 1. Determining of the constants $\alpha_j=\alpha_{k,j}$

We will use the following denotations

$$(38)\qquad \theta_{s,j}(x)=\frac{1}{x^2-j^2}x^2(x^2-1^2)\dots(x^2-s^2)$$

$$=x^{2s}-b^j_{s,1}x^{2s-2}+\dots+(-1)^s b^j_{s,s}$$

and analogously to the function (30)

$$(39)\qquad \psi_k(x)=\left(x-\frac{k-1}{2}\right)\left(x-\frac{k-3}{2}\right)\dots\left(x+\frac{k-1}{2}\right)-x^k+A$$

$$=-a_{k,1}x^{k-2}+a_{k,2}x^{k-4}-\dots+(-1)^s a_{k,s}x^{k-2s}+A,$$

where A is an arbitrary constant.

Lemma 4. *The identity*

$$(40)\qquad \sum_{j=1}^{q}\alpha_j[(x+j)^k+(x-j)^k-2x^k]=\psi_k(x)$$

is satisfied if the constants α_j *are*

$$(41)\qquad \alpha_j=\alpha_{k,j}=\frac{(-1)^{s+j}}{k!\,(s+j)!\,(s-j)!}\sum_{i=1}^{s}\theta^{(2i)}_{s,j}(0)\psi^{(k-2i)}_k(0).\ j=1,2,3,\dots,s.$$

Proof. Denote by $T_k(t)$ the polynomial

$$(42)\qquad T_k(t)=t^2(t^2-1^2)\dots(t^2-s^2)\sum_{j=1}^{s}(-1)^j\frac{(x+j)^k+(x-j)^k-2x^k}{(s+j)!\,(s-j)!\,(t^2-j^2)}$$

of the variable t and depending on the parameter x. The values of T_k for $t=l=\pm 1, \pm 2,\dots, \pm s$ are respectively

$$T_k(l)=l^2(l^2-1^2)\dots(l^2-s^2)(-1)^l\frac{(x+l)^k+(x-l)^k-2x^k}{(s-l)!\,(s+l)!}$$

$$=\frac{(-1)^s}{2}[(x+l)^k+(x-l)^k-2x^k].$$

Therefore the identity holds

$$(43)\qquad T_k(t)=\frac{(-1)^s}{2}[(x+t)^k+(x-t)^k-2x^k].$$

After the substitution of the values for α_j, defined by (41) in the left side of (40) and using (42) and (43), we get

$$\sum_{j=1}^{s}[(x+j)^k+(x-j)^k-2x^k]\frac{(-1)^{s+j}}{k!\,(s+j)!\,(s-j)!}\sum_{i=1}^{s}\theta^{(2i)}_{s,j}(0)\psi^{(k-2i)}_k(0)$$

$$=\frac{(-1)^s}{k!}\sum_{i=1}^{s}\psi^{(k-2i)}_k(0)\frac{(-1)^s}{2}\frac{k!}{(k-2i)!}2x^{k-2i}=\sum_{i=1}^{s}\frac{x^{k-2i}}{(k-2i)!}\psi^{(k-2i)}_k(0)=\psi_s(x).\ \square$$

Note! The expressions (41) for the values of α_j can be obtained by solving the system of linear algebraic equations. This system is got after comparing the coefficients of the respective degrees of x in the identity (40). For proof briefness, the solution in Lemma 4 is given in a completely ready form, and is only checked, that it is the very solution.

From Lemma 4, taking into consideration (38) and (39), we can express the constants α_j by the coefficients of ψ_k and $0_{s,j}$, namely

$$\alpha_j = \alpha_{k,j} = \frac{(-1)^j}{(s+j)!\,(s-j)!} \sum_{i=1}^{s} a_{k,i} b^j_{s,s-i} \Big/ \binom{k}{2i}. \tag{44}$$

Since the coefficients $a_{k,i}$ and $b^j_{s,i}$ are positive, from (42) and (43) it follows, that

$$|\alpha_j| = (-1)^j \alpha_j;\ j = 1,\ 2,\ 3, \ldots,\ s. \tag{45}$$

3. 2. Some particular formulae

We will present now some particular A-spline for minimal q and k taking values from 1 to 11.

Considering Table 1, we first assume the constants $\alpha_{k,j}$ for $k = 1, 2, 3, \ldots, 11$, that could be easily determined from (44), using the values of the coefficients from (38) and (39) which could be easily got using reccurent formulae.

Table 1

k \ j	$\alpha_{k,j}$-numerator						$\alpha_{k,j}$ denominator
	0	1	2	3	4	5	
1	1						1
2	1						1
3	8	−1					3!
4	34	−5					4!
5	438	−112	13				2.5!
6	11514	−3276	399				2^3.6!
7	39232	−14913	3168	−311			3.7!
8	937508	−383847	84894	−8521			2^3.8!
9	927230	−455536	135068	−24208	2021		(2/3).9!
10	2030828430	−1051611000	322427820	−58973640	4985805		2^7.10!
11	468125040	−274136634	101692512	−25315017	3894504	−281085	2.11!

Table 2

k \ j	β^k_j-numerator						β^k_j-denominator
	−5	−4	−3	−2	−1	0	
1						−1	1
2						−1	2
3					1	−9	$(3!)^2$
4					5	−39	$(4!)^2$
5				−13	125	−550	$2(5!)^2$
6				−399	3675	−14790	$2^3(6!)^2$
7			311	−3479	18081	−54145	$3(7!)^2$
8			8521	−93415	468741	−1321355	$2^3(8!)^2$
9		−2021	26229	−159276	590604	−1382766	$(2/3)(9!)^2$
10		−4985805	63959445	−381401460	1374038820	−3082439430	$2^7(10!)^2$
11	281085	−4175589	29209521	−127007529	375829146	−742261674	$2(11!)^2$

Taking into account (21), we can immediately get the constants β_j^k for which $\beta_{j+1}^k = -\beta_{-j}^k$; $j=0, 1, 2, \dots, s=\left[\dfrac{k-1}{2}\right]$. These constants for $k=1, 2, 3, \dots, 11$ are given in Table 2.

Using Table 2, we can explicitly write A-splines of minimal q, for $k=1, 2, 3, 4, 5, 6$.

(46)

1) $A_{1,0}(f;\ x)=f_i+\dfrac{x-x_i}{h}\Delta f_i$ for $x\in[x_i,\ x_{i+1})$,

2) $A_{2,0}(f;\ x)=Q_{1,i}(f;\ x)+\dfrac{(x-x_i-h/2)^2}{2h^2}\Delta^2 f_i$ for $x\in[x_i+\dfrac{h}{2},\ x_{i+1}+\dfrac{h}{2})$,

3) $$A_{3,1}(f;\ x)=Q_{2,i}(f;\ x)+\frac{1}{36h^3}\{-(x-x_{i+2})^3\Delta^3 f_{i-1}$$
$$+[9(x-x_{i+1})^3-(x-x_i)^3]\Delta^3 f_i-(x-x_{i+1})^3\Delta^3 f_{i+1}\} \text{ for } x\in[x_{i+1},\ x_{i+2}),$$

4) $$A_{4,1}(f;\ x)=Q_{3,i}(f;\ x)+\frac{1}{576h^4}\{-5(x-x_{i+2}-h/2)^4\Delta^4 f_{i-1}$$
$$+[39(x-x_{i+1}-h/2)^4-5(x-x_i-h/2)^4]\Delta^4 f_i-5(x-x_{i+1}-h/2)^4\Delta^4 f_{i+1}\}$$
$$\text{for } x\in[x_{i+1}+h/2,\ x_{i+2}+h/2),$$

5) $$A_{5,2}(f;\ x)=Q_{4,i}(f;\ x)+\frac{1}{28800h^5}\{13(x-x_{i+3})^5\Delta^5 f_{i-2}$$
$$+[-125(x-x_{i+3})^5+13(x-x_{i+4})^5]\Delta^5 f_{i-1}+[550(x-x_{i+2})^5-125(x-x_{i+1})^5$$
$$+13(x-x_i)^5]\Delta^5 f_i+[-125(x-x_{i+2})^5+13(x-x_{i+1})^5]\Delta^5 f_{i+1}$$
$$+13(x-x_{i+2})^5\Delta^5 f_{i+2}\}$$
$$\text{for } x\in[x_{i+2},\ x_{i+3}),$$

6) $$A_{6,2}(f;\ x)=Q_{5,i}(f;\ x)+\frac{1}{4147200h^6}\{399(x-x_{i+3}-h/2)^6\Delta^6 f_{i-2}$$
$$+[-3675(x-x_{i+3}-h/2)^6+399(x-x_{i+4}-h/2)^6]\Delta^6 f_{i-1}+[14790(x-x_{i+2}-h/2)^6$$
$$-3675(x-x_{i+1}-h/2)^6+399(x-x_i-h/2)^6]\Delta^6 f_i+[-3675(x-x_{i+2}-h/2)^6$$
$$+399(x-x_{i+1}-h/2)^6]\Delta^6 f_{i+1}+399(x-x_{i+2}-h/2)^6\Delta^6 f_{i+2}\}$$
$$\text{for } x\in[x_{i+2}+h/2,\ x_{i+3}+h/2).$$

The general form of the above formulae (46) is

$$(47)\qquad A_{k,s}(f;\ x)=Q_{s,i}(f;\ x)+\sum_{j=-s}^{s} P_{k,j}\left(\frac{x-x_i}{h}-\frac{k-1}{2}\right)\Delta^k f_{i+j}$$

$$\text{for } x\in\left[x_i+\frac{k-1}{2},\ x_i+\frac{k+1}{2}\right),$$

where the polynomials $P_{k,j}=P_j$ are determined from the formulae (9) and (10).

For small values of k, one can compute the exact uniform norms $\|P_{k,j}\|_{[0,1]}$ of the polynomials $P_{k,j}$ in the segment [0, 1]. These norms are given in Table 3 for $k=1, 2, 3, \ldots, 11$, taking into consideration the obvious equality

$$(48)\qquad \|P_{k_1-j}\|_{[0,1]}=\|P_{k,j}\|_{[0,1]};\ j=1, 2, 3, \ldots, s.$$

Table 3. Values of $\|P_{k,j}\|_{[0,1]}$ accurate to 10^{-6}.

k \ j	0	1	2	3	4	5
1	1					
2	0.5					
3	0.0625	0.027777				
4	0.071180	0.008680				
5	0.018010	0.010416	0.000451			
6	0.017436	0.005271	0.000095			
7	0.003496	0.003319	0.000478	0.000004		
8	0.005304	0.001972	0.000161	0.000001		
9	0.001076	0.001076	0.000302	0.000011	0.000000	
10	0.001807	0.001092	0.000155	0.000003	0.000000	
11	0.000353	0.000361	0.000156	0.000014	0.000000	0.000000

3.3. Approximation properties of A-splines of minimal parameter

For $q=s=\left[\frac{k-1}{2}\right]$ in Theorem 2 for A-splines, there could be estimated the constants of the respective modulus of smoothness.

Denote λ_k and μ_k the minimal constants, for which the estimates (49) and (50) hold

$$(49)\qquad \|f-A_{k,s}(f)\|\leq\lambda_k\omega_k(f;\ h),$$

$$(50)\qquad \|A^{(k)}_{k,s}(f)\|\leq\mu_k h^{-k}\omega_k(f;\ h).$$

From (16), (36), (37), Table 1, Table 3, and Theorem 1, we obtain the following estimates for the constants λ_k and μ_k for $k=1, 2, 3, \ldots, 11$.

According to Table 4 one can assume that

$$(51)\qquad \sum_{j=-s}^{s}\|P_{k,j}\|_{[0,1]}<1\qquad \text{for } k=1, 2, 3, \ldots$$

$$(52)\qquad \sum_{j=1}^{s}|\alpha_j|\leq 2^k\qquad \text{for } k=1, 2, 3, \ldots$$

Table 4.

k	$\lambda_k \leqq$	$\mu_k \leqq$
1	3	1
2	2	1
3	1.7361	1.6667
4	3.4098	1.8334
5	6.0261	3.0834
6	6.0228	3.5521
7	6.0146	5.8657
8	6.0149	6.9185
9	6.0050	11.1990
10	6.0042	13.3836
11	6.0009	21.3083

For the time being we are not able to prove (51), therefore we will give a certain rough estimate.

Theorem 3. *Under the conditions of Theorem 2, the following estimates hold*

$$\|f-A_{k,s}(f)\| \leqq 2^k \omega_k(f;\ h),$$

$$\|A_{k,s}^{(k)}(f)\| \leqq 2^k h^{-k} \omega_k(f;\ h).$$

Proof. According to Theorem 2, it is necessary to prove the following two estimates.

$$W_k + 1 + \frac{2}{k!} \sum_{j=1} |\alpha_j| (j+1)^k \leqq 2^k, \tag{53}$$

$$1 + 4 \sum_{j=1}^{s} |\alpha_j| \leqq 2^k. \tag{54}$$

Therefore, to establish the inequalities (53) and (54) one needs to estimate the constants α_j and according to (44) the coefficients $a_{k,i}$ and $b_{s,i}^j$ of the polynomials ψ_k and $\theta_{s,j}$ from (39) and (38) should be evaluated.

For $k=2s+1$

$$0 < a_{k,i} \leqq \binom{s}{i} \left(\frac{s!}{(s-i)!}\right)^2, \tag{55}$$

since a $a_{k,i}$ is a sum of $\binom{s}{i}$ terms, the biggest is $\left(\frac{s!}{(s-i)!}\right)^2$.

For, $k=2s+2$, we get in the same way, that

$$0 < a_{k,i} \leqq \binom{s}{i} \left(\frac{(2s+1)!!}{2^i(2s-2i+1)!!}\right)^2 \leqq \binom{s}{i} \left(\frac{(s+1)!}{(s-i+1)!}\right)^2. \tag{56}$$

On the other hand, for $j=1, 2, 3, \ldots, s$

$$0 < b_{s,i}^j \leqq b_{s,i}^c = a_{2s+1,i} \leqq \binom{s}{i} \left(\frac{s!}{(s-i)!}\right)^2. \tag{57}$$

From (44), taking into consideration (55) — (57) and the formula of Sterling $n!=n^n e^{-n}\sqrt{2\pi n}\,e^{\theta/12}$; $0<\theta<1$ for $k=2s+1$ we obtain

$$(58)\qquad |\alpha_{k,j}|\leqq\frac{(s!)^2}{(2s)!}\binom{2s}{s-j}\sum_{i=1}^{s}\binom{s}{s-i}^4\Big/\binom{2s+1}{2i}$$

$$\leqq\frac{\sqrt{\pi s}\,e^{1/6}}{2^{2s}}\binom{2s}{s-j}s\binom{s}{[s/2]}^4\Big/\binom{2s+1}{s}.$$

Taking into account that

$$\sqrt{2/\pi s}\,2^s\leqq\binom{s}{[s/2]}\leqq e^{1/12}\sqrt{2/\pi s}\,2^s$$

from (58) we get

$$(59)\qquad |\alpha_{k,j}|\leqq\frac{2\sqrt{es(2s+1)}}{\pi\sqrt{2s}}\binom{2s}{s-j}\leqq\binom{2s}{s-j}$$

and still

$$\sum_{j=1}^{k}|\alpha_{k,j}|\leqq 2^{2s-1}-\binom{2s}{s}<2^{2s-1}-\frac{1}{4}.$$

Therefore

$$(60)\qquad 1+4\sum_{j=1}^{k}|\alpha_{k,j}|\leqq 2^{2s+1}=2^k.$$

In the same way we prove the estimate (60) for $k=2s+2$, as well. Thus the inequality (54) is proved.

To prove the inequality (53) considering (59), we should estimate the sum

$$(61)\qquad D_k=\frac{1}{k!}\sum_{j=1}^{s}|\alpha_{k,j}|(j+1)^k\leqq\frac{1}{k!}\sum_{j=1}^{k}\binom{2s}{s-j}(j+1)^k.$$

We will get the estimate (61) for $k=2s+1$ only, since the estimate for $k=2s+2$ can be obtained in the same way.

Using the formula of Stirling, we get

$$(62)\qquad D_k\leqq\frac{1}{2s+1}\sum_{j=1}^{s-1}\frac{(j+1)^{2s+1}}{(s+j)!\,(s-j)!}+\frac{(s+1)^k}{k!}$$

$$\leqq\frac{e^{2s}}{\pi(2s+1)}\sum_{j=1}^{s-1}\left(\frac{(1/s+j/s)^2}{(1+j/s)^{1+j/s}(1-j/s)^{1-j/s}}\right)^s\frac{j+1}{\sqrt{(s+j)(s-j)}}+\frac{e}{\sqrt{\pi s}}\left(\frac{e}{2}\right)^k.$$

Since according to Table 4, the inequality (53) is valid for $k\leqq 11$, then we could consider $s\geqq 5$.

It can be directly calculated that

$$(63)\qquad \frac{(0.2+t)^2}{(1+t)^{1+t}(1-t)^{1-t}} \leqq 0.48023 < \frac{4}{e^2} \text{ for } t\in[0,\ 1].$$

From (62) and (63) we have for $s\geqq 5$

$$(64)\qquad D_k \leqq \left(\frac{s-1}{2\pi(2s+1)} + \frac{e}{\sqrt{\pi s}}\left(\frac{e}{4}\right)^k\right)2^{2s+1} \leqq \frac{1}{100}2^k,$$

wherefrom the inequality (53) holds. □

4. A-splines of parameters q, multiple of the minimal

We consider A-splines $A_{k,q}(f)$ for which $q=ps$, $s=\left[\frac{k-1}{2}\right]$ where p is a natural number. To obtain the unique solution for the constants $\alpha_{k,j}$ we assume

$$(65)\qquad \alpha_{k,j}=0 \text{ for those } j, \text{ that are not multiple of } p.$$

Then in the conditions for determining the constants $\alpha_{k,j}$ only the constants $\alpha_{k,p,j}=\alpha_{k,p,j}$ will be used.

From (29) it follows that to determine the constants one should use the identity

$$(66)\qquad \sum_{j=1}^{s} \alpha_{k,p,j}[(x+pj)^k+(x-pj)^k-2x^k]=\varphi_k(x),$$

where φ_k is got from (30).

Denote $\theta_{s,p,j}$ the polynomial

$$(67)\qquad \theta_{s,p,j}(x)=\frac{1}{x^2-p^2j^2}x^2(x^2-p^2)(x^2-4p^2)\dots(x^2-s^2p^2)$$

$$=x^{2s}-b^{p,j}_{s,1}x^{2s-2}+\dots+(-1)^s b^{p,j}_{s,s}.$$

In an absolutely analogous way to Lemma 4, the following assertion is proved:

The identity (66) is satisfied if the values of constants $\alpha_{k,p,j}$ are

$$(68)\qquad \alpha_{k,p,j}=\frac{(-1)^{s+j}}{p^{2s}k!(s+j)!(s-j)!}\sum_{j=1}^{s}\theta^{(2i)}_{s,p,j}(0)\,\psi_k^{(k-2i)}(0).$$

From (67) it can be seen that

$$(69)\qquad b^{p,j}_{s,i}=p^{2i}b^j_{s,i}\ ;\ i=1,\ 2,\ 3,\dots,\ s.$$

From (68) and (69) it follows directly that

$$(70)\qquad \alpha_{k,p,j}=\frac{(-1)^j}{(s+j)!(s-j)!}\sum_{i=1}^{s}\frac{a_{k,i}b^j_{s,s-i}}{p^{2i}\binom{k}{2i}}.$$

Taking into consideration (58), (64) and (70), we get the following estimates

(71) $$|\alpha_{k,p,j}| \leqq \frac{1}{p^2}\binom{2s}{s-j},$$

(72) $$1+4\sum_{j=1}^{s}|\alpha_{k,p,j}| \leqq 1+\frac{2^k}{p^2},$$

(73) $$W_k+1+\frac{2}{k!}\sum_{j=1}^{s}|\alpha_{k,p,j}|(pj+1)^k \leqq (2p)^{k-2}.$$

Thus we obtain

Theorem 4. *Under the conditions of Theorem 2, the estimates* (74) and (75) *are valid.*

(74) $$\|f-A_{k,p,s}(f)\| \leqq (2p)^{k-2}\omega_k(f;\ h),$$

(75) $$\|A^{(k)}_{k,p,s}(f)\| \leqq (1+2^k p^{-2})\, h^{-k}\omega_k(f;\ h).$$

From (75) it is obvious that the coefficient preceding $h^{-k}\omega_k(f; h)$ could be got arbitrarily close to 1, if the parameter p is chosen sufficiently big.

On the basis of the estimate (75) we can state the following

Assumption. At $k \geqq 3$ there does not exist an operator defined for the functions with a bounded k-th modulus of smoothness on the real axis, for which $P_h^{(k)}(f)$ exists almost everywhere and the following estimates hold

$$\|f-P_h(f)\| \leqq C_k\omega_k(f;\ h),$$

$$\|P_h^{(k)}(f)\| \leqq h^{-k}\omega_k(f;\ h),$$

where C_k is a constant depending only on k.

It is evident that the constants preceding $h^{-k}\omega_k(f; h)$ can not be less than 1.

4.1. Some particular formulae

In Table 5 and Table 6 the values of the constants $\alpha_{k,p,j}$ and $\beta_{k,p,j}$ for $k=3$, 4, 5, 6 are given. Let note that for $k=1$, $k=2$ it is senseless to take $q \neq 0$, since it does not lead to approvement in the approximate properties of the respective spline.

Table 5.

k \ j	$d_{k,p,j}$-numerator 0	1	2	$\alpha_{k,p,j}$-denominator
3	$6p^2+2$	-1		$3!\,p^2$
4	$24p^2+10$	-5		$4!\,p^2$
5	$240p^4+150p^2+48$	$-80p^2-32$	$5p^2+8$	$2.5!\,p^4$
6	$5760p^4+4200p^2+1554$	$-2240p^2-1036$	$140p^2+259$	$8.6!\,p^4$

Table 6.

k \ j	$\beta_{k,p,j}$-numerator −2	−1	0	$\beta_{k,p,j}$-denominator
3		1	$-6p^2-3$	$(3!)^2p^2$
4		5	$-24p^2-15$	$(4!)^2p^2$
5	$-5p^2-8$	$85p^2+40$	$-240p^4-230p^2-80$	$2(5!)^2p^4$
6	$-140p^2-259$	$2380p^2+1295$	$-5760p^4-6440p^2-2590$	$8(6!)^2p^4$

The values of $\alpha_{k,p,j}$ for $k=7$, 8, 9 are:

$$3\cdot 7!\,p^6\alpha_{7,p,1}=-7560p^4-5733p^2-1620$$

$$3\cdot 7!\,p^6\alpha_{7,p,2}=756p^4+1764p^2+648$$

$$3\cdot 7!\,p^6\alpha_{7,p,3}=-56p^4-147p^2-108$$

$$8\cdot 8!\,p^6\alpha_{8,p,1}=-181440p^4-153972p^2-48435$$

$$8\cdot 8!\,p^6\alpha_{8,p,2}=18144p^4+47376p^2+19374$$

$$8\cdot 8!\,p^6\alpha_{8,p,3}=-1344p^4-3948p^2-3229$$

$$\frac{2}{3}9!\,p^8\alpha_{9,p,1}=-161280p^6-177632p^4-95120p^2-21504$$

$$\frac{2}{3}9!\,p^8\alpha_{9,p,2}=20160p^6+61516p^4+42640p^2+10750$$

$$\frac{2}{3}9!\,p^8\alpha_{9,p,3}=-2560p^6-8736p^4-9840p^2-3072$$

$$\frac{2}{3}9!\,p^8\alpha_{9,p,4}=180p^6+637p^4+820p^2+384.$$

5. A-spline in a finite segment

Since A-splines are defined only by the values of the approximated function at the knots, when defining an A-spline of a function given in a finite segment, this function should be extended on an infinite uniform net. Let the function f be given on the finite interval $[a, b]$ and the uniform net in this interval

$$x_0=a,\ x_1=a+h,\ldots,\ x_n=a+nh=b\ ;\ h=(b-a)/n.$$

We consider the infinite uniform net $\{x_i\}$ for which $x_i=x_0+ih$; $i=\pm 1, \pm 2,\ldots$ We extended the function f on this infinite net in the following way

$$(76)\qquad \begin{array}{ll} f_i = Q_{k-1,1}(f;\ x_i) & \text{for } i=0,\ -1,\ 2,\dots \\ f_i = Q_{k-1,n-k}(f;\ x_i) & \text{for } i=n,\ n+1,\ n+2,\dots \end{array}$$

From the extention (76) altogether $f_0 \neq f(a)$ and $f_n \neq f(b)$, but this fact is not of considerable significance. Surely $f_i = f(x_i)$ for $i=1, 2, 3, \dots, n-1$. The choice of the continuation (76) provides the opportunity of applying Theorem 1.

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Institute of Mathematics
Bulgarian Academy of Sciences
1090 Sofia,
BULGARIA

Received 27. 08. 1988