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Super SMP-Riesz Spaces

B. Lavrič

Presented by Ž. Mijajlović

An Archimedean Riesz space is called a super SMP-space if the Boolean ring $\mathscr{B}_p(L)$ of all its principal projection bands is super order dense in the lattice $\mathscr{A}_p(L)$ of its principal bands. A characterization of super SMP-spaces is given and some results are generalized on these spaces.

0. Introduction

Let L be a Riesz space. L is said to have sufficiently many projections (L is an *SMP-space*) if every nonzero band in L contains a nonzero projection band ([5]). If L is Archimedean, the Boolean ring $\mathscr{B}_p(L)$ of all principal projection bands of L is order dense in the lattice $\mathscr{A}_p(L)$ of all principal bands of L iff L is an SMP-space ([3]). Following the usual terminology an Archimedean Riesz space with $\mathscr{B}_b(L)$ super order dense in $\mathscr{A}_p(L)$ will be called a *super SMP-space.* The family of all super SMP-spaces contains all Riesz spaces with the principal projection property. In this paper we will characterize super SMP-spaces and extend some results (for example, Egoroff type theorem) on this family. For terminology concerning Riesz spaces we refer the reader to [5], [7] and [2].

1. Characterizations

Let X be a distributive lattice with the smallest element 0. The subset $D\subset X$ is said to be (super) order dense in X if for every $x>0$, $x\in X$, there exists an increasing net (sequence) of elements $y_\tau\in D$ such that $\sup_\tau y_\tau=x$ (abbreviated $y_\tau\uparrow x$).

Let now L be an Archimedean Riesz space. If M is a Riesz subspace of L, then M is (super) order dense in L ([2], 14F) if M^+ is (super) order dense in L^+. Recall that L is said to be an SMP-space if the Boolean algebra $\mathscr{B}(L)$ of all its projection bands is order dense in the Boolean algebra $\mathscr{A}(L)$ of all bands in L. As shown in [3] the order denseness of $\mathscr{B}_p(L)$ in $\mathscr{A}_p(L)$ is necessary and sufficient for

L to be an SMP-space. This result is based on the following lemma. For the sake of convenience we give a short proof.

Lemma 1.1. (*Jakubik*) *Let $B \in \mathscr{B}(L)$ be contained in the principal band $\{f\}^{dd}$ generated by $f \in L^+$. Then B is a principal projection band generated by the component f_B of f in B.*

Proof. Since $\{f_B\}^{dd} \subset B$ we have to prove only the inclusion $\{f_B\}^d \subset B^d$. To this end let $0 \leqq v \in \{f_B\}^d$ and let v_B denote the component of v in B. The inequalities

$$0 \leqq f \wedge v_B \leqq \sup\{w \in B : 0 \leqq w \leqq f\} = f_B$$

shows that $0 \leqq f \wedge v_B \leqq f_B \wedge v = 0$, thus $v_B \in \{f\}^d \cap B = \{0\}$. Hence $v \in B^d$ as we claimed. □

To simplify the notation we propose the following definition. An Archimedean Riesz space L is said to be a *super SMP-space* if $\mathscr{B}_p(L)$ is super order dense in $\mathscr{A}_p(L)$.

In order to characterize super SMP-spaces we will need the next result, which can be proved by standard arguments, using Lemma 1.1.

Lemma 1.2. *Let A be a band in L, $(B_\tau : \tau \in T)$ an upward directed net of band projections in L such that $B_\tau \subset A$ for all $\tau \in T$, and let S be a complete system of positive elements in A. Then $B_\tau \uparrow A$ in $\mathscr{A}(L)$ if and only if $f_{B_\tau} \uparrow f$ for all $f \in S$.*

It is easy to see that an Archimedean Riesz space L has sufficiently many projections if and only if the cone P of its projection elements (i. e. elements $f \in L^+$ satisfying $\{f\}^{dd} \in \mathscr{B}_p(L)$) is order dense in L^+. A similar characterization holds for super SMP-spaces.

Proposition 1.3. *In an Archimedean Riesz space L the following conditions are equivalent.*

(i) *The cone P of projection elements of L is super order dense in L^+.*

(ii) *L is a super SMP-space.*

Proof. It follows easily from Lemma 1.2. that (i) implies (ii). For the reverse implication use also Lemma 1.1. □

Corollary 1.4. *Let $e \in L^+$ be a weak (order) unit in L and let $\mathscr{N}_e$ denotes the Riesz subspace of L generated by the Boolean algebra $\mathscr{P}_e = \{Pe : P$ a band projection on $L\}$. If $\mathscr{N}_e$ is super order dense in L, then L is a super SMP-space.*

Proof. For any $u \in L^+$ take an increasing sequence $p_n \in \mathscr{N}_e$ such that $p_n \uparrow u$. Put $B_n = \{p_n\}^{dd}$, observe that according to Lemma 1.1. $B_n \in \mathscr{B}(L)$, $u_{B_n} \uparrow u$ and therefore by Lemma 1.2. $B_n \uparrow \{u\}^{dd}$. □

Remark. It can be shown, that $\mathscr{N}_e$ is super order dense in L if and only if L is a super SMP-space.

Any Riesz space with the principal projection property is obviously a super SMP-space. If the Boolean algebra $\mathscr{B}(L)$ is σ-Dedekind complete, then the converse holds too.

Theorem 1.5. *If a super SMP-space L has σ-Dedekind complete Boolean algebra $\mathscr{B}(L)$, then L has the principal projection property. In a uniformly complete Archimedean Riesz space L the following statements are equivalent:*

(i) *L is super SMP-space with σ-Dedekind complete algebra $\mathscr{B}(L)$.*

(ii) *L is σ-Dedekind complete.*

Proof. Assume that L is a super SMP-space, $u \in L^+$, and let B_n be an increasing sequence of projection bands in L such that $B_n \uparrow \{u\}^{dd}$ in $\mathscr{A}(L)$, i.e., $(\bigcup_n B_n)^{dd} = \{u\}^{dd}$. If $\mathscr{B}(L)$ is σ-Dedekind complete, there exists $B = \sup\{B_n : n \in \mathsf{N}\} \in \mathscr{B}(L)$. We claim, that $B = (\bigcup_n B_n)^{dd}$, and the conclusion of the first part of the theorem follows. If this is not the case, there exists a nonzero positive element $v \in B \bigcap (\bigcap_n B_n^d)$, and a nontrivial projection band $B_0 \subset \{v\}^{dd}$. Then $B_n \subset B \bigcap B_0^d$ holds for all $n \in \mathsf{N}$, and we get the contradiction $B \subset B \bigcap B_0^d \subsetneq B$ (since $v \in B \setminus B_0^d$).

The second part of the theorem is a consequence of the first part and Theorems 42.5, 30.6 of [5]. □

The preceding theorem is a σ-analogue of [5], Thm. 30.6.

For further discussion we recall some notions. A lattice X is said to have the *σ-interpolation property* if whenever the increasing sequence x_n and the decreasing sequence y_n in X are such that $x_n \leqq y_n$ for all n, then there exists an element $z \in X$ satisfying $x_n \leqq z \leqq y_n$ for all n. An Archimedean Riesz space L is called an *SF-space*, if any two disjoint elements of L are contained in disjoint projection bands. In every SF-space the following strong form of Freudenthal's spectral theorem holds ([4], Thm. 3.8). For every $u \in L^+$ each $f \in A_u$ (A_u is the principal ideal generated by u) can be u-uniformly approximated by elements of the Riesz subspace $\mathscr{N}_u$ generated by the Boolean algebra $\mathscr{P}_u$. Since every SF-space is normal ([4]), a uniformly complete SF-space has the σ-interpolation property. Thus, we can expect an interpolation property which characterizes uniformly complete SF-spaces. We shall say that L has the interpolation property (S), if for any $u \in L^+$ and sequences f_n, g_n in L^+ satisfying $f_n \uparrow \leqq g_n \downarrow$ and $f_n \wedge (u - g_n) = 0$ for all n, there exists $h \in \mathscr{P}_u$ such that $f_n \uparrow \leqq h \leqq g_n \downarrow$.

Proposition 1.6. *Let L be a uniformly complete Archimedean Riesz space. Then the following conditions are equivalent:*
(i) *L has the interpolation property* (S).
(ii) *L is an SF-space.*

Proof. (i)$\Rightarrow$(ii). If $f, g \in L^+$ are disjoint, then the constant sequences $f_n = f$, $g_n = g$, and $u = f + g$ satisfy the conditions of (S). Hence $f \in \mathscr{P}_u$ and therefore (ii) holds. (ii)$\Rightarrow$(i). Let $u \in L^+$ and let the sequences f_n, g_n in L^+ satisfy the conditions $f_n \uparrow \leqq g_n \downarrow$, $f_n \wedge (u - g_n) = 0$ for all n. Since L is uniformly complete, there exist u-uniform limits $f = \sum_1^\infty 2^{-k} f_k$, $g = \sum_1^\infty 2^{-k} g_k$. It follows that $0 \leqq f \leqq g \leqq u$, and $f \wedge (u - g) = 0$, therefore by (ii) there exists a band projection P such that $Pf = f$ and $P(u - g) = 0$. It is easy to see that $h = Pu$ fulfils the condition $f_n \uparrow \leqq h \leqq g_n \downarrow$, and the proof is complete. □

We are prepared now for the next result.

Theorem 1.7. *Let L be an Archimedean Riesz space. Consider the following two statements:*
(i) *L is a super SMP-space with $\mathscr{B}(L)$ possessing the σ-interpolation property.*
(ii) *L is an SF-space.*

Then (i) *implies* (ii). *If in addition L is uniformly complete with a weak unit, then the conditions are equivalent.*

Proof. (i) ⇒(ii). Let $u, v \in L^+$ and $u \wedge v = 0$. Choose the increasing sequences $B_n, C_n \in \mathscr{B}(L)$ such that $B_n \uparrow \{u\}^{dd}$, $C_n \uparrow \{v\}^{dd}$, set $D_n = C_n^d$ and note that $D_n \downarrow \{v\}^d$ in $\mathscr{A}(L)$. Since $\{u\}^{dd} \subset \{v\}^d$, there exists by (i) a projection band B such that $\{u\}^{dd} \subset B \subset \{v\}^d$. Thus $u \in B$, $v \in B^d$, and (ii) follows.

Assume now that $e \in L^+$ is a weak unit in a uniformly complete SF-space L. Then the Riesz subspace $\mathscr{N}_e$ is uniformly dense in A_e, hence super order dense in L. By Corollary 1.4. L is a super SMP-space. In order to show that $\mathscr{B}(L)$ has the σ-interpolation property assume that the sequences B_n and C_n of projection bands in L satisfy the condition $B_n \uparrow \subset C_n \downarrow$. Set $f_n = e_{B_n}$, $g_n = e_{C_n}$, observe that $f_n \uparrow \leqq g_n \downarrow$, $f_n \wedge (g_1 - g_n) = 0$ for all n, and use Proposition 1.6. to get an element $h \in \mathscr{P}_{g_1}$ such that $f_n \uparrow \leqq h \leqq g_n \downarrow$. Since evidently $h \in \mathscr{P}_e$, it follows easily that $B = \{h\}^{dd} \in \mathscr{B}(L)$ and $B_n \subset B \subset C_n$ for all n. □

Remark. If a uniformly complete Archimedean Riesz space L has a weak unit $e \in L^+$ such that every component of e is contained in $\mathscr{P}_e$ (for example, if e is a topological weak unit in a Banach lattice or a ring unit in an Archimedean f-algebra), then the following condition is equivalent to (i) and (ii) of Theorem 1.7.
(iii) The Kakutani representation space K of A_e is totally disconnected and $\mathscr{B}(L)$ has the σ-interpolation property.

Theorem 1.7. generalizes a part of Theorem A in [6].

2. Egoroff theorem

Now we turn our attention to the Egoroff property. Recall that a distributive lattice X with the smallest element 0 is said to have the *Egoroff property*, if for every element $x \in X$ and any double sequence $y_{n,k}$ in X such that $y_{n,k} \uparrow_k x$ there exists a sequence z_m in X such that $z_m \uparrow x$ and that for every m and n there exists $k = k(m,n)$ satisfying $z_m \leqq y_{n,k}$ (the latter property is usually denoted by $z_m \ll y_{n,k}$). A Riesz space L is said to have the *Egoroff property* if its positive cone L^+ possesses the Egoroff property.

We shall generalize now Theorem 74.2. of [5].

Theorem 2.1. *Let L be a super SMP-space with the Egoroff property. Then the Boolean ring $\mathscr{B}_b(L)$ has the Egoroff property.*

Proof. Let $B = \{u\}^{dd} \in \mathscr{B}_p(L)$, $u \in L^+$, and let the double sequence $B_{n,k} \in \mathscr{B}_p(L)$ satisfy $B_{n,k} \uparrow_k B$. Denote by $u_{n,k}$ the component of u in $B_{n,k}$. Observe that by Lemma 1.1. $B_{n,k} = \{u_{n,k}\}^{dd}$, and by [5], Thm. 30.5 $u_{n,k} \uparrow_k u$. Since L has the Egoroff property, there exists an increasing sequence u_m in L^+ such that $u_m \uparrow u$ and $u_m \ll u_{n,k}$ for all m. For every m take an increasing sequence $(B_m^i : i \in \mathrm{N})$ in $\mathscr{B}_p(L)$ which satisfies $B_m^i \uparrow_i \{u_m\}^{dd}$. We may assume that B_m^i increases in m for any fixed i. Since for every m, n there exists $k(m,n)$ such that $u_m \leqq u_{n,k(m,n)}$, we have

$$B_m = B_m^m \subset \{u_m\}^{dd} \subset \{u_{n,k(m,n)}\}^{dd} = B_{n,k(m,n)}$$

and therefore $B_m \ll B_{n,k}$ for every m. To finish the proof we will show that $B_m \uparrow B$.

Since B_m is increasing by construction, it is enough to see that $u_{B_m} \uparrow u$. Suppose that $0 \leqq w \leqq u - u_{B_m}$ holds for all m. It follows that for band projections P_m^i on B_m^i

$$u - P_m^i u_m \geqq u - u_{B_{m+i}} \geqq w \qquad \text{holds for all } i, m$$

thus

$$u - u_m = u - \sup \{P_m^i u_m : i \in \mathsf{N}\} \geqq w \qquad \text{holds for all } m$$

and consequently $w = 0$. □

Our next result extends an abstract Egoroff type theorem [5], Thm. 74.3 on super *SMP*-spaces.

Proposition 2.2. *Let L be a super SMP-space with $\mathscr{B}_p(L)$ possessing the Egoroff property. Let $e \in L^+$ be a projection element and u_k a sequence in L^+ such that $u_k \downarrow 0$. Then there exists a sequence e_m of projection elements in L satisfying $0 \leqq e_m \uparrow e$ and $P_m u_k \underset{k}{\rightarrow} 0$ e-uniformly for all m, where P_m denotes the band projection on $\{e_m\}^{dd}$.*

Proof. Set $e_{n,k} = (u_k - (1/n)e)^-$, note that the relation $0 \leqq e_{n,k} \uparrow_k (1/n)e$ holds for all n, and take sequences $(B_{n,k}^i : i \in \mathsf{N})$ in $\mathscr{B}_p(L)$ such that

$$B_{n,k}^i \uparrow_i \{e_{n,k}\}^{dd}, \quad n, k \in \mathsf{N}, \quad B_{n,k}^i \uparrow_k, \quad n, i \in \mathsf{N}.$$

Put $B_{n,k} = B_{n,k}^{n+k}$ and denote by $P_{n,k}$ the band projection on $B_{n,k}$. It is not hard to show that $P_{n,k} e \uparrow_k e$ and $B_{n,k} \uparrow_k B$. By assumption $\mathscr{B}_p(L)$ has the Egoroff property, hence there exists an increasing sequence B_m in $\mathscr{B}_p(L)$ such that $B_m \uparrow B$ and $B_m \subset B_{n,k(m,n)}$ for appropriate $k(m,n)$. Using the relation $B_{n,k} \subset \{e_{n,k}\}^{dd}$ we can show easily that $P_m u_{k(m,n)} \leqq (1/n)e$, where P_m denotes the band projection on B_m. Therefore $P_m u_k \leqq (1/n)e$ holds for all $k \geqq k(m,n)$, hence $P_m u_k \underset{k}{\rightarrow} 0$ e-uniformly for all m. Put now $e_m = P_m e$, and the proof is finished. □

In super *SMP*-spaces with the Egoroff property the previous result holds for all positive elements. The details follow.

Theorem 2.3. *Let L be a super SMP-space with the Egoroff property, $e \in L^+$ and $u_k \downarrow 0$ in L. Then there exists an increasing sequence of projection elements $e_m \in L^+$ such that $e_m \uparrow e$ and $P_m u_k \downarrow_k 0$ e-uniformly for all m, where P_m is the band projection on $\{e_m\}^{dd}$.*

Proof. In the same way as in the proof of the preceding theorem we get the double sequence $B_{n,k} \in \mathscr{B}_p(L)$ such that $B_{n,k} \uparrow_k \{e\}^{dd}$ (or $P_{n,k} e \uparrow_k e$). Choose now a sequence f_m in L^+ satisfying $f_m \uparrow e$ and $f_m \ll P_{n,k} e$ for all m. Next, take sequences $(B_m^i : i \in \mathsf{N})$, $m \in \mathsf{N}$, in $\mathscr{B}_p(L)$ such that

$$B_m^i \uparrow_i \{f_m\}^{dd}, \quad m \in \mathsf{N}, \quad B_m^i \uparrow_m, \quad i \in \mathsf{N}.$$

It follows that $B_m = B_m^m$ satisfies $B_m \uparrow \{e\}^{dd}$. Since

$$P_m u_{k(m,n)} \leqq (1/n)e \text{ and } P_m u_k \leqq (1/n)e \text{ for all } k \geqq k(m,n)$$

we have $P_m u_k \downarrow_k 0$ e-uniformly, as we claimed. □

3. Examples

Let K be a completely regular Hausdorff topological space. Denote by $\mathscr{C}(K)$ the Riesz space of all real continuous functions on K. Recall that a subset $Z \subset K$ is called a zero set, if it is a null set of some $f \in \mathscr{C}(K)$. A subset C of K is said to be a cozero set, if its complement $K \backslash C$ is a zero set. Let us describe super SMP-spaces of the type $\mathscr{C}(K)$ by topological properties of K.

Proposition 3.1. *The following conditions are equivalent* :
(i) *$\mathscr{C}(K)$ is a super SMP-space.*
(ii) *For any cozero subset $C \subset K$ there exists a countable family W_n of open and closed subsets of C such that* $\mathrm{cl}(\bigcup_n W_n) = \mathrm{cl}(C)$.

Proof. (i)⇒(ii). Let $C = \{x \in K : f(x) > 0\}$, $f \in \mathscr{C}^+(K)$, be an arbitrary cozero set. For every $n \in \mathbf{N}$ take an increasing sequence $(B_{n,k} : k \in \mathbf{N})$ of projection bands in $\mathscr{C}(K)$ such that

$$B_{n,k} \uparrow_k \{(f-(1/n)1_K)^+\}^{dd}.$$

Each $B_{n,k}$ is generated by a characteristic function of an open and closed set $W_{n,k}$. Observe that $\bigcup_k W_{n,k}$ is dense in the cozero set which corresponds to $(f-(1/n)1_K)^+$, hence the diagonal sequence $W_n = W_{n,n}$ satisfies (ii).
(ii)⇒(i). Let $f \in \mathscr{C}^+(K)$, $C = \{x \in K : f(x) > 0\}$, and let h_n be the characteristic functions of the open and closed sets W_n satisfying (ii). It follows that $B_n = \sum_{k=1}^{n} \{h_n\}^{dd} \in \mathscr{B}_p(\mathscr{C}(K))$ and $B_n \uparrow \{f\}^{dd}$. □

Example 3.2. Let K be the union of two concentric circles C_1 and C_2 in the complex plane $\mathbf{C}$. The projection $z \to z/2$ of C_2 onto C_1 will be denoted by p. The topology on K is given by a neighbourhood system $\{\mathscr{B}(z) : z \in K\}$ as follows. $\mathscr{B}(z) = \{\{z\}\}$ for $z \in C_1$ and

$$\mathscr{B}(z) = \{U_n(z) : n \in \mathbf{N}\}, \ z \in C_2, \text{ where } U_n(z) = V_n(z) \bigcup p(V_n(z) \backslash \{z\})$$

and V_n is the arc of C_2 with centre at z and of length $1/n$. The space K is called the *Alexandroff double circle* (see [1], 3.1.26.) and is compact and Hausdorff. Since every cozero set $C \subset K$ intersects C_1 (hence, contains an isolated point of K), $\mathscr{C}(K)$ is a SMP-space. To see that it is not a super SMP-space consider the function $f \in \mathscr{C}(K)$, $f : z \to (\mathrm{Re}\, z)^+$, and observe that its cozero set does not satisfies condition (ii) of Proposition 3.1. It can be seen easily that K is not totally disconnected.

Example 3.3. Let J be the closed subspace of the Alexandroff double circle K consisting of C_2 and of points $z \in C_1$ satisfying arg $z \in [0, 2\pi) \cap \mathbb{Q}$. It is easy to see that $\mathscr{C}(J)$ is a super SMP-space. We claim that it is not totally disconnected. For if we can separate different points $u, v \in C_2$ with disjoint open and closed sets W_u, W_v, we get two disjoint open and closed subsets $C_2 \cap W_u$ and $C_2 \cap W_v$ of the circle C_2 (with usual topology). But this is a contradiction.

Example 3.4. The space c of real convergent sequences is isomorphic to $\mathscr{C}(K)$, $K = \{1/n : n \in \mathbb{N}\} \cup \{0\} \subset \mathbb{R}$, thus by Proposition 3.1 a super SMP-space (since K is totally disconnected). It can be shown that c has the Egoroff property, hence it satisfies the conditions of Thm. 2.1 and Thm. 2.3, although it does not have the principal projection property (Compare with [5], Thm. 74.2.).

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Department of Mathematics
E. K. University of Ljubljana
Jadranska 19, Ljubljana
YUGOSLAVIA

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