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An Extremal Problem for Polynomials with Nonnegative Coefficients IV

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Let $\mathscr{P}_n^+$ be the set of all algebraic polynomials of exact degree n, whose coefficients are all nonnegative. For the norm in $L^p[0, \infty)$ with generalized Laguerre weight function $w(x)=x^\alpha e^{-x}(\alpha>-1)$, the extremal problem $C_{n,p}(\alpha)=\sup_{Q\in\mathscr{P}_n^+} (\|Q'\|_p/\|Q\|_p)^p$ is solved when $p=3$.

1. Introduction

In this work we give the complete solution of an analogous problem which has been investigated recently by A. K. Varma [5] and G. V. Milovanović [1].

Let $\mathscr{P}_n^+$ be the set of all algebraic polynomials of exact degree n, all coefficients of which are nonnegative, i. e.,

$$\mathscr{P}_n^+=\left\{Q_n \mid Q_n(x)=\sum_{k=0}^{n} a_k x^k,\ a_k\geqq 0\ (k=0,\ 1,\dots,n-1), a_n>0\right\}.$$

We denote by $\mathscr{P}_n^0$ the subset of $\mathscr{P}_n^+$ for which $a_0=0$; we note that $Q(0)=0$ for each Q in $\mathscr{P}_n^0$.

Let $w(x)=x^\alpha e^{-x}(\alpha>-1)$ be a weight function on $[0, \infty)$. For $Q_n\in\mathscr{P}_n^+$, we define $\|Q_n\|_p=\left[\int_0^{+\infty} w(x)Q_n^p(x)\,dx\right]^{1/p}$, $p\geqq 1$, and consider the following extremal problem:

Determine the best constant in the inequality

$$\|Q_n'\|_p^p\leqq C_{n,p}(\alpha)\,\|Q_n\|_p^p,\quad Q_n\in\mathscr{P}_n^+, \tag{1.1}$$

i. e.,

$$(1.2) \qquad C_{n,p}(\alpha)=\sup_{Q_n\in\mathscr{P}_n^+}\frac{\|Q_n'\|_p^p}{\|Q_n\|_p^p}.$$

In the case $p=2$. G. V. Milovanović [1] proved the following result:

Theorem A. *The best constant* $C_{n,2}(\alpha)$ *defined in (1.1) is*

$$C_{n,2}(\alpha)=\begin{cases}\dfrac{1}{(2+\alpha)(1+\alpha)} & \text{if } -1<\alpha\leqq\alpha_n,\\[2ex] \dfrac{n^2}{(2n+\alpha)(2n+\alpha-1)} & \text{if } \alpha_n\leqq\alpha<+\infty,\end{cases}$$

where $\alpha_n=\dfrac{1}{2(n+1)}((17n^2+2n+1)^{1/2}-3n+1)$.

An extremal problem for higher derivatives of nonnegative polynomials with respect to the same weight was investigated in [4]. A similar problem for Freud's weight function has been done by G. V. Milovanović and R. Ž. Đorđević [3]. A survey about extremal problems of Markov's type for algebraic polynomials is given in [2].

In this work we consider the extremal problem (1.1) for $p=3$.

Firstly, we note that the supremum in our extremal problem (1.2) is attained for some $Q\in\mathscr{P}_n^0$, i.e.,

$$C_{n,p}(\alpha)=\sup_{Q\in\mathscr{P}_n^+}\left(\frac{\|Q'\|_p^p}{\|Q\|_p^p}\right)=\sup_{\substack{Q\in\mathscr{P}_n^0\\ a_0\geq 0}}\left(\frac{\|Q'\|_p^p}{\|Q\|_p^p}\right)=\sup_{Q\in\mathscr{P}_n^0}\left(\frac{\|Q'\|_p^p}{\|Q\|_p^p}\right).$$

2. Some Auxiliary Results

Lemma 1. *Let* $p\in\mathrm{N}^*$ *and* q *such that* $\dfrac{1}{p}+\dfrac{1}{q}=1$. *If* $Q_n\in\mathscr{P}_n^+$ *then for every* $x\geqq 0$ *the inequality*

$$(2.1) \qquad (xQ_n'(x))^p\leqq Q_n(x)^{p-1}\sum_{i=1}^{p}s_p^{(i)}x^iQ_n^{(i)}(x)$$

holds, where $s_p^{(i)}$ *are the Stirling's numbers of the second kind.*

Proof. Let $Q_n\in\mathscr{P}_n^+$, i.e., $Q_n(x)=\sum_{k=0}^{n}a_kx^k$ with $a_k\geqq 0$ $(k=0, 1,\dots,n)$. Using the Cauchy-Schwarz inequality

$$\left|\sum_{k=0}^{n}x_ky_k\right|\leqq\left[\sum_{k=0}^{n}|x_k|^q\right]^{1/q}\left[\sum_{k=0}^{n}|y_k|^p\right]^{1/p},$$

for $x_k=(a_kx^k)^{1/q}$ and $y_k=k(a_kx^k)^{1/p}$, we obtain

$$\text{(2.2)} \qquad \sum_{k=0}^{n} k a_k x^k \leqq \left[\sum_{k=0}^{n} a_k x^k \right]^{1/q} \left[\sum_{k=0}^{n} k^p a_k x^k \right]^{1/p}.$$

On the other hand, using the Stirling's numbers of the second kind, $s_m^{(i)}$, defined by

$$t^m = \sum_{i=0}^{m} s_m^{(i)} t^{(i)}, \quad t^{(i)} = t(t-1)(t-2)\ldots(t-i+1),$$

we find that

$$\sum_{k=0}^{n} k^p a_k x^k = \sum_{k=0}^{n} \sum_{i=0}^{p} s_p^{(i)} k^{(i)} a_k x^k = \sum_{i=0}^{p} s_p^{(i)} x^i \sum_{k=0}^{n} k^{(i)} a_k x^{k-i},$$

i. e.,

$$\sum_{k=0}^{n} k^p a_k x^k = \sum_{i=1}^{p} s_p^{(i)} x^i Q_n^{(i)}(x),$$

because $s_p^{(0)} (p \geqq 1)$. So, from (2.2) we obtain

$$x Q_n'(x) \leqq Q_n(x)^{1-1/p} \left(\sum_{i=0}^{p} s_p^{(i)} x^i Q_n^{(i)}(x) \right)^{1/p},$$

which is equivalent to (2.1). □

Remark 1. The inequality (2.1), for $p=2$, is proved in [1]. For $p=3$ it reduces to the inequality

$$\text{(2.3)} \qquad (xQ_n'(x))^3 \leqq Q_n(x)^2 (xQ_n'(x) + 3x^2 Q_n''(x) + x^3 Q_n'''(x)).$$

Remark 2. The Stirling numbers satisfy the recurrence relation

$$s_{m+1}^{(i)} = i s_m^{(i)} + s_m^{(i-1)}, \quad m \geqq i \geqq 1, \text{ where } s_m^{(0)} = 0, \ s_m^{(1)} = s_m^{(m)} = 1, \ m \geqq 1.$$

By a simple application of integration by parts we can prove:

Lemma 2. *If $Q_n \in \mathscr{P}_n^0$, then for the integrals*

$$J_n(\alpha) = \int_0^{+\infty} x^\alpha e^{-x} (Q_n'(x))^3 \, dx,$$

$$I_{n,i}(\alpha) = \int_0^{+\infty} x^\alpha e^{-x} Q_n(x)^2 Q_n^{(i)}(x) \, dx, \quad i = 0, 1, 2, 3,$$

$$K_{n,i}(\alpha) = \int_0^{+\infty} x^\alpha e^{-x} Q_n(x) Q_n'(x) Q_n^{(i)}(x) \, dx, \quad i = 0, 1, 2,$$

the following recurrence relations hold:

$$(2.4)\qquad I_{n,i}(\alpha)=I_{n,i-1}(\alpha)-\alpha I_{n,i-1}(\alpha-1)-2K_{n,i-1}(\alpha),\quad i=1,\ 2,\ 3,$$

$$(2.5)\qquad 2K_{n,2}(\alpha)=K_{n,1}(\alpha)-\alpha K_{n,1}(\alpha-1)-J_n(\alpha),$$

$$K_{n,0}(\alpha)=I_{n,1}(\alpha).$$

Using the inequality (2.3) and Lemma 2 we obtain

Lemma 3. *If* $Q_n\in\mathscr{P}_n^0$ *then for* $\alpha>-1$,

$$(2.6)\quad J_n(\alpha)\leqq\frac{1}{27}[I_{n,0}(\alpha)+3(1-\alpha)I_{n,0}(\alpha-1)+(3\alpha^2-9\alpha+7)I_{n,0}(\alpha-2)$$
$$+(-\alpha^3+6\alpha^2-12\alpha+8)I_{n,0}(\alpha-3)].$$

Proof. Firstly, using Lemma 1, for $p=2$ and $p=3$, we conclude that

$$(2.7)\qquad J_n(\alpha)\leqq K_{n,1}(\alpha-1)+K_{n,2}(\alpha)$$

and

$$(2.8)\qquad J_n(\alpha)\leqq I_{n,1}(\alpha-2)+3I_{n,2}(\alpha-1)+I_{n,3}(\alpha).$$

Because of (2.4), we have

$$I_{n,3}(\alpha)=I_{n,1}(\alpha)-2\alpha I_{n,1}(\alpha-1)+\alpha(\alpha-1)I_{n,1}(\alpha-2)-2K_{n,2}(\alpha)-2K_{n,1}(\alpha)$$
$$+2\alpha K_{n,1}(\alpha-1)$$

and

$$I_{n,2}(\alpha-1)=I_{n,1}(\alpha-1)-(\alpha-1)I_{n,1}(\alpha-2)-2K_{n,1}(\alpha-1).$$

Then (2.8) becomes

$$(2.9)\qquad J_n(\alpha)\leqq U_n(\alpha)+V_n(\alpha),$$

where

$$(2.10)\quad U_n(\alpha)=I_{n,1}(\alpha)+(3-2\alpha)I_{n,1}(\alpha-1)+(4-4\alpha+\alpha^2)I_{n,1}(\alpha-2)$$

and

$$V_n(\alpha)=(2\alpha-6)K_{n,1}(\alpha-1)-2K_{n,2}(\alpha)-2K_{n,1}(\alpha).$$

Using (2.5), the last equality reduces to

$$(2.11)\qquad V_n(\alpha)=-6(K_{n,1}(\alpha-1)+K_{n,2}(\alpha))-2J_n(\alpha).$$

Combining (2.7), (2.9) and (2.11) we obtain

$$9J_n(\alpha) \leqq U_n(\alpha). \tag{2.12}$$

Since $I_{n,1}(\alpha) = \frac{1}{3}(I_{n,0}(\alpha) - \alpha I_{n,0}(\alpha-1))$, from (2.10) and (2.12) there immediately follows (2.6). □

3. Main result

Theorem. *The best constant $C_{n,3}(\alpha)$ defined in (1.1) is*

$$C_{n,3}(\alpha) = \begin{cases} \dfrac{1}{(\alpha+1)(\alpha+2)(\alpha+3)}, & -1 < \alpha \leqq \alpha_n, \\ \dfrac{n^3}{(3n+\alpha)(3n+\alpha-1)(3n+\alpha-2)}, & \alpha_n \leqq \alpha < +\infty, \end{cases} \tag{3.1}$$

where α_n is the unique positive root of the equation

$$(n^2+n+1)\alpha^3 + 3(2n^2+2n-1)\alpha^2 + (11n^2-16n+2)\alpha - 3n(7n-2) = 0. \tag{3.2}$$

Proof. Let $Q_n \in \mathscr{P}_n^0$, i.e., $Q_n(x) = \sum_{k=1}^{n} a_k x^k$, where $a_k \geqq 0$. Then

$$Q_n(x)^3 = \sum_{k=3}^{3n} b_k x^k, \qquad (b_k \geqq 0),$$

and

$$\| Q_n \|_3^3 = I_{n,0}(\alpha) = \sum_{k=3}^{3n} b_k \Gamma(k+\alpha+1),$$

where Γ is the gamma function. Using Lemma 3 we obtain

$$J_n(\alpha) \leqq \frac{1}{27} \sum_{k=3}^{3n} b_k [\Gamma(k+\alpha+1) + 3(1-\alpha)\Gamma(k+\alpha) \tag{3.3}$$

$$+ (3\alpha^2 - 9\alpha + 7)\Gamma(k+\alpha-1) + (-\alpha^3 + 6\alpha^2 - 12\alpha + 8)\Gamma(k+\alpha-2)],$$

i.e.,

$$J_n(\alpha) \leqq \sum_{k=3}^{3n} H_\alpha(k) b_k \Gamma(k+\alpha+1),$$

where

$$H_\alpha(k) = \frac{k^3}{27(k+\alpha)(k+\alpha-1)(k+\alpha-2)}.$$

From (3.3) it follows that

$$\| Q_n' \|_3^3 \leqq (\max_{3 \leqq k \leqq 3n} H_\alpha(k)) \| Q_n \|_3^3$$

and so we have

$$C_n(\alpha) \leqq \max_{3 \leqq k \leqq 3n} H_\alpha(k),$$

where

$$\max_{3 \leqq k \leqq 3n} H_\alpha(k) = \begin{cases} H_\alpha(3) & \text{if } -1 < \alpha \leqq \alpha_n, \\ H_\alpha(3n) & \text{if } \alpha \geqq \alpha_n, \end{cases}$$

and α_n is the unique positive root of the equation (3.2).

In order to prove that $C_n(\alpha)$ defined in (3.1) is the best possible, i. e. that $C_n(\alpha) = \max_{3 \leqq k \leqq 3n} H_\alpha(k)$, we consider $\tilde{Q}_n(x) = x^n + \lambda x \, (\lambda \geqq 0)$ and set $\Phi_n(\lambda) = \| \tilde{Q}_n' \|_3^3 / \| \tilde{Q}_n \|_3^3$. Since $\Phi_n(0) = H_\alpha(3n)$ and $\lim_{\lambda \to \infty} \Phi_n(\lambda) = H_\alpha(3)$, we see that $\tilde{Q}_n(x) = x^n$ is an extremal polynomial for $\alpha \geqq \alpha_n$. When $-1 < \alpha \leqq \alpha_n$, there exists a sequence of polynomials, for example, $q_{n,k}(x) = x^n + kx$, $k = 1, 2, \ldots$, for which $\lim_{k \to \infty} \| q_{n,k}' \|_3^3 / \| q_{n,k} \|_3^3 = C_n(\alpha)$. □

Remark 3. The statement of Theorem holds if $\mathscr{P}_n^+$ is the set of all algebraic polynomials $Q \, (\neq 0)$ of degree at most n (not only of exact degree n), with nonnegative coefficients. In this case, for $-1 < \alpha \leqq \alpha_n$, we can see that $\tilde{Q}(x) = \lambda x \, (\lambda > 0)$ is an extremal polynomial.

Some numerical values for α_n are presented in the Table.

n	α_n
2	1.163603698095
3	1.146295022775
4	1.137564507240
5	1.132301569521
6	1.128782608488
7	1.126263976447
8	1.124372209100
9	1.122899174368
10	1.121719698285
11	1.120753978015
12	1.119948732995
13	1.119267031884
14	1.118682468355
15	1.118175661061
16	1.117732063211
17	1.117340543614
18	1.116992439776
19	1.116680909466
20	1.116400476454

In the limit case $n \to \infty$, the equation (3.2) reduces to

$$\alpha^3 + 6\alpha^2 + 11\alpha - 21 = 0,$$

wherefrom we find

$$\alpha_\infty = \lim_{n \to \infty} \alpha_n = \left(\frac{243 + (59037)^{1/2}}{18}\right)^{1/3} + \left(\frac{243 - (59037)^{1/2}}{18}\right)^{1/3} - 2,$$

i.e., $\alpha_\infty \cong 1.111062$.

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