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Overconvergence of Some Complex Interpolants

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1. Introduction

Let $D_\rho=\{z\in C;\ |z|<\rho\}$, $\Gamma_\rho=\{z\in C;\ |z|=\rho\}$. Let A_ρ denote the set of all functions $f(z)$ which are analytic in D_ρ but not on Γ_ρ. Let α, $\beta\in D_\rho$ and for any positive integer m and n $(m>n)$ let $L_{n-1}(z, \alpha, f)$ and $L_{m-1}(z, \beta, f)$ denote the Lagrange interpolants of f in the zeros of $z^n-\alpha^n$ and $z^n-\beta^n$ respectively. Recently L. Yuanren [1] proved

Theorem A. *If* $m=m_n=rn+q$, $q=q_n=sn+0(1)$, $0\leqq s<1$, $q\geqq 0$ *then for each* $f\in A_\rho$ *and for each* α, $\beta\in D_\rho$, *we have*

$$\limsup_{n\to\infty} |\Delta_{n,m}^{\alpha,\beta}(z,f)|=0 \quad \text{for} \quad |z|<\tau, \text{ where}$$

$$\Delta_{n,m}^{\alpha,\beta}(z,f)=L_{n-1}(z,\alpha,f)-L_{n-1}(z,\alpha,L_{m-1}(z,\beta,f)) \text{ and}$$

$$\tau=\rho/\max\{|\alpha/\rho|^r,\ \ |\beta/\rho|^{r+s}\}.$$

More precisely for any R *with* $\rho<R<\infty$, *we have*

$$\limsup_{n\to\infty} \{\max_{z\in D_R} |\Delta_{n,m}^{\alpha,\beta}(z,f)|^{1/n}\}\leqq R/\tau.$$

When $\alpha=1$, $\beta=0$ and $m=rn$, the above result yields a result of A. S. Cavaretta, A. Sharma and R. S. Varga [2], which itself is a generalization of a theorem of J. L. Walsh [3].

M. R. Akhlaghi, A. Jakimovski and A. Sharma [9] obtained analogues of Theorem A to mixed Lagrange interpolation and e_2-approximation and the following more precise theorem for the differences $\Delta_{n,m}^{\alpha,\beta}$:

Theorem B [9]. *In notation of Theorem A, if* $|\alpha/\rho|^r\neq|\beta/\rho|^{r+s}$ *and for* $s\neq 0$ *if* $|\alpha/\rho|^{r+1}\neq|\beta/\rho|^{r+s}$, *then*

$$\limsup_{n\to\infty} \{\max_{|z|=R} |\Delta_{n,m}^{\alpha,\beta}(z,f)|^{1/n}\} = K_\rho(R), \quad R>0, \textit{ where}$$

$$K_\rho(R) = \begin{cases} (R/\rho)\max\{|\sigma/\rho|^r,\ |\beta/\rho|^{r+s}\} & \text{for } |R| \geqq \rho \\ \max\{|\alpha/\rho|^{r+1},\ |\alpha/\rho|^r(R/\rho)^s,\ |\beta/\rho|^{r+s}\} & \text{for } 0<|R|<\rho \end{cases}.$$

In a special case $\alpha=1$, $\beta=0$ and $m_n=rn$, Theorem B reduces to V. Totik's theorem [4]. We shall obtain Theorem B as a corollary of Theorem 2 and Theorem 4, where the more general case of mixed Lagrange interpolation and e_2-approximation are examined.

In this paper we shall be interesting also in the points of overconvergence, i.e. the points for which is valid the inequality

$$\limsup_{n\to\infty} |\Delta_{n,m}^{\alpha,\beta}(z,f)|^{1/n} < K_\rho(|z|). \tag{1.1}$$

If there is some function $f\in A_\rho$, such that (1.1) to be true for each $z\in Z$, we shall say that Z is $(\{\Delta_{n,m}^{\alpha,\beta}\},\rho)$—distinguished set. We due the first results in this field to V. Totik ([4], Theorem 3.4). With the above notations, we may state these results as follows:

Theorem C. *The set Z is $(\{\Delta_{n,rn}^{1,0}\},\rho)$—distinguished set if and only if*

$$|Z| < \begin{cases} r & \text{for} \quad Z\in D_\rho \\ r+1 & \text{for} \quad Z\in \mathbf{C}\setminus\bar{D}_\rho. \end{cases}$$

As usual, $|Z|$ denotes the number of points in Z. (More precise result in this special case is obtained in [6].)

It is clear that the number of points in some $(\{\Delta_{n,m}^{\alpha,\beta}\},\rho)$—distinguished set depends on the behavior of the sequence $\{m_n\}_{n=0}^{\infty}$, let us denote

$$\delta(\{m_n\}) = \limsup_{n\to\infty} (m_{n+1}^* - m_n^*), \tag{1.2}$$

where $\{m_n^*\}_{n=0}^{\infty}$ is the nondecreasing rearrangement of $\{m_n\}_{n=0}^{\infty}$. The following theorem is true:

Theorem 1. *Let $m=m_n=rn+q$, $q=q_n=sn+0(1)$, $0\leqq s<1$, $q_n\geqq 0$ and $\alpha,\beta\in D_\rho$, and let $\rho_1=|\beta|\,|\beta/\alpha|^{r/s}$, $\rho_2=\rho|\alpha/\rho|^{1/s}$. Then the set $Z\subset\Omega$ is an $(\{\Delta_{n,m}^{\alpha,\beta}\},\rho)$—distinguished set if and only if*

$$\text{(a)} \qquad |Z| < \begin{cases} \delta(\{m_n\}) & \text{for} \quad \Omega=D_\rho \\ \delta(\{m_n+n\}) & \text{for} \quad \Omega=\mathbf{C}\setminus\bar{D}_\rho \end{cases}$$

in the case $|\beta/\rho|^{r+s} > |\alpha/\rho|^r$;

(b) $$|Z|<\begin{cases}\delta(\{m_n\}) & \text{for} \quad \Omega=\begin{cases} D_\rho & \text{for } q_n\equiv 0 \\ D_\rho\backslash\Gamma_{\rho_1} & \textit{otherwise}\end{cases} \\ r+1 & \text{for} \quad \Omega=\mathsf{C}\backslash\bar{D}_\rho\end{cases}$$

in the cases $|\alpha/\rho|^{r+1}<|\beta/\rho|^{r+s}<|\alpha/\rho|^r$ $s\neq 0$ *and* $|\beta/\rho|^r<|\alpha/\rho|^r$ $s=0$;

(c) $$|Z|<\begin{cases}\delta(\{m_n\}) & \text{for} \quad \Omega=D_\rho\backslash\bar{D}_{\rho_2} \\ r+1 & \text{for} \quad \Omega=D_{\rho_2}\bigcup\{\mathsf{C}\backslash\bar{D}_\rho\}\end{cases}$$

in the case $|\beta/\rho|^{r+s}<|\alpha/\rho|^{r+1}$ $s\neq 0$.

Since $\delta(\{rn\})=r$ for $r>0$, Theorem 1 yields Theorem C.

We shall obtain Theorem 1 as a corollary of Theorem 3 and Theorem 5, where the more general case of mixed Lagrange interpolation and e_2-approximation are considered.

Remark 1. Let us note, that Theorem 1 examined the $(\{\Delta^{\alpha,\beta}_{n,m}\},\rho)$—distinguished set only in the cases $|\alpha/\rho|^r\neq|\beta/\rho|^{r+s}$ and $s\neq 0$ $|\alpha/\rho|^{r+1}\neq|\beta/\rho|^{r+s}$. It would be interesting to investigate this problem when $|\alpha/\rho|^r=|\beta/\rho|^{r+s}$ or $|\alpha/\rho|^{r+1}=|\beta/\rho|^{r+s}$ $s\neq 0$. Also, we don't know whether there exists $(\{\Delta^{\alpha,\beta}_{n,m}\},\rho)$—distinguished set on the circle Γ_ρ and on the circles Γ_{ρ_1} and Γ_{ρ_2} in the cases (b) and (c) respectively.

In section 2 some auxiliary lemmas are proved. The main results about mixed Lagrange interpolation and e_2-approximation are stated and proved respectively in section 3—about $\Delta^{\alpha,\beta}_{n,m,k}(z,f)=L_{n-1}(z,\alpha,f)-L_{n-1}(z,\alpha,P_{k-1\,(m)}(z,\beta,f))$ and in section 4—about $\bar{\Delta}^{\alpha,\beta}_{n,m,k}(z,f)=P_{n-1\,(m)}(z,\alpha,f)-P_{n-1\,(m)}(z,\alpha,L_{k-1}(z,\beta,f))$. ($P_{n-1\,(m)}(z,\alpha,f)$ is the polynomial of degree $n-1$ of best e_2-approximation to $f(z)$ on the zeros of $z^m-\alpha^m$).

For more detailed information about several investigations on Walsh's theorem we refer to [5].

2. Preliminaries

We shall start with the following simple lemma:

Lemma 2.1. *Let* $\{\lambda_n\}_{n=0}^\infty$ *be the sequence of integers with* $\lambda_n=\lambda n+0(1)$, $\lambda>0$, and let $\{\lambda^*_n\}_{n=0}^\infty$ *be the nondecreasing rearrangement of* $\{\lambda_n\}_{n=0}^\infty$. *Then*

(2.1) *For any* $k\in\mathsf{N}$ *there is* $n\in\mathsf{N}$ *such that* $\lambda^*_k=\lambda_n$.

(2.2) *For any* $n\in\mathsf{N}$ *there is* $k\in\mathsf{N}$ *such that* $\lambda^*_k=\lambda_n$.

(2.3) *There exists* $C>0$ *such that if* $\lambda^*_k=\lambda_n$, *then* $|k-n|\leqq C$.

(2.4) $\lambda^*_{n+1}-\lambda^*_n\leqq\delta(\{\lambda_n\})$ *for each* n *large.*

(2.5) $\lambda^*_{n+1}-\lambda^*_n=\delta(\{\lambda_n\})$ *for infinitely many values of* n.

(*The number* $\delta(\{\lambda_n\})$ *is defined with* (1.2).)

P r o o f. The statements (2.1), (2.2), (2.4) and (2.5) are obvious since $\{\lambda_n\}_{n=0}^{\infty}$ is the sequence of integers. We shall prove (2.3). Let $A \geqq 0$ be such that $|\lambda_n - \lambda n| \leqq A$ for each $n \in \mathrm{N}$. Then

$$\lambda_{m_2} \geqq \lambda m_2 - A = \lambda m_1 + A + \lambda(m_2 - m_1) - 2A > \lambda_{m_1} \quad \text{for } m_2 - m_1 > 2A/\lambda. \tag{2.6}$$

Let n and k be such that $\lambda_n = \lambda_k^*$. From (2.6) we have $\lambda_m > \lambda_n = \lambda_k^*$ for $m > n + 2A/\lambda$ and therefore

$$\{\lambda_0^*, \lambda_1^*, \ldots, \lambda_k^*\} \subset \{\lambda_0, \lambda_1, \ldots, \lambda_{n+s_0}\}, \quad (s_0 = [2A/\lambda]), \text{ i.e.}$$

$$k \leqq n + [2A/\lambda]. \tag{2.7}$$

Similar, from (2.6) we have $\lambda_m < \lambda_n = \lambda_k^*$ for $0 \leqq m < n - 2A/\lambda$, and therefore

$$\{\lambda_0, \lambda_1, \ldots, \lambda_{n-s_0-1}\} \subset \{\lambda_0^*, \lambda_1^*, \ldots, \lambda_{k-1}^*\}, \text{ i.e.}$$

$$n - [2A/\lambda] - 1 \leqq k - 1. \tag{2.8}$$

From (2.7) and (2.8) we get (2.3) with $C = [2A/\lambda]$.

Let $f \in A_\rho$. Then $f(z)$ may be expressed in D_ρ as

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad \text{where} \tag{2.9}$$

$$\limsup_{n \to \infty} |a_n|^{1/n} = \rho^{-1}. \tag{2.10}$$

Everywhere further we shall assume that $f \in A_\rho$ and $f(z)$ has the representation (2.9) in D_ρ; ε will be a sufficiently small positive number and σ will be a number with $0 < \sigma < 1$. These numbers may be different in each case.

Let us note, that from (2.10) it follows

$$a_n = 0((\rho - \varepsilon)^{-n}) \quad \text{when } n \to \infty \text{ for each } \varepsilon > 0. \tag{2.11}$$

Let the sequence of operators $\{\Delta_n\}_{n=0}^{\infty}$ be such that $\sup_{f \in A_\rho} \limsup_{n \to \infty} |\Delta_n(z, f)|^{1/n} = K_\rho(z, \{\Delta_n\})$. We shall say that Z is $(\{\Delta_n\}, \rho)$—distinguished set (or $(\{\Delta_n\}, \rho)$—set) if there exists some function $f(z) \in A_\rho$ with $\limsup_{n \to \infty} |\Delta_n(z, f)|^{1/n} < K_\rho(z, \{\Delta_n\})$ for any $z \in Z$. If there is no risk of mistake, we shall write $K_\rho(z)$ instead of $K_\rho(z, \{\Delta_n\})$.

Lemma 2.2. *Let* $f \in A_\rho$ *has the representation* (2.9) *in* D_ρ *and let* $S_n(z, f) = t^{\varphi_0(n)} \sum_{\nu=\varphi_1(n)}^{\varphi_2(n)} z^\nu a_{\nu+\varphi_0(n)}$, *where* $t \neq 0$, $\varphi_i(n) = \varphi_i n + 0(1)$, $\varphi_i(n) \in \mathrm{N}$, $\varphi_i \geqq 0$,

$i=0, 1, 2$, $\varphi_0+\varphi_1>0$ and $\varphi_2>\varphi_1$. The set $Z\subset \mathbf{C}\setminus\bar{D}_\rho$ is $(\{S_n\},\rho)$—distinguished set iff $|Z|<\delta(\{\varphi_o(n)+\varphi_2(n)\})$, where $|Z|$ denotes the number of points in Z and $\delta(\{\varphi_0(n)+\varphi_2(n)\})$ is the number, defined with (1.2).

Proof. Using (2.11) it is not difficult to see, that

$$(2.12)\quad K_\rho(z,\{S_n\})=\begin{cases}|t/\rho|^{\varphi_0}|z/\rho|^{\varphi_2} & \text{for} \quad |z|\geqq\rho\\ |t/\rho|^{\varphi_0}|z/\rho|^{\varphi_1} & \text{for} \quad 0<|z|<\rho\\ |t/\rho|^{\varphi_0} & \text{for} \quad z=0 \text{ and } \varphi_1(n)\equiv 0.\end{cases}$$

Let us denote $\lambda_n=\varphi_0(n)+\varphi_1(n)$, $\mu_n=\varphi_0(n)+\varphi_2(n)$, $\lambda=\varphi_0+\varphi_1$, $\mu=\varphi_0+\varphi_2$, $N=\delta(\{\varphi_0(n)+\varphi_2(n)\})$ and let $\{\mu_n^*\}_{n=0}^\infty$ be the nondecreasing rearrangement of $\{\mu_n\}_{n=0}^\infty$.

(a) Necessity. Let us suppose that there exists the set $Z\subset \mathbf{C}\setminus\bar{D}_\rho$ which is $(\{S_n\},\rho)$—set but $|Z|\geqq N$. Then there exist points $z_1,\dots,z_N$ ($z_i\neq z_j$ for $i\neq j$ and $|z_j|>\rho$ $1\leqq j\leqq N$) and function $f(z)\in A_\rho$ such that $S_n(z_j,f)=0((\sigma K_\rho(z_j))^n)$, $1\leqq j\leqq N$ $(0<\sigma<1)$, which gives with (2.12)

$$(2.13)\qquad \sum_{\nu=\lambda_n}^{\mu_n} z_j^\nu a_\nu=0((\sigma|z_j|/\rho)^{\mu_n}),\quad 1\leqq j\leqq N.$$

Let k be an arbitrary nonnegative integer. Because of (2.1), we may select n_0 and n_1 with $\mu_k^*=\mu_{n_0}$ and $\mu_{k+1}^*=\mu_{n_1}$. From (2.13) (with $n=n_0$ and $n=n_1$) and (2.3) we get

$$(2.14)\qquad \sum_{\nu=\lambda_{n_0}}^{\mu_k^*} z_j^\nu a_\nu-\sum_{\nu=\lambda_{n_1}}^{\mu_{k+1}^*} z_j^\nu a_\nu=-\sum_{\nu=0}^{\lambda_{n_0}-1} z_j^\nu a_\nu+\sum_{\nu=0}^{\lambda_{n_1}-1} z_j^\nu a_\nu$$
$$-\sum_{\nu=\mu_k^*+1}^{\mu_{k+1}^*} z_j^\nu a_\nu=0(|\sigma z_j/\rho|^{\mu k}).$$

Let $\varepsilon>0$ be too small that $|z_j/(\rho-\varepsilon)|^\lambda<|z_j/\rho|^\mu$, $1\leqq j\leqq N$. Then (2.3) and (2.11) give

$$\sum_{\nu=0}^{\lambda_{n_i}-1} z_j^\nu a_\nu=0\left(\sum_{\nu=0}^{\lambda_{n_i}-1}|z_j/(\rho-\varepsilon)|^\nu\right)=0(|z_j/(\rho-\varepsilon)|^{\lambda k})=0(|\sigma z_j/\rho|^{\mu k}),\quad 1\leqq j\leqq N,\ i=0,1.$$

Using the last expression in (2.14) we obtain

$$\sum_{\nu=\mu_k^*+1}^{\mu_{k+1}^*} z_j^\nu a_\nu=0(|\sigma z_j/\rho|^{\mu k}),\ \text{i.e}$$

(2.15) $$\sum_{\nu=0}^{\mu_{k+1}^*-\mu_k^*-1} z_j^{\nu} a_{\nu+\mu_k^*+1} = 0((\sigma/\rho)^{\mu k}), \quad 1 \leqq j \leqq N.$$

We may regard (2.15) as the system of equations for the unknowns a_s, $\mu_k^* < s \leqq \mu_{k+1}^*$. Because of (2.4), the solution of this system must satisfy $a_s = 0((\sigma/\rho)^{\mu k}) = 0((\sigma/\rho)^s)$ $(\mu_k^* < s \leqq \mu_{k+1}^*, \; k=0, 1, 2, \ldots)$, i.e. $\limsup_{n\to\infty} |a_n|^{1/n} \leqq \sigma/\rho < \rho^{-1}$, which contradicts the choice of $f(z)$.

(b) Sufficiency. We shall prove that any set $Z = \{z_1, z_2, \ldots, z_{N-1}\} \subset \mathbf{C} \setminus \bar{D}_\rho$ is $(\{S_n\}, \rho)$—set. Indeed, let $d_0, d_1, \ldots d_{N-1}$ be such that

$$\sum_{\nu=0}^{N-1} d_\nu z_j^\nu = 0, \quad 1 \leqq j \leqq N-1, \quad d_{N-1} = 1.$$

We define the sequence $\{a_n\}_{n=0}^{\infty}$ as follows

$$a_{\mu_k^*+\nu+1} = \begin{cases} d_\nu \rho^{-\mu_k^*} & \text{if} \quad \mu_{k+1}^* - \mu_k^* = N \\ 0 & \text{if} \quad \mu_{k+1}^* - \mu_k^* \neq N \end{cases},$$

$$0 \leqq \nu \leqq \mu_{k+1}^* - \mu_k^* - 1, \quad k = 0, 1, \ldots$$

Because of (2.5) we have $\limsup_{n\to\infty} |a_n|^{1/n} = \rho^{-1}$ and therefore $f_0(z) := \sum_{n=0}^{\infty} a_n z^n \in A_\rho$. Moreover it is clear that

(2.16) $$\sum_{\nu=0}^{\mu_{k+1}^*-\mu_k^*-1} z_j^\nu a_{\nu+\mu_k^*+1} = 0, \quad 1 \leqq j \leqq N-1, \quad k = 0, 1, \ldots$$

Let n be an arbitrary positive integer. We may choose k such that $\mu_n = \mu_k^*$ (see (2.2)). And let k_1 be such that $\mu_{k_1}^* + 1 \leqq \lambda_n$. (If n is sufficiently large, k_1 exists.) Then if $\varepsilon > 0$ is too small that $|t/(\rho-\varepsilon)|^{\varphi_0} |z_j/(\rho-\varepsilon)|^{\varphi_1} < |t/\rho|^{\varphi_0} |z_j/\rho|^{\varphi_2}$ for $1 \leqq j \leqq N-1$, with (2.12) and (2.16) we get

$$S_n(z_j, f_0) = t^{\varphi_0(n)} \left\{ \sum_{\nu=\mu_{k_1}^*+1}^{\mu_k^*} a_\nu z_j^{\nu-\varphi_0(n)} - \sum_{\nu=\mu_{k_1}^*+1}^{\lambda_n-1} a_\nu z_j^{\nu-\varphi_0(n)} \right\}$$

$$= t^{\varphi_0(n)} \sum_{s=k_1}^{k-1} z_j^{\mu_s^*+1-\varphi_0(n)} \sum_{\nu=0}^{\mu_{s+1}^*-\mu_s^*-1} z_j^\nu a_{\nu+\mu_s^*+1}$$

$$+ 0(|t/z_j|^{\varphi_0(n)} \sum_{\nu=0}^{\lambda_n-1} |z_j/(\rho-\varepsilon)|^\nu) = 0((|t/(\rho-\varepsilon)|^{\varphi_0} |z_j/(\rho-\varepsilon)|^{\varphi_1})^n)$$

$$= 0((\sigma K_\rho(z_j))^n), \quad 1 \leqq j \leqq N-1, \quad 0 < \sigma < 1$$

and therefore $\{z_1, z_2, \ldots, z_{N-1}\}$ is $(\{S_n\}, \rho)$—set.

Lemma 2.3. *In the hypothesis of Lemma 2.2 the set* $Z \in \Omega$ *is* $(\{S_n\}, \rho)$*—distinguished set iff* $|Z| < \delta(\{\varphi_0(n) + \varphi_1(n)\})$ *where*

$$\Omega = \begin{cases} D_\rho & \text{if } \varphi_1(n) = 0 \\ D_\rho \setminus \{0\} & \textit{otherwise.} \end{cases}$$

Proof. As in the proof of Lemma 2.2, we denote $\lambda_n = \varphi_0(n) + \varphi_1(n)$, $\mu_n = \varphi_0(n) + \varphi_2(n)$, $\lambda = \varphi_0 + \varphi_1$, $\mu = \varphi_0 + \varphi_2$. Let $M = \delta(\{\varphi_0(n) + \varphi_1(n)\})$ and let $\{\lambda_n^*\}_{n=0}^{\infty}$ be the nondecreasing rearrangement of $\{\lambda_n\}_{n=0}^{\infty}$.

(a) Necessity. Let us suppose that there exists the set $Z \subset \Omega$ which is $(\{S_n\}, \rho)$—set but $|Z| \geqq M$. Then there exist points $z_1, \ldots, z_M \in \Omega$ ($z_i \neq z_j$ for $i \neq j$) and function $f(z) \in A_\rho$ with

$$S_n(z_j, f) = 0((\sigma K_\rho(z_j))^n), \quad 1 \leqq j \leqq M, \quad (0 < \sigma < 1) \tag{2.17}$$

which gives with (2.12)

$$\sum_{\nu=\lambda_n}^{\mu_n} z_j^\nu a_\nu = 0(|\sigma z_j/\rho|^{\lambda n}), \quad 1 \leqq j \leqq M, \quad z_j \neq 0. \tag{2.18}$$

Let k be an arbitrary nonnegative integer. Because of (2.1), we may select n_0 and n_1 with $\lambda_k^* = \lambda_{n_0}$ and $\lambda_{k+1}^* = \lambda_{n_1}$. From (2.3) and (2.18) we get

$$\sum_{\nu=\lambda_k^*}^{\mu_{n_0}} z_j^\nu a_\nu - \sum_{\nu=\lambda_{k+1}^*}^{\mu_{n_1}} z_j^\nu a_\nu = \sum_{\nu=\lambda_k^*}^{\lambda_{k+1}^*-1} z_j^\nu a_\nu - \sum_{\nu=\mu_{n_0}+1}^{\infty} z_j^\nu a_\nu \tag{2.19}$$

$$+ \sum_{\nu=\mu_{n_1}+1}^{\infty} z_j^\nu a_\nu = 0(|\sigma z_j/\rho|^{\lambda k}).$$

Let $\varepsilon > 0$ be too small that $|z_j/(\rho-\varepsilon)|^\mu < |z_j/\rho|^\lambda$, $1 \leqq j \leqq M$, $z_j \neq 0$. Then (2.3) and (2.11) give

$$\sum_{\nu=\mu_{n_i}+1}^{\infty} z_j^\nu a_\nu = 0\left(\sum_{\nu=\mu_{n_i}+1}^{\infty} |z_j/(\rho-\varepsilon)|^\nu\right) = 0(|z_j/(\rho-\varepsilon)|^{\mu k})$$

$$= 0(|\sigma z_j/\rho|^{\lambda k}), \quad 1 \leqq j \leqq M, \quad i = 0, 1, \quad z_j \neq 0.$$

Using the last expression in (2.19) we obtain

$$\sum_{\nu=\lambda_k^*}^{\lambda_{k+1}^*-1} z_j^\nu a_\nu = 0(|\sigma z_j/\rho|^{\lambda k}), \text{ i.e.}$$

$$\sum_{\nu=0}^{\lambda_{k+1}^*-\lambda_k^*-1} z_j^\nu a_{\nu+\lambda_k^*} = 0((\sigma/\rho)^{\lambda k}), \quad 1 \leqq j \leqq M, \quad z_j \neq 0. \tag{2.20}$$

But if $z_{j_0}=0$ (then $\varphi_1(n)\equiv 0$) from (2.12) and (2.17) it is clear that (2.20) remains true for $j=j_0$, and as in the proof of Lemma 2.2 we obtain that $\limsup_{n\to\infty}|a_n|^{1/n}<\rho^{-1}$, which contradicts to the choice of $f(z)$.

(b) Sufficiency. We shall prove that any set $Z=\{z_1,z_2,\dots,z_{M-1}\}\subset\Omega$ is $(\{S_n\},\rho)$ — set. Indeed, let $d_0,d_1,\dots,d_{M-1}$ be such that

$$\sum_{\nu=0}^{M-1} d_\nu z_j^\nu=0,\quad 1\leqq j\leqq M-1,\quad d_{M-1}=1.$$

We define the sequence $\{a_n\}_{n=0}^\infty$ as follows

$$a_{\lambda_k^*+\nu}=\begin{cases} d_\nu\rho^{-\lambda_k^*} & \text{if}\quad \lambda_{k+1}^*-\lambda_k^*=M\\ 0 & \text{if}\quad \lambda_{k+1}^*-\lambda_k^*\neq M\end{cases}$$

$$0\leqq\nu\leqq\lambda_{k+1}^*-\lambda_k^*-1,\quad k=0,\ 1,\dots$$

Because of (2.5) we have $\limsup_{n\to\infty}|a_n|^{1/n}=\rho^{-1}$ and therefore $f_0(z):=\sum_{n=0}^\infty a_n z^n\in A_\rho$. Moreover it is clear that

$$\sum_{\nu=0}^{\lambda_{k+1}^*-\lambda_k^*-1} z_j^\nu a_{\nu+\lambda_k^*}=0,\quad 1\leqq j\leqq M-1,\quad k=0,\ 1,\dots \tag{2.21}$$

Let n be arbitrary positive integer. We may choose k such that $\lambda_n=\lambda_k^*$ (see (2.2)). And let k_1 be such that $\lambda_{k_1}^*>\mu_n$. Then if $\varepsilon>0$ is too small that $|t/(\rho-\varepsilon)|^{\varphi_0}|z_j/(\rho-\varepsilon)|^{\varphi_2}<|t/\rho|^{\varphi_0}|z_j/\rho|^{\varphi_1}$ for $1\leqq j\leqq M-1$, $z_j\neq 0$, with (2.11), (2.12) and (2.21) we get

$$S_n(z_j,f_o)=t^{\varphi_0(n)}\sum_{\nu=\lambda_n}^{\mu_n} a_\nu z_j^{\nu-\varphi_0(n)}$$

$$=t^{\varphi_0(n)}\left\{\sum_{\nu=\lambda_k^*}^{\lambda_{k_1}^*-1} a_\nu z_j^{\nu-\varphi_0(n)}-\sum_{\nu=\mu_n+1}^{\lambda_{k_1}^*-1} a_\nu z_j^{\nu-\varphi_0(n)}\right\}$$

$$=t^{\varphi_0(n)}\sum_{s=k}^{k_1-1} z_j^{\lambda_s^*-\varphi_0(n)}\sum_{\nu=0}^{\lambda_{s+1}^*-\lambda_s^*-1} z_j^\nu a_{\nu+\lambda_s^*}$$

$$+0\left(|t|^{\varphi_0(n)}\sum_{\nu=\mu_n+1}^{\infty}|z_j|^{\nu-\varphi_0(n)}/(\rho-\varepsilon)^\nu\right)$$

$$=0((|t/(\rho-\varepsilon)|^{\varphi_0}|z_j/(\rho-\varepsilon)|^{\varphi_2})^n)=0((\sigma K_\rho(z_j))^n),\quad 1\leqq j\leqq M-1$$

and therefore $\{z_1,z_2,\dots,z_{M-1}\}$ is $(\{S_n\},\rho)$ — set.

3. Mixed Lagrange interpolation and l_2—approximation

Let $f\in A_\rho$ and let k, m be integers ($k\leqq m$). We shall be interested in the unique polynomial of degree $\leqq k-1$ for which

$$(3.1)\qquad \min_{P\in\pi_{k-1}} \sum_{j=0}^{m-1} |f(\beta\omega^j)-P(\beta\omega^j)|^2, \quad (\omega=\exp\{2\pi i/m\},\quad \beta\in D_\rho)$$

is attained. Let $P_{k-1\,(m)}(z,\beta,f)$ denote the polynomial which minimizes (3.1). We shall consider the difference

$$(3.2)\qquad \Delta^{\alpha,\beta}_{n,m,k}(z,f)=L_{n-1}(z,\alpha,f)-L_{n-1}(z,\alpha,P_{k-1\,(m)}(z,\beta,f))$$

$(\alpha,\beta\in D_\rho)$. It is proved [9] that if $m=rn+s_n$, $s_n=sn+0(1)$, $0\leqq s<1$, $k=pn+q_n$, $0\leqq q_n<n$, $q_n=qn+0(1)$, $0\leqq q<1$, and $|z|<\tau$, then $\Delta^{\alpha,\beta}_{n,m,k}(z,f)$ tends to zero as $n\to\infty$, where $\tau=\rho/\max\{|\alpha/\rho|^p,\ |\beta/\rho|^{r+s}\}$. In this section the following more quantitative result is obtained:

Theorem 2. *Let* $f\in A_\rho$, *and let* $\Delta^{\alpha,\beta}_{n,m,k}(z,f)$ *be defined with* (3.2). *If* $m=m_n=rn+s_n$, $s_n=sn+0(1)$, $0\leqq s<1$, $k=k_n=pn+q_n$, $0\leqq q_n<n$, $q_n=qn+0(1)$, $0\leqq q<1$ and $|\alpha/\rho|^p\neq|\beta/\rho|^{r+s}$, and $|\alpha/\rho|^{p+1}\neq|\beta/\rho|^{r+s}$ *for* $q\neq 0$, *then*

$$f_\rho(R):=\limsup_{n\to\infty}\{\max_{|z|=R}|\Delta^{\alpha,\beta}_{n,m,k}(z,f)|^{1/n}\}=K_\rho(R)\ \text{for}\ R>0,\ \textit{where}$$

$$K_\rho(z)=\begin{cases}\max\{|\alpha/\rho|^{p+1},\ |\alpha/\rho|^p|z/\rho|^q,\ |\beta/\rho|^{r+s}\} & \text{for}\quad 0<|z|<\rho\\ |z/\rho|\max\{|\alpha/\rho|^p,\ |\beta/\rho|^{r+s}\} & \text{for}\quad |z|\geqq\rho.\end{cases}$$

Remark 2. Let us note that $P_{m-1\,(m)}(z,\beta,f)$ is exactly the Lagrange interpolant to $f(z)$ on the zeros of $z^m-\beta^m$, i.e. $P_{m-1\,(m)}(z,\beta,f)\equiv L_{m-1}(z,\beta,f)$ and respectively $\Delta^{\alpha,\beta}_{n,m,m}(z,f)\equiv\Delta^{\alpha,\beta}_{n,m}(z,f)$. So, in a special case $m_n\equiv k_n$ Theorem 2 yields Theorem C.

Now to prove Theorem 2, we shall need the following representation for $\Delta^{\alpha,\beta}_{n,m,k}(z,f)$.

Lemma 3.1. *Let* $f\in A_\rho$, *and let* $\Delta^{\alpha,\beta}_{n,m,k}(z,f)$ *be defined with* (3.2). *If* $m=m_n=rn+s_n$, $s_n=sn+0(1)$, $0\leqq s<1$, $k=k_n=pn+q_n$, $0\leqq q_n<n$, $q_n=qn+0(1)$, $0\leqq q<1$ *then we have*

$$\Delta^{\alpha,\beta}_{n,m,k}(z,f)=\alpha^{(p+1)n}\sum_{\nu=0}^{q_n-1} z^\nu a_{\nu+(p+1)n}+\alpha^{pn}\sum_{\nu=q_n}^{n-1} z^\nu a_{\nu+pn}$$

$$-\beta^m\sum_{\nu=0}^{n-1} z^\nu a_{\nu+m}+0((\sigma K_\rho(z))^n)$$

where $0<\sigma<1$.

Proof. Let us denote $\Delta^{\alpha,\beta}_{n,m,k}(z,f)=\sum_{\nu=0}^{n-1} A_{\nu,n} z^{\nu}$. We have (see (4.11) in [9])

$$A_{\nu,n}=\begin{cases}\sum_{j=0}^{\infty} \alpha^{jn} a_{\nu+jn}-\sum_{j=0}^{\infty}\sum_{i=0}^{p} \beta^{jm}\alpha^{in} a_{\nu+jm+in} & \text{for} \quad 0\leqq\nu\leqq q_n-1.\\ \sum_{j=0}^{\infty} \alpha^{jn} a_{\nu+jn}-\sum_{j=0}^{\infty}\sum_{i=0}^{p-1} \beta^{jm}\alpha^{in} a_{\nu+jm+in} & \text{for} \quad q_n\leqq\nu\leqq n-1.\end{cases}$$

Let $0\leqq\nu\leqq q_n-1$ and let $\varepsilon>0$ be too small that

$$(\rho/(\rho-\varepsilon))^q \max\{|\alpha/(\rho-\varepsilon)|^{p+2},\ |\alpha/(\rho-\varepsilon)|\ |\beta/(\rho-\varepsilon)|^{r+s},\ |\beta/(\rho-\varepsilon)|^{2(r+s)}\}$$

$<\max\{|\alpha/\rho|^{p+1},\ |\beta/\rho|^{r+s}\}$. Then the above expression for $A_{\nu,n}$ $(0\leqq\nu<q_n)$ and (2.11) yield

$$\begin{aligned}(3.4)\qquad A_{\nu,n}&=\sum_{j=0}^{\infty}\alpha^{jn}a_{\nu+jn}-\sum_{i=0}^{p}\alpha^{in}a_{\nu+in}-\beta^m\sum_{i=0}^{p}\alpha^{in}a_{\nu+in+m}\\ &-\sum_{j=2}^{\infty}\sum_{i=0}^{p}\beta^{jm}\alpha^{in}a_{\nu+jm+in}=\alpha^{(p+1)n}a_{\nu+(p+1)n}\\ &+0\left(\sum_{j=p+2}^{\infty}|\alpha/(\rho-\varepsilon)|^{jn}(\rho-\varepsilon)^{-\nu}\right)-\beta^m a_{\nu+m}\\ &+0\left(|\beta/(\rho-\varepsilon)|^m\sum_{j=1}^{\infty}|\alpha/(\rho-\varepsilon)|^{jn}(\rho-\varepsilon)^{-\nu}\right)\\ &+0\left(\sum_{j=2}^{\infty}|\beta/(\rho-\varepsilon)|^{jm}\sum_{i=0}^{p}|\alpha/(\rho-\varepsilon)|^{in}(\rho-\varepsilon)^{-\nu}\right)\\ &=\alpha^{(p+1)n}a_{\nu+(p+1)n}-\beta^m a_{\nu+m}+\rho^{-\nu}0((\sigma\Lambda_1)^n),\quad(0\leqq\nu<q_n),\end{aligned}$$

where $0<\sigma<1$, $\Lambda_1=\max\{|\alpha/\rho|^{p+1},\ |\beta/\rho|^{r+s}\}$.

And similar

$$(3.5)\qquad A_{\nu,n}=\alpha^{pn}a_{\nu+pn}-\beta^m a_{\nu+m}+\rho^{-\nu}0((\sigma\Lambda_2)^n),\quad q_n\leqq\nu\leqq n-1,$$

where $\Lambda_2=\max\{|\alpha/\rho|^p,\ |\beta/\rho|^{r+s}\}$. From (3.3)—(3.5) we obtain

$$\begin{aligned}(3.6)\qquad \Delta^{\alpha,\beta}_{n,m,k}(z,f)&=\alpha^{(p+1)}\sum_{\nu=0}^{q_n-1}z^{\nu}a_{\nu+(p+1)n}+\alpha^{pn}\sum_{\nu=q_n}^{n-1}z^{\nu}a_{\nu+pn}\\ &-\beta^m\sum_{\nu=0}^{n-1}z^{\nu}a_{\nu+m}+R_n(z),\end{aligned}$$

where

$$R_n(z)=0\left(\sigma^n\left(\Lambda_1^n\sum_{\nu=0}^{q_n-1}|z/\rho|^\nu+\Lambda_2^n\sum_{\nu=q_n}^{n-1}|z/\rho|^\nu\right)\right)$$

$$=\begin{cases}0((\sigma\max\{\Lambda_1,\ |z/\rho|^q\Lambda_2\})^n) & \text{for} \quad |z|<\rho\\ 0((\sigma\max\{|z/\rho|^q\Lambda_1,\ |z/\rho|\Lambda_2\})^n) & \text{for} \quad |z|\geqq\rho.\end{cases}$$

Replacing the last estimates in (3.6) we obtain the desired reprezentation for $\Delta_{n,m,k}^{\alpha,\beta}(z,f)$.

Proof of Theorem 2.

(i) We shall first prove that

$$(3.7) \qquad f_\rho(R)\leqq K_\rho(R).$$

(a) Let $R<\rho$. Let $\varepsilon>0$ be too small that $\alpha,\beta\in D_{\rho-\varepsilon}$ and $R-\varepsilon<\rho$. Then with Lemma 3.1 and (2.11) we obtain

$$\max_{|z|=R}|\Delta_{n,m,k}^{\alpha,\beta}(z,f)|=0((\sigma K_\rho(R))^n)+0\left(\left|\frac{\alpha}{\rho-\varepsilon}\right|^{(p+1)n}\sum_{\nu=0}^{q_n-1}\left|\frac{R}{\rho-\varepsilon}\right|^\nu\right.$$

$$\left.+\left|\frac{\alpha}{\rho-\varepsilon}\right|^{pn}\sum_{\nu=q_n}^{n-1}\left|\frac{R}{\rho-\varepsilon}\right|^\nu+\left|\frac{\beta}{\rho-\varepsilon}\right|^{m}\sum_{\nu=0}^{n-1}\left|\frac{R}{\rho-\varepsilon}\right|^\nu\right)$$

$$=0\left(\left(\max\left\{\left|\frac{\alpha}{\rho-\varepsilon}\right|^{p+1},\ \left|\frac{\alpha}{\rho-\varepsilon}\right|^{p}\left|\frac{R}{\rho-\varepsilon}\right|^{q},\ \left|\frac{\beta}{\rho-\varepsilon}\right|^{r+s}\right\}\right)^n\right)+0((\sigma K_\rho(R))^n)$$

$$=0((K_{\rho-\varepsilon}(R))^n)+0((\sigma K_\rho(R))^n)$$

and therefore $f_\rho(R)\leqq K_{\rho-\varepsilon}(R)$. From the last inequality, with $\varepsilon\to 0$ we obtain (3.7) for $R<\rho$.

(b) Let now $R\geqq\rho$. As above, we get

$$\max_{|z|=R}|\Delta_{n,m,k}^{\alpha,\beta}(z,f)|=0((\sigma K_\rho(R))^n)$$

$$+0\left(\left(\max\left\{\left|\frac{\alpha}{\rho-\varepsilon}\right|^{p+1}\left|\frac{R}{\rho-\varepsilon}\right|^{q},\ \left|\frac{\alpha}{\rho-\varepsilon}\right|^{p}\frac{R}{\rho-\varepsilon},\ \left|\frac{\beta}{\rho-\varepsilon}\right|^{r+s}\frac{R}{\rho-\varepsilon}\right\}\right)^n\right)$$

$$=0((K_{\rho-\varepsilon}(R))^n)+0((\sigma K_\rho(R))^n)$$

and therefore $f_\rho(R)\leqq K_\rho(R)$ for $R\geqq\rho$ because of the arbitrariness of ε.
(ii) Now we shall prove the inverse inequality

(3.8) $$K_\rho(R) \leqq f_\rho(R).$$

From (3.3) with Coushi's formula we have

$$A_{\nu,n} = \frac{1}{2\pi i} \int_{|z|=R} \frac{\Delta^{\alpha,\beta}_{n,m,k}(z,f)}{z^{\nu+1}} dz$$

and therefore

(3.9) $$R^\nu |A_{\nu,n}| \leqq \max_{|z|=R} |\Delta^{\alpha,\beta}_{n,m,k}(z,f)|, \quad 0 \leqq \nu \leqq n-1, \quad R>0.$$

Since $m_n = rn + sn + 0(1)$, it is clear that there exists an integer $C>0$ such that the sequences $\{pn+\nu\}$ and $\{m_n+\nu\}$, $n-C \leqq \nu \leqq n-1$ assume all positive integer values. But $n-C>q_n$ for n sufficiently large. And because $|\alpha/\rho|^p \neq |\beta/\rho|^{r+s}$, (3.5) yields

$$\limsup_{n\to\infty} \{ \max_{n-C \leqq \nu \leqq n-1} |A_{\nu,n}|\}^{1/n} = \rho^{-1}\, max\, \{|\alpha/\rho|^p, |\beta/\rho|^{r+s}\}$$

which yields with (3.9)

(3.10) $$(R/\rho) \max \{|\alpha/\rho|^p, |\beta/\rho|^{r+s}\} \leqq f_\rho(R).$$

On the other hand, we may choose $C>0$ such that the sequences $\{pn+\nu\}$ and $\{m_n+\nu\}$, $q_n \leqq \nu \leqq q_n + C$ assume all positive integer values. And because $q_n + C < n$ for n sufficiently large, we obtain from (3.5)

$$\limsup_{n\to\infty} \{ \max_{q_n \leqq \nu \leqq q_n + C} |A_{\nu,n}|\}^{1/n} = \rho^{-q} \max \{|\alpha/\rho|^p, |\beta/\rho|^{r+s}\}$$

which yields with (3.9)

(3.11) $$(R/\rho)^q \max \{|\alpha/\rho|^p, |\beta/\rho|^{r+s}\} \leqq f_\rho(R).$$

From (3.10) and (3.11) we get (3.8) in the case $q=0$.

Let now $q>0$. As above we take $C>0$ such that the sequences $\{(p+1)n+\nu\}$ and $\{m_n+\nu\}$, $0<\nu<C$ assume all positive integer values. But if n is sufficiently large, we have $C<q_n$, and since $|\alpha/\rho|^{p+1} \neq |\beta/\rho|^{r+s}$, (3.4) yields

$$\limsup_{n\to\infty} \{ \max_{0 \leqq \nu \leqq C} |A_{\nu,n}|\}^{1/n} = \max \{|\alpha/\rho|^{p+1}, |\beta/\rho|^{r+s}\}$$

which yields with (3.9)

(3.12) $$\max \{|\alpha/\rho|^{p+1}, |\beta/\rho|^{r+s}\} \leqq f_\rho(R).$$

On the other hand, if $C>0$ is such that the sequences $\{(p+1)n+v\}$ and $\{m_n+v\}$, $q_n-C\leqq v\leqq q_n-1$ assume all positive integer values, from (3.4) we obtain

$$\limsup_{n\to\infty}\{\max_{q_n-C\leqq v\leqq q_n-1}|A_{v,n}|\}^{1/n}=\rho^{-q}\max\{|\alpha/\rho|^{p+1},|\beta/\rho|^{r+s}\}$$

which yields with (3.9)

$$(R/\rho)^q\max\{|\alpha/\rho|^{p+1},|\beta/\rho|^{r+s}\}\leqq f_\rho(R). \tag{3.13}$$

From (3.10)—(3.13) it follows (3.8) for $0<q<1$, and this completes the proof.

Now we may investigate the question about overconvergence of the difference $\Delta^{\alpha,\beta}_{n,m,k}(z,f)$.

Theorem 3. *Let $m=m_n=rn+s_n$, $s_n=sn+0(1)$, $0\leqq s<1$, $k=k_n=pn+q_n$, $q_n=qn+0(1)$, $0\leqq q_n<n$, $0\leqq q<1$ and $\alpha,\beta\in D_\rho$, and let $\rho_1=\rho(|\beta/\rho|^{r+s}|\alpha/\rho|^{-p})^{1/q}$, $\rho_2=\rho|\alpha/\rho|^{1/q}$. Then the set $Z\subset\Omega$ is $(\{\Delta^{\alpha,\beta}_{n,m,k}\},\rho)$—distinguished set if and only if*

$$\text{(a)}\qquad |Z|<\begin{cases}\delta(\{m_n\}) & \text{for}\quad \Omega=D_\rho\\ \delta(\{m_n+n\}) & \text{for}\quad \Omega=\mathbf{C}\setminus\bar D_\rho\end{cases}$$

in the case $|\beta/\rho|^{r+s}>|\alpha/\rho|^p$;

$$\text{(b)}\ |Z|<\begin{cases}\delta(\{m_n\}) & \text{for}\quad \Omega=D_{\rho_1}\\ \delta(\{k_n\}) & \text{for}\quad \Omega=\begin{cases}D_\rho & \text{if } q_n\equiv 0\\ \{z\in\mathbf{C};\ \rho_1<|z|<\rho\} & \text{otherwise}\end{cases}\\ p+1 & \text{for}\quad \Omega=\mathbf{C}\setminus\bar D_\rho\end{cases}$$

in the cases $|\alpha/\rho|^{p+1}<|\beta/\rho|^{r+s}<|\alpha/\rho|^p$, $q\neq 0$, and $|\beta/\rho|^{r+s}<|\alpha/\rho|^p$, $q=0$;

$$\text{(c)}\qquad |Z|<\begin{cases}\delta(\{k_n\}) & \text{for}\quad \Omega=D_\rho\setminus\bar D_{\rho_2}\\ p+1 & \text{for}\quad \Omega=D_{\rho_2}\cup\{\mathbf{C}\setminus\bar D_\rho\}\end{cases}$$

in the case $|\beta/\rho|^{r+s}<|\alpha/\rho|^{p+1}$, $q\neq 0$.

Proof. (a) In the case $|\beta/\rho|^{r+s}>|\alpha/\rho|^p$ we have $K_\rho(z)=|\beta/\rho|^{r+s}\max\{1,|z/\rho|\}$. Therefore, if $\varepsilon>0$ is too small that

$$\left|\frac{\alpha}{\rho-\varepsilon}\right|^p\max\left\{1,\left|\frac{z}{\rho-\varepsilon}\right|\right\}<K_\rho(z)\ \text{and}\ \left|\frac{\alpha}{\rho-\varepsilon}\right|^{p+1}\max\left\{1,\left|\frac{z}{\rho-\varepsilon}\right|^q\right\}<K_\rho(z),$$

using (2.11) we have

$$(3.14)\qquad \alpha^{(p+1)n}\sum_{\nu=0}^{q_n-1} z^\nu a_{\nu+(p+1)n}=0\left(\left|\frac{\alpha}{\rho-\varepsilon}\right|^{(p+1)n}\max\left\{1,\left|\frac{z}{\rho-\varepsilon}\right|^{q_n}\right\}\right)$$

$$=0\left((\sigma K_\rho(z))^n\right),\quad (0<\sigma<1)$$

and

$$(3.15)\qquad \alpha^{pn}\sum_{\nu=q_n}^{n-1} z^\nu a_{\nu+pn}+0\left(\left|\frac{\alpha}{\rho-\varepsilon}\right|^{pn}\max\left\{\left|\frac{z}{\rho-\varepsilon}\right|^{q_n},\left|\frac{z}{\rho-\varepsilon}\right|^{n}\right\}\right)$$

$$=0\left((\sigma K_\rho(z))^n\right).$$

From (3.14), (3.15) and Lema 3.1 we have

$$\Delta^{\alpha,\beta}_{n,m,k}(z,f)=-\beta^m\sum_{\nu=0}^{n-1} z^\nu a_{\nu+m}+0\left((\sigma K_\rho(z))^n\right).$$

From the above expression it is clear that the set Z is $(\{\Delta^{\alpha,\beta}_{n,m,k}\},\rho)$ — distinguished set iff Z is an $(\{T_n\},\rho)$ — distinguished set, where $T_n(z,f)=-\beta^m\sum_{\nu=0}^{n-1} z^\nu a_{\nu+m}$. To complete the proof in case (a) it remains to apply Lemma 2.3 and Lemma 2.2 with $t=\beta$, $\varphi_0(n)=m_n$, $\varphi_1(n)=0$, $\varphi_2(n)=n-1$.
(b) Now we shall prove the statement (b).

(i) In the case $|\beta/\rho|^{r+s}<|\alpha/\rho|^p$, $q=0$, we have $K_\rho(z)=|\alpha/\rho|^p\max\{1,|z/\rho|\}$, $z\neq 0$. (We are interesting on $K_\rho(0)$ only if $q_n\equiv 0$. But in this case $K_\rho(0)=|\alpha/\rho|^p$, and we may assume that $K_\rho(0)=|\alpha/\rho|^p$ everywhere.) Now from Lemma 3.1 we get

$$\Delta^{\alpha,\beta}_{n,m,k}(z,f)=\alpha^{pn}\sum_{\nu=q_n}^{n-1} z^\nu a_{\nu+pn}+0\left((\sigma K_\rho(z))^n\right).$$

To complete the proof in this case it remains to apply Lemma 2.2 and Lemma 2.3 with $t=\alpha$, $\varphi_0(n)=pn$, $\varphi_1(n)=q_n$, $\varphi_2(n)=n-1$.
(ii) In the case $|\alpha/\rho|^{p+1}<|\beta/\rho|^{r+s}<|\alpha/\rho|^p$, $q\neq 0$ we have

$$K_\rho(z)=\begin{cases}|\beta/\rho|^{r+s} & |z|<\rho_1\\ |\alpha/\rho|^p\,|z/\rho|^q & \rho_1\leqq|z|<\rho\\ |\alpha/\rho|^p\,|z/\rho| & |z|\geqq\rho\end{cases}$$

$(\rho_1=\rho(|\beta/\rho|^{r+s}\,|\alpha/\rho|^{-p})^{1/q})$.

Now, if $|z|<\rho_1$ we have $|\alpha/\rho|^p\,|z/\rho|^q<|\beta/\rho|^{r+s}$, and we may take $\varepsilon>0$ too small that $\left|\frac{\alpha}{\rho-\varepsilon}\right|^p\left|\frac{z}{\rho-\varepsilon}\right|^q<\left|\frac{\beta}{\rho}\right|^{r+s}$ and $\left|\frac{\alpha}{\rho-\varepsilon}\right|^{p+1}<\left|\frac{\beta}{\rho}\right|^{r+s}$. Then (2.11) yields

$$\alpha^{(p+1)n}\sum_{\nu=0}^{q_n-1} z^\nu a_{\nu+(p+1)n}=0\left(\left|\frac{\alpha}{\rho-\varepsilon}\right|^{(p+1)n}\right)=0((\sigma K_\rho(z))^n),\quad |z|<\rho_1$$

and

$$\alpha^{pn}\sum_{\nu=q_n}^{n-1} z^\nu a_{\nu+pn}=0\left(\left|\frac{\alpha}{\rho-\varepsilon}\right|^{pn}\left|\frac{z}{\rho-\varepsilon}\right|^{q_n}\right)=0((\sigma K_\rho(z))^n),\quad |z|<\rho_1$$

which gives with Lemma 3.1

$$\Delta^{\alpha,\beta}_{n,m,k}(z,f)=-\beta^m\sum_{\nu=0}^{n-1} z^\nu a_{\nu+m}+0((\sigma K_\rho(z))^n),\quad |z|<\rho_1. \tag{3.16}$$

Similarly, if $\rho_1<|z|<\rho$, $|\beta/\rho|^{r+s}<|\alpha/\rho|^p\,|z/\rho|^q$ and if $\varepsilon>0$ is sufficiently small we have $|z|<\rho-\varepsilon$ and $\left|\frac{\alpha}{\rho-\varepsilon}\right|^{p+1}<\left|\frac{\beta}{\rho}\right|^{r+s}<\left|\frac{\beta}{\rho-\varepsilon}\right|^{r+s}<\left|\frac{\alpha}{\rho}\right|^{p}\left|\frac{z}{\rho}\right|^{q}$. Now (2.11) implies

$$\alpha^{(p+1)n}\sum_{\nu=0}^{q_n-1} z^\nu a_{\nu+(p+1)n}=0\left(\left|\frac{\alpha}{\rho-\varepsilon}\right|^{(p+1)n}\right)=0((\sigma K_\rho(z))^n),\quad \rho_1<|z|<\rho$$

and

$$\beta^m\sum_{\nu=0}^{n-1} z^\nu a_{\nu+m}=0\left(\left|\frac{\beta}{\rho-\varepsilon}\right|^{m}\right)=0((\sigma K_\rho(z))^n),\quad \rho_1<|z|<\rho$$

which yields with Lemma 3.1

$$\Delta^{\alpha,\beta}_{n,m,k}(z,f)=\alpha^{pn}\sum_{\nu=q_n}^{n-1} z^\nu a_{\nu+pn}+0((\sigma K_\rho(z))^n),\quad \rho_1<|z|<\rho. \tag{3.17}$$

At last, if $|z|>\rho$, as above we get

$$\Delta^{\alpha,\beta}_{n,m,k}(z,f)=\alpha^{pn}\sum_{\nu=q_n}^{n-1} z^\nu a_{\nu+pn}+0((\sigma K_\rho(z))^n),\quad |z|>\rho. \tag{3.18}$$

The desired statement about the domain D_ρ follows from (3.16) and Lemma 2.3 with $t=\beta$, $\varphi_0(n)=m_n$, $\varphi_1(n)=0$, $\varphi_2(n)=n-1$; *about* $D_\rho\backslash\bar{D}_{\rho_1}$ – from (3.17) and Lemma 2.3 with $t=\alpha$, $\varphi_0(n)=pn$, $\varphi_1(n)=q_n$, $\varphi_2(n)=n-1$ $(k_n=pn+q_n)$; and about $\mathbf{C}\backslash\bar{D}_\rho$ – from (3.18) and Lemma 2.2 with $t=\alpha$, $\varphi_0(n)=pn$, $\varphi_1(n)=q_n$, $\varphi_2(n)=n-1$.
(c) Now we shall consider the case $|\beta/\rho|^{r+s}<|\alpha/\rho|^{p+1}$, $q\neq 0$. In this case

$$K_\rho(z)=\begin{cases}|\alpha/\rho|^{p+1} & |z|<\rho_2\\ |\alpha/\rho|^{p}\,|z/\rho|^{q} & \rho_2\leqq|z|<\rho\\ |\alpha/\rho|^{p}\,|z/\rho| & \rho\leqq|z|\end{cases}$$

($\rho_2=\rho\,|\alpha/\rho|^{1/q}$). Using (2.11) and Lemma 3.1, it is not difficult to see that

$$(3.19)\quad \Delta^{\alpha,\beta}_{n,m,k}(z,f)=\begin{cases}\alpha^{(p+1)n}\sum\limits_{\nu=0}^{q_n-1} z^\nu a_{\nu+(p+1)n} \quad +0((\sigma K_\rho(z))^n) & |z|<\rho_2\\ \alpha^{pn}\sum\limits_{\nu=q_n}^{n-1} z^\nu a_{\nu+pn} \quad +0((\sigma K_\rho(z))^n) & |z|>\rho_2\end{cases}$$

The statement about the domain $D_\rho\backslash\bar{D}_{\rho_2}$ follows, as above, from (3.19) and Lemma 2.3 with $t=\alpha$, $\varphi_0(n)=pn$, $\varphi_1(n)=q_n$, $\varphi_2(n)=n-1$ $(k_n=pn+q_n)$.

We shall prove the statement about the domain $D_{\rho_2}\bigcup\{\mathbf{C}\backslash\bar{D}_\rho\}$. Let us suppose that there exists some set $Z\subset\Omega=D_{\rho_2}\bigcup\{\mathbf{C}\backslash\bar{D}_\rho\}$, which is $(\{\Delta^{\alpha,\beta}_{n,m,k}\},\rho)$ — distinguished set, but $|Z|\geqq p+1$. Then there exist points $z_1, z_2,\dots,z_{p+1}\subset\Omega$ $(z_i\neq z_j$ for $i\neq j)$ and function $f\in A_\rho$ such that

$$|\Delta^{\alpha,\beta}_{n,m,k}(z_j,f)|=0((\sigma K_\rho(z_j))^n),\quad 1\leqq j\leqq p+1$$

which yields with (3.19):

$$(3.20)\quad \begin{cases}\sum\limits_{\nu=0}^{q_n-1} z_j^\nu a_{\nu+(p+1)n}=0((\sigma\rho^{-p-1})^n) & z_j\in D_{\rho_2}\\ \sum\limits_{\nu=q_n}^{n-1} z_j^\nu a_{\nu+pn}=0(|\sigma z\rho^{-p-1}|^n) & z_j\in\mathbf{C}\backslash\bar{D}_\rho.\end{cases}$$

But if $z_j\in D_{\rho_2}$, from (3.20) it is clear that

$$(3.21)\quad \sum_{\nu=0}^{p} z_j^\nu a_{\nu+(p+1)n}=\sum_{\nu=0}^{q_n-1} z_j^\nu a_{\nu+(p+1)n}-z_j^{p+1}\sum_{\nu=0}^{q_{n+1}-1} z_j^\nu a_{\nu+(p+1)(n+1)}$$

$$+\sum_{\nu=q_n}^{q_{n+1}+p} z_j^\nu a_{\nu+(p+1)n}=0((\sigma\rho^{-p-1})^n)\quad (z_j\in D_{\rho_2})$$

and for $z_j\in\mathbf{C}\backslash\bar{D}_\rho$:

$$(3.22)\qquad \sum_{\nu=0}^{p} z_j^{\nu} a_{\nu+(p+1)n} = z_j^{-n}\left\{ z_j^{p} \sum_{\nu=q_{n+1}}^{n} z_j^{\nu} a_{\nu+p(n+1)} - \sum_{\nu=q_n}^{n-1} z_j^{\nu} a_{\nu+pn} + \sum_{\nu=q_n}^{q_{n+1}+p-1} z_j^{\nu} a_{\nu+pn} \right\} = 0((\sigma\rho^{-p-1})^n) \quad z_j \in \mathbf{C}\setminus\bar{D}_\rho.$$

We may regard (3.21) — (3.22) as a system of equations for a_i, $(p+1)n \leqq i < (p+1)(n+1)$ $(n=0, 1, \ldots)$. Solving this system we obtain $\limsup_{n\to\infty} |a_n|^{1/n} < \rho^{-1}$, which contradicts to the choice of $f(z)$, i.e. if $Z \in \Omega$ is $(\{\Delta^{\alpha,\beta}_{n,m,k}\}, \rho)$ — distinguished set, then $|Z| < p+1$.

Now we shall prove that any $Z = \{z_1, z_2, \ldots, z_p\} \subset \Omega$ is $(\{\Delta^{\alpha,\beta}_{n,m,k}\}, \rho)$ — distinguished set. Indeed, let $d_0, d_1, \ldots, d_p$ be such that $\Sigma_{\nu=0}^{p} z_j^{\nu} d_\nu = 0$, $1 \leqq j \leqq p$, $d_p = 1$. We set $a_{\nu+(p+1)n} = \rho^{-(p+1)n} d_\nu$, $0 \leqq \nu \leqq p$, $n = 0, 1, \ldots$ It is clear that $f_0(z) = \sum_0^\infty a_n z^n \in A_\rho$ and from (3.19) we obtain that $\Delta^{\alpha,\beta}_{n,m,k}(z_j, f) = 0((\sigma K_\rho(z_j))^n)$, $1 \leqq j \leqq p$ since $\Sigma_{\nu=0}^{p} z_j^{\nu} a_{\nu+(p+1)n} = 0$, $1 \leqq j \leqq p$, $n = 0, 1, \ldots$. This completes the proof.

4. A variant of mixed e_2 – approximation and Lagrange interpolation

In section 3 we have considered the difference $\Delta^{\alpha,\beta}_{n,m,k}(z, f)$, where the Lagrange operator was applied to an e_2 – operator. Here we interchange their roles. Let $L_{k-1}(z, \beta, f)$ denotes the Lagrange interpolant to $f(z)$ on the roots of $z^k - \beta^k$ $(f \in A_\rho, \beta \in D_\rho)$, and, let $P_{n-1(m)}(z, \alpha, f)$ denotes the polynomial of degree $\leqq n-1$ of best e_2 – approximation to $f(z)$ on the roots of $z^m - \alpha^m$, i.e. $P_{n-1(m)}(z, \alpha, f)$ minimizes the sum

$$\sum_{\nu=0}^{m-1} |f(\alpha\omega^\nu) - P(\alpha\omega^\nu)|^2, \quad (\omega = \exp\{2\pi i/m\})$$

over all polynomials $P \in \pi_{n-1}$ $(f \in A_\rho, \alpha \in D_\rho)$. We shall consider the difference

$$(4.1)\qquad \bar{\Delta}^{\alpha,\beta}_{n,m,k}(z, f) = P_{n-1(m)}(z, \alpha, f) - P_{n-1(m)}(z, \alpha, L_{k-1}(z, \beta, f)).$$

It is proved in [9] that if $m = m_n = rn + s_n$, $s_n = sn + 0(1)$, $0 \leqq s < 1$, $k = k_n = \lambda m_n + \mu_n$, $0 \leqq \mu_n < m_n$, $\mu_n = \mu n + 0(1)$ then $\bar{\Delta}^{\alpha,\beta}_{n,m,k}(z, f)$ tends to zero as $n \to \infty$ for $|z| < \rho/\tau$, where

$$(4.2)\qquad \tau = \begin{cases} \max\{|\alpha/\rho|^{\lambda(r+s)}, |\beta/\rho|^{\lambda(r+s)+\mu}\} & \text{if} \quad \mu_n < n \\ \max\{|\alpha/\rho|^{(\lambda+1)(r+s)}, |\beta/\rho|^{\lambda(r+s)+\mu}\} & \text{if} \quad \mu_n \geqq n. \end{cases}$$

Here we shall prove the following more quantitative result:

Theorem 4. *Let* $f\in A_\rho$ *and let* $\bar{\Delta}^{\alpha,\beta}_{n,m,k}(z,f)$ *be the difference, defined with* (4.1), *where* $m=m_n=rn+s_n$, $s_n=sn+0(1)$, $0\leqq s<1$; $k=k_n=\lambda m_n+\mu_n$, $0\leqq\mu_n<m_n$, $\mu_n=\mu n+0(1)$ and $\mu<1$ or $\mu_n\geqq n$. *If* $|\alpha/\rho|^{(\lambda+1)(r+s)}\neq|\beta/\rho|^{\lambda(r+s)+\mu}$ for $\mu\neq 0$, *and for* $\mu<1$ $|\alpha/\rho|^{\lambda(r+s)}\neq|\beta/\rho|^{\lambda(r+s)+\mu}$, *then*

$$f_\rho(R):=\limsup_{n\to\infty}\{\max_{|z|=R}|\bar{\Delta}^{\alpha,\beta}_{n,m,k}(z,f)|\}^{1/n}=K_\rho(R),\quad R>0$$

where

$$(4.3)\qquad K_\rho(z)=\begin{cases}\tau|z|/\rho & \text{for}\quad |z|\geqq\rho\\ \tau_0 & \text{for}\quad |z|<\rho,\quad \mu\geqq 1\\ \max\{\tau_0,|\alpha/\rho|^{\lambda(r+s)}|z/\rho|^\mu\} & \text{for}\quad 0<|z|<\rho,\quad \mu<1\end{cases}$$

and $\tau_0=\max\{|\alpha/\rho|^{(\lambda+1)(r+s)},|\beta/\rho|^{\lambda(r+s)+\mu}\}$ (τ *is defined with* (4.2)).

In a special case $\alpha=1$, $\beta=0$, $\mu_n=n$ and $s_n=C$. Theorem 4 yields Theorem 5 in [7]. Also since $P_{n-1(n)}(z,\alpha,f)\equiv L_{n-1}(z,\alpha,f)$, we have $\bar{\Delta}^{\alpha,\beta}_{n,n,k}(z,f)\equiv\Delta^{\alpha,\beta}_{n,k}(z,f)$, i.e. Theorem 4 reduces to Theorem *C* in a special case $m_n\equiv n$.

To prove the above theorem, we shall use the following representation for $\bar{\Delta}^{\alpha,\beta}_{n,m,k}(z,f)$.

Lemma 4.1. *Let* $f\in A_\rho$ *and let* $\bar{\Delta}^{\alpha,\beta}_{n,m,k}(z,f)$ *be defined with* (4.1), *where* $m=m_n=rn+s_n$, $s_n=sn+0(1)$, $0\leqq s<1$; $k=k_n=\lambda m_n+\mu_n$, $0\leqq\mu_n<m_n$, $\mu_n=\mu n+0(1)$ and $\mu<1$ or $\mu_n\geqq n$. *Then*

$$\bar{\Delta}^{\alpha,\beta}_{n,m,k}(z,f)=0((\sigma K_\rho(R))^n)$$
$$+\begin{cases}\alpha^{(\lambda+1)m}\sum_{v=0}^{\mu_n-1}z^v a_{(\lambda+1)m+v}-\beta^k\sum_{v=0}^{n-1}z^v a_{k+v}+\alpha^{\lambda m}\sum_{v=\mu_n}^{n-1}z^v a_{\lambda m+v} & \text{for}\quad \mu<1\\ \alpha^{(\lambda+1)m}\sum_{v=0}^{n-1}z^v a_{(\lambda+1)m+v}-\beta^k\sum_{v=0}^{n-1}z^v a_{k+v} & \text{for}\quad \mu\geqq 1\end{cases}$$

where $K_\rho(z)$ *is defined with* (4.3).

Proof. The following representation for $P_{n-1(m)}(z,\alpha,f)$ is easily obtained from Rivlin's Theorem 1 in [8]

$$(4.4)\qquad P_{n-1(m)}(z,\alpha,f)=\sum_{v=0}^{n-1}z^v\sum_{j=0}^{\infty}\alpha^{jm}a_{jm+v}\qquad n\leqq m.$$

Moreover, it is clear that

$$(4.5)\qquad L_{k-1}(z,\beta,f)=\sum_{v=0}^{k-1}z^v\sum_{j=0}^{\infty}\beta^{jk}a_{jk+v}.$$

Let us denote $b_\nu=\sum_{j=0}^{\infty}\beta^{jk}a_{jk+\nu}$, $0\leqq\nu\leqq k-1$ and $b_\nu=0$ for $\nu\geqq k$. Then (4.4) and (4.5) yield

$$(4.6)\qquad P_{n-1(m)}(z,\alpha,L_{k-1}(z,\beta,f))=\sum_{\nu=0}^{n-1}z^\nu\sum_{j=0}^{\infty}\alpha^{jm}b_{jm+\nu}$$

$$=\begin{cases}\sum\limits_{\nu=0}^{\mu_n-1}z^\nu\sum\limits_{j=0}^{\lambda}\alpha^{jm}b_{jm+\nu}+\sum\limits_{\nu=\mu_n}^{n-1}z^\nu\sum\limits_{j=0}^{\lambda-1}\alpha^{jm}b_{jm+\nu} & \text{for}\quad \mu_n<n\\ \sum\limits_{\nu=0}^{n-1}z^\nu\sum\limits_{j=0}^{\lambda}\alpha^{jm}b_{jm+\nu} & \text{for}\quad \mu\geqq n\end{cases}.$$

Let us denote

$$(4.7)\qquad \bar{\Delta}^{\alpha,\beta}_{n,m,k}(z,f)=\sum_{\nu=0}^{n-1}z^\nu A_{\nu,n}.$$

(a) Let $\mu\geqq1$ (i.e. $\mu_n\geqq n$ for any n). Then if $\varepsilon>0$ is too small that $\left|\frac{\rho}{\rho-\varepsilon}\right|\max\left\{\left|\frac{\alpha}{\rho-\varepsilon}\right|^{(\lambda+2)(r+s)},\left|\frac{\beta}{\rho-\varepsilon}\right|^{\lambda(r+s)+\mu}\left|\frac{\alpha}{\rho-\varepsilon}\right|^{r+s},\left|\frac{\beta}{\rho-\varepsilon}\right|^{2(\lambda(r+s)+\mu)}\right\}<\tau$, from (4.4), (4.6) and (2.11) we get

$$(4.8)\ A_{\nu,n}=\sum_{j=0}^{\infty}\alpha^{jm}a_{jm+\nu}-\sum_{j=0}^{\lambda}\alpha^{jm}\sum_{i=0}^{\infty}\beta^{ik}a_{ik+jm+\nu}=\alpha^{(\lambda+1)m}a_{(\lambda+1)m+\nu}$$

$$+\sum_{j=\lambda+2}^{\infty}\alpha^{jm}a_{jm+\nu}-\beta^k a_{k+\nu}-\beta^k\sum_{j=1}^{\lambda}\alpha^{jm}a_{jm+k+\nu}-\sum_{j=0}^{\lambda}\sum_{i=2}^{\infty}\alpha^{jm}\beta^{ik}a_{ik+jm+\nu}$$

$$=\alpha^{(\lambda+1)m}a_{(\lambda+1)m+\nu}-\beta^k a_{k+\nu}+O\Bigg((\rho-\varepsilon)^{-\nu}\Bigg\{\sum_{j=\lambda+2}^{\infty}\left|\frac{\alpha}{\rho-\varepsilon}\right|^{jm}$$

$$+\left|\frac{\beta}{\rho-\varepsilon}\right|^{k}\sum_{j=1}^{\lambda}\left|\frac{\alpha}{\rho-\varepsilon}\right|^{jm}+\sum_{j=0}^{\lambda}\left|\frac{\alpha}{\rho-\varepsilon}\right|^{jm}\sum_{i=2}^{\infty}\left|\frac{\beta}{\rho-\varepsilon}\right|^{ik}\Bigg\}\Bigg)$$

$$=\alpha^{(\lambda+1)m}a_{(\lambda+1)m+\nu}-\beta^k a_{k+\nu}+\rho^{-\nu}O((\sigma\tau)^n),\quad (0\leqq\nu\leqq n-1)$$

(τ is the number given with (4.2) and $0<\sigma<1$).

Replacing the last expression for $A_{\nu,n}$ in (4.7) we obtain a desired representation for $\bar{\Delta}^{\alpha,\beta}_{n,m,k}(z,f)$ in the case $\mu\geqq1$.

(b) Let now $\mu<1$. As above we obtain .

$$(4.9)\ A_{\nu,n}=\begin{cases}\alpha^{(\lambda+1)m}a_{(\lambda+1)m+\nu}-\beta^k a_{k+\nu}+\rho^{-\nu}0((\sigma\tau_0)^n) & 0\leqq\nu<\mu_n\\ \alpha^{\lambda m}a_{\lambda m+\nu}-\beta^k a_{k+\nu}+\rho^{-\nu}0((\sigma\tau)^n) & \mu_n\leqq\nu<n.\end{cases}$$

To complete the proof it remains to replace the last expression in (4.7) and use (4.2) and (4.3).

Proof of Theorem 4. As in the proof of Theorem 2 using Lemma 4.1 we see easily that

$$(4.10)\qquad f_\rho(R)\leqq K_\rho(R)\quad\text{for}\quad R>0.$$

We shall prove the inverse inequality

$$(4.11)\qquad K_\rho(R)\leqq f_\rho(R)\quad\text{for}\quad R>0.$$

Coushi's integral formula gives

$$(4.12)\qquad |A_{\nu,n}|R^\nu\leqq\max_{|z|=R}|\bar{\Delta}^{\alpha,\beta}_{n,m,k}(z,f)|,\qquad 0\leqq\nu\leqq n-1.$$

(a) $\mu\geqq1$ (i.e. $\mu_n\geqq n$). Let $C>0$ be such that the sequences $\{k_n+\nu\}$ and $\{(\lambda+1)m_n+\nu\}$, $0\leqq\nu\leqq C$ attain all positive integer values. Since $|\alpha/\rho|^{(\lambda+1)(r+s)}\neq|\beta/\rho|^{\lambda(r+s)+\mu}$, from (4.8) and (4.12) we get

$$(4.13)\qquad \max\{|\alpha/\rho|^{(\lambda+1)(r+s)},\quad |\beta/\rho|^{\lambda(r+s)+\mu}\}$$
$$=\limsup_{n\to\infty}\{\max_{0\leqq\nu\leqq C}|A_{\nu,n}|\}^{1/n}\leqq f_\rho(R),\qquad(\mu\geqq1).$$

On the other hand, if we take $C>0$ such that the sequences $\{k_n+\nu\}$ and $\{(\lambda+1)m_n+\nu\}$ $n-C\leqq\nu\leqq n-1$ attain all positive integer values we get

$$(4.14)\qquad \rho^{-1}\max\{|\alpha/\rho|^{(\lambda+1)(r+s)},\quad |\beta/\rho|^{\lambda(r+s)+\mu}\}$$
$$=\limsup_{n\to\infty}\{\max_{n-C\leqq\nu\leqq n-1}|A_{\nu,n}|\}^{1/n}\leqq R^{-1}f_\rho(R),\qquad(\mu\geqq1).$$

From (4.13), (4.14) and the definition of $K_\rho(R)$ we obtain (4.11) in the case $\mu\geqq1$.

(b) $\mu<1$. Since $|\alpha/\rho|^{(\lambda+1)(r+s)}\neq|\beta/\rho|^{\lambda(r+s)+\mu}$ for $\mu\neq0$ and $|\alpha/\rho|^{\lambda(r+s)}\neq|\beta/\rho|^{\lambda(r+s)+\mu}$, if $C>0$ is sufficiently large, as above from (4.9) and (4.12) we get

$$(4.15)\qquad \rho^{-\mu}\max\{|\alpha/\rho|^{\lambda(r+s)},\quad |\beta/\rho|^{\lambda(r+s)+\mu}\}$$
$$=\limsup_{n\to\infty}\{\max_{\mu_n\leqq\nu\leqq\mu_n+C}|A_{\nu,n}|\}^{1/n}\leqq R^{-\mu}f_\rho(R),\qquad(\mu<1)$$

and

$$\text{(4.16)} \qquad \rho^{-1} \max\{|\alpha/\rho|^{\lambda(r+s)}, \quad |\beta/\rho|^{\lambda(r+s)+\mu}\}$$
$$=\limsup_{n\to\infty}\{\max_{n-C\leqq\nu\leqq n-1}|A_{\nu,n}|\}^{1/n}\leqq R^{-1}f_\rho(R), \qquad (\mu<1).$$

From (4.15) and (4.16) we obtain (4.11) in the case $\mu=0$. But if $0<\mu<1$, from (4.9) and (4.12) we obtain

$$\text{(4.17)} \qquad \max\{|\alpha/\rho|^{(\lambda+1)(r+s)}, \quad |\beta/\rho|^{\lambda(r+s)+\mu}\}$$
$$=\limsup_{n\to\infty}\{\max_{0\leqq\nu\leqq C}|A_{\nu,n}|\}^{1/n}\leqq f_\rho(R), \qquad (0<\mu<1).$$

$$\text{(4.18)} \qquad \rho^{-\mu}\max\{|\alpha/\rho|^{(\lambda+1)(r+s)}, \quad |\beta/\rho|^{\lambda(r+s)+\mu}\}$$
$$=\limsup_{n\to\infty}\{\max_{\mu_n-C\leqq\nu\leqq\mu_n-1}|A_{\nu,n}|\}^{1/n}\leqq R^{-\mu}f_\rho(R), \qquad (0<\mu<1).$$

From (4.15) — (4.18) we obtain (4.11) in the case $0<\mu<1$, and the proof is completed.

Now we may state the overconvergence result.

Theorem 5. *Let* $m=m_n=rn+s_n$, $s_n=sn+O(1)$, $0\leqq s<1$; $k=k_n=\lambda m_n+\mu_n$, $0\leqq\mu_n<m_n$, $\mu_n=\mu n+O(1)$ *and* $\mu<1$ *or* $\mu_n\geqq n$ *and* $\alpha,\beta\in D_\rho$, *and let* $\rho_1=|\beta|\,|\beta/\alpha|^{\lambda(r+s)/\mu}$, $\rho_2=\rho\,|\alpha/\rho|^{(r+s)/\mu}$. *Then the set* $Z\subset\Omega$ *is an* $(\{\bar\Delta^{\alpha,\beta}_{n,m,k}(z,f)\},\rho)$ — *distinguished set if and only if*

$$\text{(a)} \qquad |Z|<\begin{cases}\delta(\{k_n\}) & \textit{for} \quad \Omega=D_\rho\\ \delta(\{k_n+n\}) & \textit{for} \quad \Omega=\mathbf{C}\setminus\bar D_\rho\end{cases}$$

in the cases $|\beta/\rho|^{\lambda(r+s)+\mu}>|\alpha/\rho|^{(\lambda+1)(r+s)}$ $\mu\geqq1$ *and* $|\beta/\rho|^{\lambda(r+s)+\mu}>|\alpha/\rho|^{\lambda(r+s)}$ $\mu<1$;

$$\text{(b)} \qquad |Z|<\begin{cases}\delta(\{(\lambda+1)m_n\}) & \textit{for} \quad \Omega=D_\rho\\ \delta(\{(\lambda+1)m_n+n\}) & \textit{for} \quad \Omega=\mathbf{C}\setminus\bar D_\rho\end{cases}$$

in the case $|\beta/\rho|^{\lambda(r+s)+\mu}<|\alpha/\rho|^{(\lambda+1)(r+s)}$ $\mu\geqq1$;

$$\text{(c)} \qquad |Z|<\begin{cases}\delta(\{k_n\}) & \textit{for} \quad \Omega=\begin{cases}D_\rho & \textit{for } \mu_n\equiv0\\ D_\rho\setminus\Gamma_{\rho_1} & \textit{otherwise}\end{cases}\\ \delta(\{\lambda m_n+n\}) & \textit{for} \quad \Omega=\mathbf{C}\setminus\bar D_\rho\end{cases}$$

in the cases $|\alpha/\rho|^{(\lambda+1)(r+s)}<|\beta/\rho|^{\lambda(r+s)+\mu}<|\alpha/\rho|^{\lambda(r+s)}$, $0<\mu<1$ *and* $|\beta/\rho|^{\lambda(r+s)}<|\alpha/\rho|^{\lambda(r+s)}$ $\mu=0$;

$$\text{(d)} \qquad |Z| < \begin{array}{lll} \delta(\{(\lambda+1)m_n\}) & \textit{for} & \Omega = D_{\rho_2} \\ \delta(\{k_n\}) & \textit{for} & \Omega = D_\rho \setminus \bar{D}_{\rho_2} \\ \delta(\{\lambda m_n + n\}) & \textit{for} & \Omega = \mathbf{C} \setminus \bar{D}_\rho \end{array}$$

in the case $|\beta/\rho|^{\lambda(r+s)+\mu} < |\alpha/\rho|^{(\lambda+1)(r+s)}$, $0<\mu<1$.

Proof. We will regard the case $|\alpha/\rho|^{(\lambda+1)(r+s)} < |\beta/\rho|^{\lambda(r+s)+\mu} < |\alpha/\rho|^{\lambda(r+s)}$, $0<\mu<1$, $\Omega = D_\rho \setminus \Gamma_{\rho_1}$. In this case we have

$$(4.19) \qquad K_\rho(z) = \begin{cases} |\beta/\rho|^{\lambda(r+s)+\mu} & \text{for} \quad |z| < \rho_1 \\ |z/\rho|^\mu \, |\alpha/\rho|^{\lambda(r+s)} & \text{for} \quad \rho_1 \leqq |z| < \rho \\ |z/\rho| \, |\alpha/\rho|^{\lambda(r+s)} & \text{for} \quad |z| \geqq \rho \end{cases}$$

and using the representation for $\bar{\Delta}^{\alpha,\beta}_{n,m,k}(z,f)$, given in Lemma 4.1, we obtain

$$(4.20) \quad \bar{\Delta}^{\alpha,\beta}_{n,m,k}(z,f) = \begin{cases} -\beta^k \sum\limits_{v=0}^{n-1} z^v a_{k+v} + 0((\sigma K_\rho(z))^n), & |z| < \rho_1 \\ \alpha^{\lambda m} \sum\limits_{v=\mu_n}^{n-1} z^v a_{\lambda m+v} + 0((\sigma K_\rho(z))^n), & |z| > \rho_1. \end{cases}$$

Let us denote $T_n(z) = \sum\limits_{v=0}^{n-1} z^v a_{k+v}$. Since $k_n = \lambda m_n + \mu_n$, $0<\mu<1$, it is clear from (4.19) and (4.20) that the set $Z \subset \Omega$ is $(\{\bar{\Delta}^{\alpha,\beta}_{n,m}\}, \rho)$ — distinguished set iff Z is an $(\{T_n\}, \rho)$ — distinguished set $(\Omega = D_\rho \setminus \Gamma_{\rho_1})$. But from Lemma 2.3 with $t=1$, $\varphi_0(n) = k_n$, $\varphi_1(n) = 0$, $\varphi_2(n) = n-1$ we obtain a desired statement in this case. The proofs of the other statements are based on Lemma 4.1, Lemma 2.2 and Lemma 2.3. We omit them, since they are not essentially different from the proof of Theorem 3.

Remark 3. As was noted above $\Delta^{\alpha,\beta}_{n,m,m}(z,f) \equiv \Delta^{\alpha,\beta}_{n,m}(z,f)$ and $\bar{\Delta}^{\alpha,\beta}_{n,n,k}(z,f) \equiv \Delta^{\alpha,\beta}_{n,k}(z,f)$. Using this fact, we get Theorem 1 as a corollary from Theorem 3 and Theorem 5. Indeed in the case $m_n \equiv k_n$ the statements (a) and (c) of Theorem 3 reduce respectively to statements (a) and (c) of Theorem 1 ; and in the case $m_n \equiv n$ the statement (c) of Theorem 5 reduces to the statement (b) of Theorem 1.

In the special case $\alpha = 1$, $\beta = 0$, $\mu_n = n$ and $s_n = c$ of Theorem 5, a more precise result is obtained in [7].

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