Provided for non-commercial research and educational use.
Not for reproduction, distribution or commercial use.

Mathematica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

---

The attached copy is furnished for non-commercial research and education use only. Authors are permitted to post this version of the article to their personal websites or institutional repositories and to share with other researchers in the form of electronic reprints.
Other uses, including reproduction and distribution, or selling or licensing copies, or posting to third party websites are prohibited.

For further information on Mathematica Balkanica visit the website of the journal
http://www.mathbalkanica.info
or contact:
Mathematica Balkanica - Editorial Office;
Acad. G. Bonchev str., Bl. 25A, 1113 Sofia, Bulgaria
Phone: +359-2-979-6311, Fax: +359-2-870-7273,
E-mail: balmat@bas.bg

Mathematica Balkanica

Now Series Vol. 3, 1989, Fasc. 2

A Note on Quasitriangular Subalgebras of Von Neumann Algebras

Florin Pop

Presented by M. Putinar

Every quasitriangular operator in a von Neumann algebra M is expressed as the sum between a triangular operator in M (with respect to the same increasing sequence of projections (p_n)) and an operator in the norm-closed right ideal generated by the projections (p_n).

Quasitriangular algebras, introduced by W. Arveson in [2], are the algebras of all bounded operators x on a Hilbert space H satisfying

$$\lim \|(1-p_n)xp_n\|=0$$

where (p_n) is a fixed increasing sequence of finite dimensional (selfadjoint) projections on H.

The study of quasitriangular operators had been initiated earlier by P. R. Halmos. We refer the reader to [5], [1], [3] and [6] for more information about these operators.

A major result of W. Arveson was that the algebra of quasitriangular operators is the sum between the algebra of triangular operators (i.e. the nest algebras) and the ideal of compact operators on H. The remarkable fact is that the latter algebra is norm-closed.

Complete characterisations of quasitriangularity with respect to arbitrary nests were obtained by T. Fall, W. Arveson and P. Muhly ([4]). (A nest is a totally ordered strongly closed family of projections on H, containing 0 and 1.)

This paper originates in our attempt to extend quasitriangularity to arbitrary von Neumann algebras. Our main result is the following:

If M is a von Neumann algebra and (p_n) is an increasing sequence of projections in M converging strongly to the identity, then every quasitriangular operator in M is the sum between a triangular operator in M and an operator in the norm-closed right ideal generated by the projections (p_n). In particular, if $M=B(H)$ we obtain W. Arveson's results.

Let M be a von Neumann algebra and $L=(p_n)$ be an increasing sequence of projections in M, converging strongly to the identity. Define

$$QT=\{x\in M\,;\quad \lim \|(1-p_n)xp_n\|=0\}$$

(the algebra of quasitriangular operators with respect to L),

$$T=\{x\in M\,;\qquad (1-p_n)xp_n=0\qquad (\forall)\ n\geqq 1\}$$

(the nest-subalgebra of M with respect to L),

$$S=\{x\in M\,;\qquad \lim \|(1-p_n)x(1-p_n)\|=0\}$$

$$R=\{x\in M\,;\qquad \lim \|(1-p_n)x\|=0\}$$

(the norm-closed right ideal generated by L).
It is easy to see that $QT\bigcap S=R$.

Lemma 1. *For every x in M*

$$\operatorname{dist}(x,S)-\inf \|(1-p_n)x(1-p_n)\|.$$

Proof. Note that the sequence $(\|(1-p_n)x(1-p_n)\|)$ is decreasing, hence the above infimum is actually equal to the limit of the sequence.

$$\|(1-p_n)x(1-p_n)\|=\|x-(p_nx+xp_n-p_nxp_n)\|\geqq \operatorname{dist}(x,S),\ \text{hence}$$

(1) $$\inf \|(1-p_n)x(1-p_n)\|\geqq \operatorname{dist}(x,S).$$

Let now $\varepsilon>0$ be given and $s\in S$ be such that

$$\|x-s\|\leqq \operatorname{dist}(x,S)+\varepsilon/2.$$

Choose n_0 such that

$$\|(1-p_n)s(1-p_n)\|\leqq\varepsilon/2\qquad (\forall)\ n\geqq n_0.$$ It follows that

$$\|(1-p_n)x(1-p_n)\|-\|(1-p_n)s(1-p_n)\|\leqq \|(1-p_n)(x-s)(1-p_n)\|$$

(2) $$\leqq\|x-s\|\ \text{hence}\ \inf\|(1-p_n)x(1-p_n)\|\leqq \operatorname{dist}(x,S)+\varepsilon.$$

Now (1) and (2) lead to the desired conclusion.

Lemma 2. *The subspace $T+S$ is uniformly closed.*

Proof. Define

$$\varphi: T/T\bigcap S\to T+S/S$$

$$\varphi(a+S\bigcap T)=a+S\ \text{for every}\ a\ \text{in}\ T.$$

This mapping is easily seen to be norm-decreasing. We claim that it is isometric. For this purpose let $a \in T$ and $\varepsilon > 0$ be arbitrary and choose $n_0 \geqq 1$ such that every $n \geqq n_o$ satisfies

$$\operatorname{dist}(a, S) \geqq \|(1-p_n)a(1-p_n)\| - \varepsilon.$$

$$\text{Now } \|(1-p_n)a(1-p_n)\| = \|a-(p_n a + a p_n - p_n a p_n)\|$$

$$\geqq \operatorname{dist}(a, S \cap T), \text{ hence}$$

$\operatorname{dist}(a, S) \geqq \operatorname{dist}(a, S \cap T) - \varepsilon \ (\forall)\, \varepsilon > 0$ and consequently the equality holds for every a in T. This establishes our claim.

It follows that $T+S/S$ is closed in the quotient norm and since S is also closed, $T+S$ will be (uniformly) closed, which concludes the proof.

Theorem 1. $QT = T+R$.
In particular $T+R$ is a closed subalgebra.

Proof. The nontrivial inclusion to be proved is

$$QT \subset T+R.$$

Let then $x \in QT$ and $\varepsilon > 0$ be arbitrary.
By hypothesis, there is $n_0 \geqq 1$ such that for every $n \geqq n_o$

$$\|(1-p_n) x p_n\| \leqq \varepsilon.$$

By the distance formula for nest-subalgebras of von Neumann algebras ([7]) there is an operator a in M satisfying

$$(1-p_i) a p_i = 0 \quad (\forall)\, i \geqq n_0 \quad \textit{and} \quad \|x-a\| \leqq \varepsilon.$$

There are operators $b \in T$ and $c \in S$ such that $a = b + c$ (b is obtained from a after finitely many steps, the first one being $a \to a - (1-p_{n_0-1}) a p_{n_0-1}$).

It follows that x belongs to the uniform closure of $T+S$. By Lemma 2, x belongs to $T+S$.
Since x is in QT, it follows that actually $x \in T+R$, which concludes the proof.

Remarks. In particular, let M be a semifinite von Neumann algebra and $L = (p_n)$ be an increasing sequence of finite projections in M, converging strongly to the identity.

Denote by J the norm-closed two-sided ideal generated by the finite projections in M. Recall that

$$S=\{x\in M; \quad \lim \|(1-p_n)x(1-p_n)\|=0\} \qquad \text{and that } R=S\bigcap QT.$$

Define $J_0=J\bigcap QT$. Theorem 1 shows that

$$QT=T+J_0=T+R.$$

If $M=B(H)$ then $J=J_0=K(H)$ (the ideal of compact operators on the Hilbert space H) and we obtain W. Arveson's classical result $QT=T+K(H)$. Note that in this case

$$T+K(H)=T+R.$$

Proposition 2. *For every x in M*

$$\operatorname{dist}(x,QT)=\limsup \|(1-p_n)xp_n\|.$$

Proof. For every operator a in QT

$$\limsup \|(1-p_n)xp_n\|=\limsup \|(1-p_n)(x-a)p_n\|\leqq \|x-a\|$$

$$\text{hence } \limsup \|(1-p_n)xp_n\|\leqq \operatorname{dist}(x,QT).$$

For the opposite inequality, let $\varepsilon>0$ be arbitrary and choose $n_0\geqq 1$ such that for every $n\geqq n_0$

$$\|(1-p_n)xp_n\|\leqq \limsup \|(1-p_n)xp_n\|+\varepsilon/2.$$

As in the proof of Theorem 1, there is an operator a in M such that $(1-p_i)ap_i=0$ $(\forall)\, i\geqq n_0$ and

$$\|x-a\|\leqq \limsup \|(1-p_n)xp_n\|+\varepsilon.$$

It follows that there are operators $b\in T$, $c\in R$ such that $a=b+c$, hence

$$\|x-b-c\|\leqq \limsup \|(1-p_n)xp_n\|+\varepsilon, \text{ so that}$$

$$\operatorname{dist}(x,QT)\leqq \limsup \|(1-p_n)xp_n\|+\varepsilon.$$

Since ε was arbitrary, the proposition is proved.

References

1. C. Apostol, C. Foias, L. Zsido. Some results on non-quasitriangular operators. *Indiana U. Math. J.* **22**, 1973, 1151-1162 and C. Apostol, C. Foias, D. Voiculescu. Some results on non-quasitriangular operators II–VI. *Revue Roum. Math. Pures Appl.* **18**, 1973, 159-182, 309-324, 487-514, 1133-1150, 1473-1494.

2. W. Arveson. Interpolation problems in nest algebras. *J. Functional Analysis*, **20**, 1975, 208-233.
3. R. G. Douglas, C. Pearcy. A note on quasitriangular operators. *Duke Math. J.*, **37**, 1970, 177-188.
4. T. Fall, W. Arveson, P. Muhly. Perturbations of nest algebras. *J. Operator Theory*, **1**, 1979, 137-150.
5. P. R. Halmos. Quasitriangular operators. *Acta Sci. Math.*, **29**, 1968, 283-293.
6. C. Pearcy, N. Salinas. An invariant subspace theorem. *Michigan Math. J.*, **20**, 1973, 21-31.
7. F. Pop. Perturbations of nest-subalgebras of von Neumann algebras. *J. Operator Theory*, **21**, 1989, 165-170.

Department of Mathematics
INCREST
Bdul Pacii 220
79622 Bucharest
ROMANIA

Received 17. 08. 1988