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Some Notes on Locally Convex Topologies on $L(X)$

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Some results on locally convex topologies on algebras of bounded linear operators acting on Banach and Hilbert space are proved.

Let X be a Hausdorff locally convex space. Its topology t is usually given by some family of seminorms P. Define for a given P the following set

$$X_P = \{x \in X : \sup_{p \in P} p(x) < \infty\}.$$

It is a normed subspace of X with respect to the norm

$$\|x\|_P = \sup_{p \in P} p(x), \qquad x \in X_P. \tag{1}$$

The set X_P depends on the choice of the family P. For example if X is a normed space, then $P = \{\|\cdot\|\}$ and $P' = \{n\|\cdot\|,\ n \in \mathrm{N}\}$ define the same topology, but $X_P = X$ and $X_{P'} = \{0\}$. We will consider the situation where $X_P = X$ for some P. In this case the norm (1) induces on X a topology t' finer than the original one. As a simple consequence of the open mapping theorem we obtain

Theorem 1. *If X is an infrabarrelled, normcomplete locally convex space and $X_P = X$ for some family P of seminorms on X then X is normable.*

Proof. Since X is infrabarrelled and normcomplete it is actually barrelled ([5]). Let us prove that X is also complete with respect to the norm $\|\cdot\|_P$. The normcompleteness of X means that for each closed, convex, balanced and bounded set B in X its linear span $V(B)$ is complete with respect to the Minkowski functional $p_B(x) = \inf\{\lambda > 0 : x \in \lambda B\}$ (see [6]). The unit sphere $S_P = \{x \in X : \|x\|_P \leqq 1\}$ is clearly convex, balanced, bounded and closed with respect to the original topology t and its Minkowski functional is equal to $\|\cdot\|_P$. Since $V(S_P) = X$ the $\|\cdot\|_P$ — completeness of X follows immediately. Since $t \subset t'$, the identity map from the Banach space $(X, \|\cdot\|_P)$ onto the barrelled space (X, P) is continuous and by the open mapping theorem ([8]) it follows $t=t'$.

Corollary 2. *A Fréchet locally convex space X for which $X_P=X$ for some family P, is normable.*

We will consider in the following the above situation on some operator algebras. Let X be a normed space and $L(X)$ the algebra of bounded linear operators on X. We can introduce several locally convex topologies in $L(X)$. The most significant are the weak and the strong operator topology (wo- resp. so-topology). Let us define for $A\in L(X): q_{x,f}(A)=|f(Ax)|$ and $p_x(A)=\|Ax\|$, where $x\in X$ and $f\in X'$. Then the wo- and the so- topology can be defined with the following families of seminorms respectively:

$$P^{\mathrm{WO}}=\{q_{x,f}: x\in X,\ f\in X',\ \|x\|=\|f\|=1\}, \tag{2}$$

$$P^{\mathrm{SO}}=\{p_x: x\in X,\ \|x\|=1\}. \tag{3}$$

Lemma 3. *Let X be a Banach space, then $L(X)$ is sequentially complete with respect to the so-topology and it is normcomplete with respect to the wo-topology.*

P r o o f. Let $\{A_n\}$ be a so-Cauchy sequence, then $\{A_n x\}$ is a Cauchy sequence in X for each $x\in X$, and hence it is for each $x\in X$ convergent to some $y\in X$. Defining $y=Ax$, A is clearly linear. Since the sequence $\{A_n x\}$ converges it is pointwise bounded and from the uniform boundedness theorem there exists $M>0$ such that $\|A_n\|\leqq M$ for all $n\in \mathrm{N}$. The inequality $\|Ax\|\leqq\|(A-A_n)x\|+\|A_n x\|$, implies $\|A\|\leqq M$, thus $A_n \xrightarrow{\mathrm{SO}} A$ in $L(X)$. Since the wo-topology is one of the weak topologies of the dual pair $(L(X),\ L(X)')$ it is normcomplete (see [1]).

Lemma 4. *Let X be a normed space. If on $L(X)$ there exists a norm which is so-continuous then X is finite-dimensional.*

P r o o f. Suppose that $\|\cdot\|'$ is a so-continuous norm on $L(X)$. Then there exists $x_1, x_2, \dots, x_n\in X$ and $c>0$ such that

$$\|A\|'\leqq c.\max\{\|Ax_i\|: i=1, 2, \dots, n\}, \qquad A\in L(X). \tag{4}$$

Let us denote by Y the linear span of $x_1, x_2, \dots, x_n$ and let $m=\dim Y$. By the Auerbach theorem (see [2]) there are linearly independent $e_1, e_2, \dots, e_m\in Y$ and $f_1, f_2, \dots, f_m\in X'$ such that $\|e_i\|=\|f_i\|=1$, $i=1, 2, \dots, m$ and $f_i(e_j)=\delta_{ij}$; $i,j=1, 2, \dots, m$. Consequently, we have $x_i=\sum_{j=1}^{m} f_j(x_i)e_j$, $i=1, 2, \dots, n$. Denoting $P=\sum_1^m e_j\otimes f_j$, it is a projector on Y and let a subspace $Z\subset X$ be such that $X=Y\oplus Z$ is a decomposition of X made by P. For $Q=I-P$ we have $Qx_i=0$, $i=1, 2, \dots, n$ and by (4) it follows $\|Q\|'=0$ and $X=Y$.

Theorem 5. *Let X be a Banach space. If $L(X)$ is infrabarrelled with respect to the wo- or the so-topology then $\dim X<\infty$.*

Proof. Since $q_{x,f}(A)=|f(Ax)|\leqq\|A\|$ for all $q_{x,f}\in P^{WO}$, it follows $L(X)_{P_{WO}}=L(X)$, and by lemma 3 it is wo-normcomplete. By assumption it is infrabarrelled with respect to this topology, hence by Theorem 1 it is normable and by lemma 4, which holds also for wo-continuous norms, X is finite-dimensional. Similarly, we have $L(X)_{P_{SO}}=L(X)$, by lemma 3 it is so-sequentially complete and so-normcomplete too. We assumed infrabarrelledness, so this topology is by Theorem 1 normable, and by lemma 4 dim $X<\infty$.

Since infrabarrelled locally convex spaces contain barrelled and also bornological spaces, we have

Corollary 6. *Let X be an infinite-dimensional Banach space. Then $L(X)$ is neither barrelled nor bornological with respect to the wo- and the so-topology.*

Let now X be again a locally convex space with the system of seminorms P. An operator $T\in L(X)$ is called P-finite ([6]) if there exists $M>0$ such that $p(Tx)\leqq M.p(x)$ for all $p\in P$ and $x\in X$. The set of all such operators $B_P(X)$ is normed algebra with respect to the norm $\|T\|_P=\sup\{p(Tx):x\in X,\ p\in P,\ p(x)\leqq 1\}$. Let us address the question under what condition an operator on $L(X)$ is P^{SO}- or P^{WO}-finite, where X is a normed space. Let $\hat{A}$ denote a linear operator from $L(X)$ into $L(X)$.

Theorem 7. *Let X be a normed space, then for the family P^{SO} from* (2) *the following two assertions hold*

(i) *each operator of the form $\hat{A}S=AS$, where $A,S\in L(X)$, A fixed, is P^{SO}-finite on $L(X)$ and $\|\hat{A}\|_{P^{SO}}=\|A\|$.*

(ii) *if $\hat{A}$ is P^{SO}-finite on $L(X)$ and $\hat{A}(ST)=(\hat{A}S)T$ for all $S,T\in L(X)$, then there is an $A\in L(X)$ such that $\hat{A}S=AS$, $S\in L(X)$ and $\|\hat{A}\|_{p^{SO}}=\|A\|$.*

Proof. (i): Clearly $p_x(\hat{A}S)\leqq\|A\|\,p_x(S)$, for all $p_x\in P^{SO}$ and $S\in L(X)$. Thus A is P^{SO}-finite and $\|\hat{A}\|_{p^{SO}}\leqq\|A\|$. On the other hand $\|\hat{A}\|_{p^{SO}}=\sup\{p_x(AS):x\in X,\ S\in L(X),\ p_x(S)\leqq 1\}\geqq\sup\{p_x(A):x\in X,\ p_x(I)\leqq 1\}=\|A\|$.

(ii): Choose arbitrary $z\in X$, $\|z\|=1$ and $f\in X'$ such that $f(z)=1$ and define $A\in L(X)$ with the equation $Ax=(\hat{A}(x\otimes f))z$, $x\in X$. For any $S\in L(X)$ it follows: $(AS)x=(\hat{A}(Sx\otimes f))z=(\hat{A}(S(x\otimes f)))z=(\hat{A}S)x$. Clearly, A is linear and continuous because of $\|Ax\|=\|(\hat{A}(x\otimes f))z\|=p_z(\hat{A}(x\otimes f))\leqq\|\hat{A}\|_{p^{SO}}\cdot p_z(x\otimes f)=\|\hat{A}\|_{p^{SO}}\cdot\|x\|$. By (i) the equality of norms follows immediately.

Theorem 8. *Let X be a normed space. Then an operator $\hat{A}$ is P^{WO}-finite on $L(X)$ if and only if it is of the form $\hat{A}=\alpha\hat{I}$, where $\alpha\in\mathbb{C}$ and $\hat{I}$ is the identity $:L(X)\to L(X)$.*

Proof. Let $\hat{A}$ be P^{WO}-finite, then there exists $M>0$ such that

$$|f(\hat{A}Sx)|\leqq M\,|f(Sx)|;\quad S\in L(X),\quad x\in X,\quad f\in X'.$$

We first prove that for each $S\in L(X)$ there is a λ (depending on S) such that $\hat{A}S=\lambda S$. If this were not be the case, one could find $S_0\in L(X)$, $\lambda_0\in\mathbb{C}$ and $x_0\in X$ such that $\hat{A}Sx_0\neq\lambda_0 S_0 x_0$. One may suppose that $\lambda_0\neq 0$. By the Hahn-Banach theorem there exists $g\in X'$ such that $g(S_0x_0)=0$ and $g(\hat{A}S_0x_0)\neq 0$. Inserting g, S_0

and x_0 in the above inequality we obtain a contradiction. Now it is not difficult to see that there is a $\alpha > 0$ independent of $S \in L(X)$ such that $\hat{A}S = \alpha S$, $S \in L(X)$. If $\hat{A}$ is of this type it is clearly P^{WO}-finite.

A locally convex space X is said to be H-locally convex with respect to a family P, if P consists of Hilbertian seminorms. That is, each $p \in P$ satisfies the "parallelogram equality", hence there is a semi-inner product $(,)_p$ (it needs not be positive definite) such that $p^2(x) = (x, x)_p$, $x \in X$ (see e. g. [7] and [3]). There arises a question if $L(X)$ is an H-locally convex space with respect to the families (2) and (3) for a normed space X.

Theorem 9. *Let X be a normed space. Then $L(X)$ is*
i) with respect to the family P^{WO} an H-locally convex space,
ii) with respect to the family P^{SO} an H-locally convex space if and only if X is a pre-Hilbert space.

Proof. i): It is easy to see that $q_{x,f} \in P^{WO}$ are Hilbertian seminorms. ii): Let $p_x \in P^{SO}$ be a Hilbertian seminorm. This implies $\| Ax + Bx \|^2 + \| Ax - Bx \|^2 = 2 \| Ax \|^2 + 2 \| Bx \|^2$ for all A, $B \in L(X)$ and $x \in X$. Choose arbitrary $x, y \in X$ and let $f \in X'$ be such that $f(x) = 1$. Letting $A = I$ and $B = y \otimes f$ in the above equality we obtain that $\| \cdot \|$ is a Hilbertian norm on X. The "only if" part is immediate.

If X is an H-locally convex space with respect to a family P then two elements $x, y \in X$ is said to be *P-orthogonal* provided $(x, y)_p = 0$ for all semi-inner products generated by the seminorms $p \in P$ ([3]). Note that there may be some vectors which are in relation of orthogonality with no other nontrivial vector. Let us denote by $V(A, B)$ the linear span of operators A and B in the vector space $L(X)$.

Theorem 10. *Let X be a normed space. Then for each pair $A, B \in L(X)$ there is a P^{WO}-orthogonal element in $V(A, B)$ to operator A if and only if X is finite-dimensional.*

Proof. By Theorem 2.5 of [4] an H-locally convex space is normable if and only if in arbitrary twodimensional subspace each vector has a nontrivial orthogonal vector in the above sense. Thus, by the assumption of theorem, wo-topology on $L(X)$ is normable and by lemma 4 X is finite-dimensional.

Let now X be a Hilbert space. Besides the wo- and the so-topogy we will consider also the following locally convex topologies in $L(X)$: the strong operator* topology (so$_*$), the ultrastrong (s), the ultrastrong* topology (s$_*$) and the ultraweak topology (w) which can be defined with the following families of seminorms successively

$$P^{SO_*} = \{p'_x : x \in X,\ \| x \| = 1\}, \text{ where } p'_x(A) = (\| Ax \|^2 + \| A^* x \|^2)^{1/2},$$

$$P^S = \left\{ p_{\underline{x}} : \underline{x} = (x_j) \subset X,\ \sum_1^\infty \| x_j \|^2 = 1 \right\}, \text{ where } p^2_{\underline{x}}(A) = \sum_1^\infty \| Ax_j \|^2,$$

$$P^{S_*} = \left\{ p'_{\underline{x}} : \underline{x} = (x_j) \subset X,\ \sum_1^\infty \| x_j \|^2 = 1 \right\}, \text{ where } p'_{\underline{x}}(A)^2 = p^2_{\underline{x}}(A) + p^2_{\underline{x}}(A^*),$$

$$P^W = \left\{ q_{\underline{x},\underline{y}} : \underline{x} = (x_j), \quad \underline{y} = (y_j) \subset X, \quad \sum_1^\infty \|x_j\|^2 = \sum_1^\infty \|y_j\|^2 = 1 \right\}, \text{ where}$$

$$q_{\underline{x},\underline{y}}(A) = \left| \sum_1^\infty (Ax_j, y_j) \right|.$$

It is well known that for the infinite dimensional Hilbert space X all the locally convex topologies in $L(X)$ introduced above are nonmetrizable ([9]).

Theorem 11. *Let X be a Hilbert space and let $L(X)$ be infrabarrelled with respect to one of the following topologies: wo, so, w, s, so_* or s_*. Then X is finite-dimensional.*

Proof. We have to consider only the last four cases. It is easily seen that also $L(X)_P = L(X)$ for $P = P^W$, P^S, P^{SO_*} or P^{S_*}. Let us suppose that $L(X)$ is infrabarrelled with respect to the w-topology. Since this topology is one of the topologies of the dual pair $(L(X), L(X)')$ it is normcomplete ([1]). By Theorem 1 the w-topology is normable and X is not infinite-dimensional. Let us prove that $L(X)$ is sequentially complete with respect to the s-topology. If $\{A_n\}$ is any Cauchy sequence with respect to the s-topology, then it is also a so-Cauchy sequence and by lemma 3 it is so-convergent to some A in $L(X)$. From the proof of this lemma it follows also $\|A\| \leqq M$ and $\|A_n\| \leqq M$ for all $n \in \mathrm{N}$. Let us prove that $\{A_n\}$ is also s-convergent to A. Choose arbitrary $\varepsilon > 0$ and $\underline{x} = (x_j) \subset X$ such that $\sum_1^\infty \|x_j\|^2 = 1$. Then there exists some $m \in \mathrm{N}$ such that $\sum_{j=m+1}^\infty \|x_j\|^2 < \varepsilon$ and from the so-convergence $A_n \to A$ there is some $n_0 \in \mathrm{N}$ such that $\|(A_n - A)x_j\| < \sqrt{\varepsilon/m}$, for $j = 1, 2, \ldots, m$ and all $n \geqq n_0$. For $n \geqq n_0$ we can estimate

$$\sum_{j=1}^\infty \|(A_n - A)x_j\|^2 = \sum_{j=1}^m \|(A_n - A)x_j\|^2 + \sum_{j=m+1}^\infty \|(A_n - A)x_j\|^2 < \varepsilon + 4M^2\varepsilon.$$

Thus, $L(X)$ is s-sequentially complete and hence s-normcomplete. If it is infrabarrelled with respect to the s-topology it is, by Theorem 1, normable and $\dim X < \infty$. Let us take the so_*-topology. To prove the so_*-sequentially completeness of $L(X)$ let $\{A_n\}$ be a so_*-Cauchy sequence. Then $\{A_n\}$ and $\{A_n^*\}$ are also so-Cauchy sequences and by lemma 3 they are so-convergent to A respectively to B in $L(X)$. For arbitrary $x, y \in X$ we have

$$|(Ax, y) - (x, By)| = |((A - A_n)x, y) + (x, (A_n^* - B)y)|$$

$$\leqq \|(A - A_n)x\| \cdot \|y\| + \|x\| \cdot \|(A_n^* - B)y\|$$

and by $n \to \infty$ it follows $B = A^*$. If $L(X)$ is infrabarrelled with respect to the so_*-topology, this topology is, by Theorem 1, normable. In the same manner we prove that $L(X)$ is sequentially complete also in the s_*-topology and the conclusion follows immediately for this topology too.

Corollary 12. *Let X be an infinite-dimensional Hilbert space. Then $L(X)$ is neither barrelled nor bornological with respect to all the locally convex topologies considered above.*

As for the family P^{SO} in the normed space we have

Theorem 13. *Let X be a pre-Hilbert space, then for the family P, where $P = P^{SO}$ or P^S, the following two statements hold*
i) each operator of the form $\hat{A}S = AS$, $A, S \in L(X)$, A fixed, is P-finite on $L(X)$ and $\|\hat{A}\|_P = \|A\|$,
ii) if $\hat{A}$ is P-finite on $L(X)$ and $\hat{A}(ST) = (\hat{A}S)T$ for all $S, T \in L(X)$, then there exists an $A \in L(X)$ such that $\hat{A}S = AS$, $S \in L(X)$ and $\|\hat{A}\|_P = \|A\|$.

The proof is the same as for Theorem 7. Analogously as Theorem 8 one can prove

Theorem 14. *Let X be a pre-Hilbert space, then an operator on $L(X)$ is P-finite, where $P = P^{WO}$ or P^W, if and only if it is proportional to the identity.*

As is easily seen all above defined families of seminorms in the Hilbert space are Hilbertian. By Theorem 2.5 of [4] the following result holds immediately

Theorem 15. *Let X be a pre-Hilbert space and P one of the families P^{SO}, P^{WO}, P^S, P^W, P^{SO_*} or P^{S_*}. Then for each pair $A, B \in L(X)$ there is a P-orthogonal element in $V(A, B)$ to the operator A if and if X is finite-dimensional.*

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