Mathematica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

Mathematica
Balkanica

Now Series Vol. 3, 1989, Fasc. 2

A Geometric Question Concerning the Strong Duality of Neutral Subspaces

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Introduction

The inner product $[\cdot,\cdot]$ of a Krein space $\mathscr{K}$ can also be considered as a duality. Since $\mathscr{K}$ has a natural strong topology, one can consider the duality of two different subspaces of $\mathscr{K}$, regarded as Banach spaces. This is what we mean by strong duality; if no topology is taken into account, as in [3, I.§ 10], then we refer to weak duality.

For the case of Pontryagin spaces, two neutral subspaces $\mathscr{M}$ and $\mathscr{N}$ are called skew linked (see [6], [2]) if they are in duality (strong or weak, this is the same in this case). In this situation one can decompose the space $\mathscr{K}$ as follows

$$\mathscr{K} = \mathscr{M} + \hat{\mathscr{K}} + \mathscr{N},$$

where $\hat{\mathscr{K}}$ is a regular subspace of $\mathscr{K}$. This proved to be a useful instrument in the analysis of operators. In the case of a Krein space, if $\mathscr{M}$ and $\mathscr{N}$ are two neutral subspaces in strong duality then the above decomposition also holds (e. g. this was used in [5]).

A "special" kind of the construction described above is the situation when $\mathscr{N}$ is a neutral subspace and $\mathscr{M} = J\mathscr{N}$ where J is a fundamental symmetry of $\mathscr{K}$ (beginning with [7] this was intensively used in the literature). In this paper we show that this is the typical case, more precisely (see theorem 2.4), if $\mathscr{M}$ and $\mathscr{N}$ are two neutral subspaces in strong duality then there exists a fundamental symmetry J of $\mathscr{K}$ such that $\mathscr{M} = J\mathscr{N}$.

Our proof uses certain duality operators that we investigate in the second section. Then we are interested in getting an explicit formula for the fundamental symmetry which maps $\mathscr{M}$ onto $\mathscr{N}$, in terms of angular operators. In order to do so, in the third section we relate the duality operators with the angular operators of two maximal non-negative subspaces and then, in the fourth section, we obtain such a formula for the fundamental symmetry obtained in the proof of theorem 2.4.

The problem we considered in this paper is equivalent with solving a certain non-linear operatorial equation. Once we obtained a solution for this equation, in the last section we can find a parametrization of all solutions (see corollary 5.2).

§1. In this section we fix some terminology and notation to be used in this paper.

Let $(\mathscr{K}, [\cdot,\cdot])$ be a (complex) Krein space. If J is a fundamental symmetry (shortly f.s.) of $\mathscr{K}$ and $J=J^+-J^-$ is its Jordan decomposition then $\mathscr{K}=\mathscr{K}^++\mathscr{K}^-$ is the corresponding fundamental decomposition (shortly f.d.) of $\mathscr{K}$. The J-inner product

$$(x, y)_J=[Jx, y], \qquad x, y\in\mathscr{K},$$

determines the corresponding unitary norm $\|\cdot\|$.

The strong topology on $\mathscr{K}$ is determined by an arbitrary unitary norm. A subspace $\mathscr{L}$ of $\mathscr{K}$ is by definition a closed linear manifold of $\mathscr{K}$. We denote by

$$\mathscr{L}^\perp=\{x\in\mathscr{K}\ ;\ [x, y]=0, \qquad y\in\mathscr{L}\}$$

the orthogonal companion of $\mathscr{L}$. The subspace $\mathscr{L}$ is called regular if $\mathscr{L}+\mathscr{L}^\perp=\mathscr{K}$ holds. Non-negative subspaces, maximal non-negative subspaces or neutral subspaces are used with the usual meaning.

Throughout this paper the involution $*$ will be used only with respect to a positive definite inner product associated to a f.s. J which will be fixed in advance or will be clear from the context. Also, the term linear contraction will be used only with respect to some unitary norm.

We will use the following fact, which can be proved either by using a f.s. or by observing that in the corresponding proposition in the Hilbert space context the positive definiteness plays no role.

1.1. Proposition. *Let $\mathscr{M}$ and $\mathscr{N}$ be subspaces of the Krein space $\mathscr{K}$.* Then *$\mathscr{M}+\mathscr{N}$ is a subspace if and only if $\mathscr{M}^\perp+\mathscr{N}^\perp$ is a subspace.*

§2. Let $\mathscr{L}_1$ and $\mathscr{L}_2$ be two subspaces of the Krein space $(\mathscr{K}, [\cdot,\cdot])$. We fix on $\mathscr{K}$ a unitary norm $\|\cdot\|$ and denote by $(\cdot,\cdot)$ the corresponding positive definite inner product. Consider the Hilbert spaces $(\mathscr{L}_i, (\cdot,\cdot))$, $i=1, 2$. For any $x\in\mathscr{L}_1$ the mapping

$$\mathscr{L}_2\ni y\rightarrow[y, x]$$

is a bounded linear form on $\mathscr{L}_2$ hence the Riesz representation theorem implies the existence of a bounded linear operator $T_1\in\mathscr{L}(\mathscr{L}_1, \mathscr{L}_2)$ such that

$$[x, y]=(T_1 x, y), \qquad x\in\mathscr{L}_1, y\in\mathscr{L}_2. \tag{2.1}$$

Denote

(2.2) $$p_1(x)=\|T_1x\|=\sup_{\substack{y\in\mathcal{L}_2\\ \|y\|\leqq 1}}|[x,y]|, \qquad x\in\mathcal{L}_1.$$

Then p_1 is a semi-norm on $\mathcal{L}_1$ and if the non-negative inner product $\langle\cdot,\cdot\rangle_1$ on $\mathcal{L}_1$ is defined by

(2.3) $$\langle x,y\rangle_1=(T_1^*T_1x,y), \qquad x,y\in\mathcal{L}_1,$$

then p_1 is exactly the semi-norm associated with $\langle\cdot,\cdot\rangle_1$, i.e.

(2.4) $$p_1(x)=(\langle x,x\rangle_1)^{1/2}, \qquad x\in\mathcal{L}_1.$$

Moreover,

(2.5) $$\ker p_1=\ker T_1=\mathcal{L}_1\cap\mathcal{L}_2^{\perp}.$$

Similarly, there exists a bounded linear operator $T_2=T_1^*\in\mathcal{L}(\mathcal{L}_2,\mathcal{L}_1)$ such that

(2.6) $$[x,y]=(x,T_2y), \qquad x\in\mathcal{L}_1, y\in\mathcal{L}_2,$$

and the non-negative inner product on $\mathcal{L}_2$

(2.7) $$\langle x,y\rangle_2=(T_2^*T_2x,y)=(T_1T_1^*x,y), \qquad x,y\in\mathcal{L}_2,$$

gives rise to a semi-norm

(2.8) $$p_2(x)=\|T_2x\|=\sup_{\substack{y\in\mathcal{L}_1\\ \|y\|\leqq 1}}|[x,y]|, \qquad x\in\mathcal{L}_2,$$

which satisfies

(2.9) $$\ker p_2=\ker T_2=\mathcal{L}_2\cap\mathcal{L}_1^{\perp}.$$

From (2.2) and (2.8) it is clear that T_i are contractions, or equivalently

(2.10) $$p_i(x)\leqq\|x\|, \qquad x\in\mathcal{L}_i, \qquad i=1,2,$$

in particular the topology induced by p_i on $\mathcal{L}_i$ is weaker than the strong topology.

2.1. Proposition. *The following statements are equivalent:*

(i) *The semi-norm p_i on $\mathcal{L}_i$ is equivalent with the unitary norm $\|\cdot\|$, $i=1,2$.*

(ii) *T_1 (or equivalently, $T_2=T_1^*$) is boundedly invertible.*

(iii) $\mathcal{L}_1\cap\mathcal{L}_2^{\perp}=0$ *and* $\mathcal{L}_1+\mathcal{L}_2^{\perp}=\mathcal{K}$.

(iv) $\mathcal{L}_2\cap\mathcal{L}_1^{\perp}=0$ *and* $\mathcal{L}_2+\mathcal{L}_1^{\perp}=\mathcal{K}$.

Proof. (i)$\Rightarrow$(ii) Taking account of (2.10) it follows that the statement (i) is equivalent with the existence of $\alpha>0$ such that

$$(2.11) \qquad \alpha \| x \| \leqq p_i(x), \qquad x \in \mathscr{L}_i, \qquad i=1, 2.$$

But (2.11) for $i=1$ means that T_1 is one-to-one and has closed range while the same inequality for $i=2$ yields $T_2 = T_1^*$ one-to-one, hence T_1 has dense range.

(ii)⇒(iii) If T_1 is boundedly invertible then $\mathscr{L}_1 \cap \mathscr{L}_2^{\perp} = 0$ follows from (2.5). Let now $z \in \mathscr{K}$ be arbitrary. Considering the linear form

$$\mathscr{L}_2 \ni y \to [y, z]$$

we get a vector $t \in \mathscr{L}_2$ such that

$$[y, z] = (y, t), \qquad y \in \mathscr{L}_2.$$

Taking $z_1 = T_1^{-1} t \in \mathscr{L}_1$ by (2.1) it follows that

$$[y, z] = [y, z_1], \qquad y \in \mathscr{L}_2,$$

hence $z_2 = z - z_1 \in \mathscr{L}_2$. Thus we have proved $\mathscr{L}_1 + \mathscr{L}_2^{\perp} = \mathscr{K}$.

(iii)⇒(iv) If $\mathscr{L}_1 \cap \mathscr{L}_2^{\perp} = 0$ holds then, considering the orthogonal complement, it follows that the linear manifold $\mathscr{L}_2 + \mathscr{L}_1^{\perp}$ is dense in $\mathscr{K}$. From $\mathscr{L}_1 + \mathscr{L}_2^{\perp} = \mathscr{K}$ we first obtain $\mathscr{L}_2 \cap \mathscr{L}_1^{\perp} = 0$ and then by proposition 1.1 it follows that $\mathscr{L}_2 + \mathscr{L}_1^{\perp}$ is closed, hence $\mathscr{L}_2 + \mathscr{L}_1^{\perp} = \mathscr{K}$.

(iv)⇒(i) Let P_2 be the projection of $\mathscr{K}$ onto $\mathscr{L}_2$ along $\mathscr{L}_1^{\perp}$. By assumption P_2 is bounded hence for any $x \in \mathscr{L}_1$ we have

$$\| x \| = \sup_{\| y \| \leqq 1} |[x, y]| \leqq \sup_{\substack{y \in \mathscr{L}_2 \\ \| y \| \leqq \| P_2 \|}} |[x, y]| \leqq \| P_2 \| \cdot p_1(x).$$

We have to remark now that by proving (iii)⇒(iv) we proved also (iv)⇒(iii) interchange the roles of $\mathscr{L}_1$ and $\mathscr{L}_2$). Thus (2.11) holds with

$$\alpha = \min \{1/\| P_1 \|, 1/\| P_2 \|\},$$

where P_1 denotes the projection of $\mathscr{K}$ onto $\mathscr{L}_1$ along $\mathscr{L}_2^{\perp}$ (of course, the case $\mathscr{L}_1 = \mathscr{L}_2 = 0$ should be treated separately). ■

2.2. Definition. The subspaces $\mathscr{L}_1$ and $\mathscr{L}_2$ are in strong duality with respect to the inner product $[\cdot, \cdot]$, or equivalently, $\mathscr{L}_1$ and $\mathscr{L}_2$ form a strongly dual pair, if one (hence all) of the statements from proposition 2.1 is (are) satisfied.

Clearly, strong duality of subspaces does not depend on the particular unitary norm that we considered, it depends only on the strong topology of $\mathscr{K}$. Also, if $\mathscr{L}_1$ and $\mathscr{L}_2$ are two subspaces in strong duality then their dimensions (as Hilbert spaces) coincide. The subspaces $\mathscr{L}_1$ and $\mathscr{L}_2$ form a strongly dual pair if and only if $\mathscr{L}_1^{\perp}$ and $\mathscr{L}_2^{\perp}$ form a strongly dual pair.

If the subspaces $\mathscr{L}_1$ and $\mathscr{L}_2$ are in strong duality then they are also in weak duality. If $\mathscr{L}_1$ and $\mathscr{L}_2$ are either of finite dimension or of finite codimension then their strong duality is equivalent with their weak duality.

Let $\mathscr{L}$ be a subspace of $\mathscr{K}$. Then $\mathscr{L}$ is a regular subspace if and only if $\mathscr{L}$ is in strong duality with itself. In connection with this fact we note that if $\mathscr{L}=\mathscr{L}_1=\mathscr{L}_2$ then the operator T_1 defined at (2.1) is the Gram operator of the inner product space $(\mathscr{L}, [\cdot,\cdot])$ with respect to the positive definite inner product $(\cdot,\cdot)$.

2.3. Example. Let J be a f.s. of the Krein space $\mathscr{K}$ and $\mathscr{N}$ an arbitrary subspace. Then $\mathscr{N}$ and $J\mathscr{N}$ form a strongly dual pair of subspaces. Here we want to point out that, in general, strongly dual pairs of subspaces are not of this kind.

If $\mathscr{M}$ is a maximal uniformly positive subspace of $\mathscr{K}$ and $\mathscr{N}$ is an arbitrary maximal non-negative subspace then $\mathscr{M}\cap\mathscr{N}^{\perp}=0$ and $\mathscr{M}+\mathscr{N}^{\perp}\emptyset\mathscr{K}$ (e.g., by [1, Corollary 1.5.2]). But if $\mathscr{N}$ is degenerate then $\mathscr{M}\neq J\mathscr{N}$ for any f.s. J of K, since $J\mathscr{M}$ is always maximal uniformly positive.

2.4. Theorem. *Let $\mathscr{L}_1$ and $\mathscr{L}_2$ be two neutral subspaces of the Krein space $\mathscr{K}$. The following statements are equivalent:*

(a) *$\mathscr{L}_1$ and $\mathscr{L}_2$ are in strong duality.*

(b) *$\mathscr{L}_1+\mathscr{L}_2$ (the algebraic sum) is a regular subspace.*

(c) *There exists a f.s. S of K such that $\mathscr{L}_2=S\mathscr{L}_1$ (or, equivalently, $\mathscr{L}_1=S\mathscr{L}_2$).*

Proof. (a)⇒(b) Assume that the neutral subspaces $\mathscr{L}_1$ and $\mathscr{L}_2$ are in strong duality. From proposition 2.1 we have $\mathscr{L}_1\cap\mathscr{L}_2^{\perp}=\mathscr{L}_2\cap\mathscr{L}_1^{\perp}=0$ and

$$\mathscr{L}_1+\mathscr{L}_2^{\perp}=\mathscr{L}_2+\mathscr{L}_1^{\perp}=\mathscr{K}. \tag{2.12}$$

Since $\mathscr{L}_1\subseteq\mathscr{L}_1^{\perp}$, from (2.12) we get $\mathscr{L}_1^{\perp}+\mathscr{L}_2^{\perp}=\mathscr{K}$, hence by proposition 1.1 it follows that $\mathscr{L}_1+\mathscr{L}_2$ is closed. On the other hand, from (2.12) it is easy to derive the equality

$$\mathscr{L}_1^{\perp}=\mathscr{L}_1+\mathscr{L}_1^{\perp}\cap\mathscr{L}_2^{\perp} \tag{2.13}$$

hence

$$\mathscr{L}_1+\mathscr{L}_2+\mathscr{L}_1^{\perp}\cap\mathscr{L}_2^{\perp}=\mathscr{K}, \tag{2.14}$$

which shows that the subspace $\mathscr{L}_1+\mathscr{L}_2$ is regular.

(b)⇒(a). If $\mathscr{L}_1+\mathscr{L}_2$ is a regular subspace then the representation (2.14) holds and from here it follows easily (2.13), hence $\mathscr{L}_2\cap\mathscr{L}_1^{\perp}=0$ holds and using once more (2.14) we get $\mathscr{L}_1+\mathscr{L}_2^{\perp}=\mathscr{K}$, i.e. $\mathscr{L}_1$ and $\mathscr{L}_2$ form a strogly dual pair.

(b)⇒(c). Let the subspace $\mathscr{L}=\mathscr{L}_1+\mathscr{L}_2$ be regular. From above we know that $\mathscr{L}_1$ and $\mathscr{L}_2$ are in strong duality, in particular the operators T_1 and T_2 defined at (2.1) and (2.6) are boundedly invertible. We define a linear operator G on $\mathscr{L}$ by

(2.15) $$G=\begin{bmatrix} 0 & T_1^{-1} \\ T_2^{-1} & 0 \end{bmatrix} \quad \text{w.r.t. } \mathscr{L}=\mathscr{L}_1+\mathscr{L}_2.$$

On $\mathscr{L}$ we consider a positive definite inner product $\langle\cdot,\cdot\rangle$ defined by

(2.16) $$\langle x_1+x_2, y_1+y_2\rangle=\langle x_1, y_1\rangle_1+\langle x_2, y_2\rangle_2, \qquad x_i, y_i\in\mathscr{L}_i, \qquad i=1, 2,$$

where the positive definite inner products $\langle\cdot,\cdot\rangle_i$, $i=1, 2$ are defined at (2.3) and (2.7). The corresponding norm is

$$s(x_1+x_2)=(p_1(x_1)^2+p_2(x_2)^2)^{1/2}, \qquad x_i\in\mathscr{L}_i, \qquad i=1, 2.$$

$\mathscr{L}_1$ and $\mathscr{L}_2$ are orthogonal with respect to the inner product $\langle\cdot,\cdot\rangle$ and the operator G is continuous with respect to s, or equivalently to $\|\cdot\|$.

Let $x_1, y_1\in\mathscr{L}_1$ and $x_2, y_2\in\mathscr{L}_2$ be arbitrary. Then

$$\langle G(x_1+x_2), y_1+y_2\rangle=\langle T_2^{-1}x_1+T_1^{-1}x_2, y_1+y_2\rangle$$
$$=\langle T_1^{-1}x_2, y_1\rangle_1+\langle T_2^{-1}x_1, y_2\rangle_2=(T_1^*T_1T_1^{-1}x_2, y_1)+(T_2^*T_2T_2^{-1}x_1, y_2)$$
$$=(T_1^*x_2, y_1)+(T_2^*x_1, y_2)=[x_2, y_1]+[x_1, y_2]=[x_1+x_2, y_1+y_2],$$

which means

(2.17) $$\langle Gx, y\rangle=[x, y], \qquad x, y\in\mathscr{L},$$

hence G is the Gram operator of the inner product space $(\mathscr{L}, [\cdot,\cdot])$ with respect to $\langle\cdot,\cdot\rangle$. In particular, G is a selfadjoint operator on the Hilbert space $(\mathscr{L}, \langle\cdot,\cdot\rangle)$.

With respect to the Hilbert spaces $(\mathscr{L}_i, \langle\cdot,\cdot\rangle)$, $i=1, 2$ we consider the left polar decompositions

$$T_i^{-1}=U_i|T_i^{-1}|, \qquad i=1, 2.$$

Then it is easy to see that

$$G=\begin{bmatrix} 0 & U_1 \\ U_2 & 0 \end{bmatrix} \begin{bmatrix} |T_2^{-1}| & 0 \\ 0 & |T_1^{-1}| \end{bmatrix}$$

is the left polar decomposition of G with respect to the Hilbert space $(\mathscr{L}, \langle\cdot,\cdot\rangle)$. Denote

$$S=\begin{bmatrix} 0 & U_1 \\ U_2 & 0 \end{bmatrix}, \quad |G|=\begin{bmatrix} |T_2^{-1}| & 0 \\ 0 & |T_1^{-1}| \end{bmatrix}.$$

Then, (2.17) can be written

$$\langle S|G|x, y\rangle = [x, y], \qquad x, y \in \mathscr{L},$$

and since G, hence also $|G|$, are boundedly invertible on $\mathscr{L}$ it follows that S is a f. s. of the Krein space $(\mathscr{L}, [\cdot, \cdot])$. Also $\mathscr{L}_2 = S\mathscr{L}_1$ and S can be extended to a f. s. of $\mathscr{K}$. ■

(c)⇒(a) Obvious.

§3. Let $\mathscr{K} = \mathscr{K}^+ + \mathscr{K}^-$ be a f. d. of the Krein space $\mathscr{K}$ and $\mathscr{L}_1$ and $\mathscr{L}_2$ two maximal non-negative subspaces of $\mathscr{K}$. We let K_1 and respectively K_2 denote the angular operators of the subspaces $\mathscr{L}_1$ and $\mathscr{L}_2$ with respect to this f. d.. Recall that for $i=1, 2$, $K_i \in \mathscr{L}(\mathscr{K}^+, \mathscr{K}^-)$ is the unique linear contraction satisfying

$$\mathscr{L}_i = \{x + K_i x \mid x \in \mathscr{K}^+\}. \tag{3.1}$$

Then, recall also

$$\mathscr{L}_i^\perp = \{y + K_i^* y \mid y \in \mathscr{K}^-\}, \qquad i=1, 2. \tag{3.2}$$

The next lemma is a result from [2, Theorem I.8.15]. We give a short proof.

3.1. Lemma. *The maximal non-negative subspaces $\mathscr{L}_1$ and $\mathscr{L}_2$ are in strong duality if and only if the operator $I_+ - K_2^* K_1$ (or, equivalently, the operator $I_- - K_1 K_2^*$) is invertible in $\mathscr{L}(\mathscr{K}^+)$ (respectively, in $\mathscr{L}(\mathscr{K}^-)$).*

Proof. Let us consider the linear operator $X \in \mathscr{L}(\mathscr{K})$ defined by

$$X = \begin{bmatrix} I_+ & K_2^* \\ K_1 & I_- \end{bmatrix} \quad \text{w. r. t. } \mathscr{K} = \mathscr{K}^+ + \mathscr{K}^-. \tag{3.3}$$

From (3.1) and (3.2) follow $\mathscr{L}_1 \cap \mathscr{L}_2^\perp = J \ker X$ and $\mathscr{L}_1 + \mathscr{L}_2^\perp = \mathscr{R}(X)$, where J is the f. s. associated to the f. d. $\mathscr{K} = \mathscr{K}^+ + \mathscr{K}^-$ and $\mathscr{R}(X)$ denotes the range of X. Taking account of Proposition 2.1 (iii), from here we infer that $\mathscr{L}_1$ and $\mathscr{L}_2$ are in strong duality if and only if X is invertible in $\mathscr{L}(\mathscr{K})$. But from the factorization

$$X = \begin{bmatrix} I_+ & K_2^* \\ 0 & I \end{bmatrix} \begin{bmatrix} I_+ - K_2^* K_1 & 0 \\ 0 & I_- \end{bmatrix} \begin{bmatrix} I_+ & K_1 \\ 0 & I_- \end{bmatrix} \tag{3.4}$$

it follows that X is invertible in $\mathscr{L}(\mathscr{K})$ if and only if $I_+ - K_2^* K_1$ is invertible in $\mathscr{L}(\mathscr{K}^+)$. ■

We consider now the operators T_1 and T_2 defined at (2.1) and (2.6) and we want to calculate these operators in terms of the angular operators K_1 and K_2.

3.2. Proposition. *Let $\mathscr{L}_1$ and $\mathscr{L}_2$ be two maximal non-negative subspaces of $\mathscr{K}$ and K_1 and respectively K_2 their angular operators. An operator $\tilde{T}_1 \in \mathscr{L}(\mathscr{K})$ is an extension of the operator $T_1 \in \mathscr{L}(\mathscr{L}_1, \mathscr{L}_2)$ defined at (2.1) if and only if it has the following block-matrix representation*

$$(3.5)\qquad \tilde{T}_1=\begin{bmatrix}(I_+ + K_2^* K_2)^{-1}(I_+ - K_2^* K_1)-\Gamma_1 K_1 & \Gamma_1\\ K_2(I_+ + K_2^* K_2)^{-1}(I_+ - K_2^* K_1)-\Gamma_2 K_1 & \Gamma_2\end{bmatrix},$$

where $\Gamma_1\in\mathscr{L}(\mathscr{K}^-,\mathscr{K}^+)$ and $\Gamma_2\in\mathscr{L}(\mathscr{K}^-)$ are arbitrary.

Proof. Let $\tilde{T}_1\in\mathscr{L}(K)$ be an extension of T_1, i.e. $\tilde{T}_1|\mathscr{L}_1=T_1$. This means

$$(3.6)\qquad \tilde{T}_1\mathscr{L}_1\subseteq\mathscr{L}_2$$

and

$$(3.7)\qquad (\tilde{T}_1 x, y)=[x, y],\qquad x\in\mathscr{L}_1,\qquad y\in\mathscr{L}_2,$$

where $(\cdot,\cdot)$ denotes the positive definite inner product corresponding to the f. d. $\mathscr{K}=\mathscr{K}^+ + \mathscr{K}^-$. Let us consider the block-matrix representation of $\tilde{T}_1$

$$(3.8)\qquad \tilde{T}_1=\begin{bmatrix}A & B\\ C & D\end{bmatrix}\ \text{w.r.t. } \mathscr{K}=\mathscr{K}^+ + \mathscr{K}^-.$$

From (3.6) we get

$$C+DK_1=K_2(A+BK_1)$$

and then from (3.7) it follows

$$(3.9)\qquad (A+BK_1)=(I_+ + K_2^* K_2)^{-1}(I_+ - K_2^* K_1),$$

hence denoting $\Gamma_1=B$ and $\Gamma_2=D$ we obtain the formula (3.5). ■

3.3. Corollary. *With the notation from proposition 3.2, the operator $\tilde{T}_2\in\mathscr{L}(\mathscr{K})$ is an extension of the operator T_2 defined at (2.6) if and only if it has the following block-matrix representation*

$$(3.10)\qquad \tilde{T}_2=\begin{bmatrix}(I_+ + K_1^* K_1)^{-1}(I_+ - K_1^* K_2)-\Delta_1 K_2 & \Delta_1\\ K_1(I_+ + K_1^* K_1)^{-1}(I_+ - K_1^* K_2)-\Delta_2 K_2 & \Delta_2\end{bmatrix},$$

where $\Delta_1\in\mathscr{L}(\mathscr{K}^-,\mathscr{K}^+)$ and $\Delta_2\in\mathscr{L}(\mathscr{K}^-)$ are arbitrary.

3.4. Remark. Let us consider the notation from proposition 3.2. The operators T_1 and T_2 are related by the identity $T_2^*=T_1$, so it is natural to ask if there exists any extension of T_1 such that its adjoint is an extension of T_2. A straightforward calculation shows that if $\tilde{T}_1\in\mathscr{L}(\mathscr{K})$ is represented by (3.5) then $\tilde{T}_1^*$ is an extension of T_2 if and only if

$$(3.11)\qquad \Gamma_1=(I_+ - K_2^* K_1)(I_+ + K_1^* K_1)^{-1}K_1^* - K_2^*\Gamma_2.$$

3.5. Remark. Assume that the maximal non-negative subspaces $\mathscr{L}_1$ and $\mathscr{L}_2$ are in strong duality. Then, from proposition 2.1 (ii) we know that T_1 is invertible.

Let $\tilde{T}_1$ be an extension of T_1 and assume that $\tilde{T}_1$ is also invertible (in $\mathscr{L}(\mathscr{K})$). Then $\tilde{T}_1^{-1}$ is an extension of the operator T_1^{-1}.

In order to prove this let K_1 and K_2 be the angular operators of $\mathscr{L}_1$ and respectively $\mathscr{L}_2$. From lemma 3.1 we know that $I_+ - K_2^* K_1$ is invertible in $\mathscr{L}(\mathscr{K}^+)$. If $\tilde{T}_1$ is represented by (3.8) then from (3.9) it follows that $\tilde{T}_1 \mathscr{L}_1 = \mathscr{L}_2$ hence $\tilde{T}_1^{-1}$ is an extension of T_1^{-1}.

§4. Let $\mathscr{L}_1$ and $\mathscr{L}_2$ be two neutral subspaces of the Krein space $\mathscr{K}$. Assuming that $\mathscr{L}_1$ and $\mathscr{L}_2$ are in strong duality then, it follows from the theorem 2.4(b) that $\mathscr{L}_1 + \mathscr{L}_2$ is a regular subspace, hence we can suppose, without restricting the generality, that $\mathscr{L}_1 + \mathscr{L}_2 = \mathscr{K}$. Then, it is easy to see that $\mathscr{L}_i^{\perp} = \mathscr{L}_i$, $i = 1, 2$, i.e. $\mathscr{L}_1$ and $\mathscr{L}_2$ are *hypermaximal neutral* subspaces of $\mathscr{K}$ (cf. [3]). If $\mathscr{K} = \mathscr{K}^+ + \mathscr{K}^-$ is a fixed f.d. and K_1 and K_2 denote the angular operators of $\mathscr{L}_1$ and respectively $\mathscr{L}_2$, with respect to this f.d., then K_1 and K_2 are unitary operators, in the sense $K_i^* K_i = I_+$, $K_i K_i^* = I_-$, $i = 1, 2$. Also, from lemma 3.1 we know that the operators $I_+ - K_1^* K_2$, $I_+ - K_2^* K_1 \in \mathscr{L}(\mathscr{K}^+)$ and $I_- - K_1 K_2^*$, $I_- - K_2 K_1^* \in \mathscr{L}(\mathscr{K}^-)$ are all invertible.

Consider now the operator G defined at (2.15) and denote $F = G^{-1}$, i.e.

$$F = \begin{bmatrix} 0 & T_2 \\ T_1 & 0 \end{bmatrix} \quad \text{w.r.t. } \mathscr{K} = \mathscr{L}_1 + \mathscr{L}_2,$$

in particular F is an extension of both T_1 and T_2. Taking account of proposition 3.2 and corollary 3.3, it follows by simple calculations that, representing F as in (3.5), we must have

$$\Gamma_1 = \frac{1}{2}(K_1^* + K_2^*), \qquad \Gamma_2 = -I_-,$$

hence

$$(4.1)\quad F = \begin{bmatrix} I_+ & -\frac{1}{2}(K_1^* + K_2^*) \\ \frac{1}{2}(K_1 + K_2) & -I_- \end{bmatrix} \quad \text{w.r.t. } \mathscr{K} = \mathscr{K}^+ + \mathscr{K}^-.$$

Now the block-matrix representation of G with respect to the f.d. $\mathscr{K} = \mathscr{K}^+ + \mathscr{K}^-$ can be calculated quite easy (e.g. using a factorization of F similar to that used in (3.4)) but since the formula is a bit longer and we will not use it, we leave this to the reader.

Further, according to the proof of the theorem 2.4, we have to consider the positive definite inner product $\langle \cdot, \cdot \rangle$

$$\langle x, y \rangle = (JFx, y), \qquad x, y \in \mathscr{K},$$

where J is the f. s. corresponding to the f. d. $\mathscr{K}=\mathscr{K}^+ + \mathscr{K}^-$ and $(\cdot,\cdot)$ is the positive definite inner product determined by J. The polar decomposition of G with respect to the Hilbert space $(\mathscr{K}, \langle\cdot,\cdot\rangle)$ will produce a f. s. S such that $S\mathscr{L}_1=\mathscr{L}_2$. But, doing so, we encounter the obstruction of calculating a square-root. However, we can find an explicit formula for S, using a geometric reasoning as follows.

The block-matrix representation of the f. s. S is

$$(4.2)\quad S=\begin{bmatrix}(I_+-K^*K)^{-1/2} & -K^*(I_--KK^*)^{-1/2}\\ K(I_+-K^*K)^{-1/2} & -(I_--KK^*)^{-1/2}\end{bmatrix} \quad \text{w. r. t. } \mathscr{K}=\mathscr{K}^+ + \mathscr{K}^-,$$

(e. g. see [4, Proposition 4.5]), where K is the angular operator of the maximal uniformly positive subspace $S\mathscr{K}^+$. On the other hand, the f. s. S is also produced by the polar decomposition of F with respect to the Hilbert space $(\mathscr{K}, \langle\cdot,\cdot\rangle)$. Since, from (4.1), the geometric interpretation of F is that it changes the coordinates $\mathscr{K}=\mathscr{K}^+ + \mathscr{K}^-$ into the coordinates $\mathscr{K}=\mathscr{M}+\mathscr{M}^\perp$, where $\mathscr{M}=G(K)$ with

$$(4.3)\qquad K=\frac{1}{2}(K_1+K_2),$$

this suggests that $S\mathscr{K}^+=\mathscr{M}$. Inserting K given at (4.3) in (4.4), and taking account of

$$(4.4)\qquad I_+-K^*K=\frac{1}{4}(K_1^*-K_2^*)(K_1-K_2)=\frac{1}{4}(I_+-K_2^*K_1)(I_+-K_1^*K_2)$$

$$=\frac{1}{4}(I_+-K_1^*K_2)(I_+-K_2^*K_1),$$

and similarly

$$(4.5)\qquad I_--KK^*=\frac{1}{4}(K_1-K_2)(K_1^*-K_2^*)=\frac{1}{4}(I_--K_2K_1^*)(I_--K_1K_2^*)$$

$$=\frac{1}{4}(I_--K_1K_2^*)(I_--K_2K_1^*),$$

we get

$$(4.6)\quad S=\begin{bmatrix}2|K_1-K_2|^{-1} & -(K_1^*+K_2^*)\,|K_1^*-K_2^*|^{-1}\\ (K_1+K_2)\,|K_1-K_2|^{-1} & -|K_1^*-K_2^*|^{-1}\end{bmatrix}.$$

$$\text{w. r. t. } \mathscr{K}=\mathscr{K}^+ + \mathscr{K}^-$$

Before starting to prove that the operator S given at (4.6) is indeed the f.s. produced by the polar decomposition of F, let us recall the well-known "defect relations"

$$K(I_+ - K^* K)^{1/2} = (I_- - KK^*)^{1/2} K, \qquad K^*(I_- - KK^*)^{1/2} = (I_+ - K^* K)^{1/2} K^*,$$

which in our case yield, via (4.4) and (4.5), the following identities

$$(4.7) \qquad (K_1 + K_2)\,|K_1 - K_2|^{-1} = |K_1^* - K_2^*|^{-1}(K_1 + K_2),$$

$$(4.8) \qquad (K_1^* + K_2^*)\,|K_1^* - K_2^*|^{-1} = |K_1 - K_2|^{-1}(K_1^* + K_2^*).$$

Now, S is isometric with respect to the inner product $\langle \cdot, \cdot \rangle$ if and only if

$$(4.9) \qquad S^* JFS = JF.$$

Since S is a symmetry in the Krein space $\mathcal{K}$, i.e. $JS^* J = S^{-1} = S$ it follows that (4.9) is equivalent with

$$(4.10) \qquad FS = SF.$$

Making use of (4.7) and (4.8) the proof of (4.10) is immediate. Also, in order to prove that SF is positive with respect to the inner product $\langle \cdot, \cdot \rangle$ we must prove that JFSF is positive with respect to $(\cdot, \cdot)$. But, a direct calculation gives

$$JFSF = \begin{bmatrix} 2|K_1 - K_2| & -|K_1 - K_2|(K_1^* + K_2^*) \\ -|K_1^* - K_2^*|(K_1 + K_2) & 2|K_1^* - K_2^*| \end{bmatrix} \text{ w.r.t. } \mathcal{K} = \mathcal{K}^+ + \mathcal{K}_-,$$

hence, the positivity of $JFSF$ is equivalent with the positivity of the following operator

$$2|K_1^* - K_2^*| - \frac{1}{2}|K_1^* - K_2^*|(K_1 + K_2)|K_1 - K_2|^{-1}|K_1 - K_2|(K_1^* + K_2^*)$$

$$= 2|K_1^* - K_2^*|[I_- - \frac{1}{4}(K_1 + K_2)(K_1^* + K_2^*)] = \frac{1}{2}|K_1^* - K_2^*|^{3/2},$$

which is clear.

From what we have proved above and from the proof of the implication (b)⇒(c) in theorem 2.4 we conclude that the f.s. S given at (4.6) satisfies $S\mathcal{L}_1 = \mathcal{L}_2$.

§5. In this section we continue to keep the notation and the assumptions considered in section four. It is easy to see that the f.s. S given in the block representation (4.2) satisfies $S\mathcal{L}_1 = \mathcal{L}_2$ if and only if the following identity holds

$$(5.1) \quad K_2(I_+ - K^* K)^{-1/2}(I_+ - K^* K_1) = (I_- - KK^*)^{-1/2}(K - K_1).$$

Thus, the problem of finding the f. s. which maps $\mathscr{L}_1$ onto $\mathscr{L}_2$ is equivalent with the problem of solving the operatorial non-linear equation (5.1), where the solutions $K \in \mathscr{L}(\mathscr{K}^+, \mathscr{K}^-)$ are required to be uniform contractions, i. e. $\|K\| < 1$. In the course of the preceding section we found a solution of the equation (5.1), this is the arithmetic mean of the unitary operators K_1 and K_2 (see (4.3)), provided that the hypermaximal subspaces $\mathscr{L}_1$ and $\mathscr{L}_2$ are in strong duality.

We are now interested in finding all the solutions of the equation (5.1), or equivalently in describing all the f. s. S which maps $\mathscr{L}_1$ onto $\mathscr{L}_2$. From theorem 2.4 it follows that we can choose the f. s. J, which was fixed at the beginning of section four, such that $J\mathscr{L}_1 = \mathscr{L}_2$. Assuming this it follows $K_1 = -K_2$. Denoting $U = K_1 = -K_2$. $U \in \mathscr{L}(\mathscr{K}^+, \mathscr{K}^-)$ unitary operator, the equation (5.1) becomes

$$U(I_+ - K^*K)^{-1/2}(K^* - U^*)U = (I_- - KK^*)^{-1/2}(K - U). \tag{5.2}$$

In the following, we use the convention: if $A, B \in \mathscr{L}(\mathscr{H})$, $\mathscr{H}$ Hilbert space, then we write $A > B$ if $A \geqq B$ and $A - B$ is invertible in $\mathscr{L}(\mathscr{H})$.

5.1. Proposition. *Let $U \in \mathscr{L}(\mathscr{K}^+, \mathscr{K}^-)$ be unitary operator. Then, the identity*

$$K = AU \tag{5.3}$$

establishes a bijective correspondence between the class of all solutions of equation (5.2) and the class of operators $A \in \mathscr{L}(\mathscr{K}^-)$ such that $-I_- < A < I_-$.

Proof. Let $K \in \mathscr{L}(\mathscr{K}^+, \mathscr{K}^-)$, $\|K\| < 1$ be a solution of (5.2). Using the defect relations for K it follows that

$$U(I - K^*K)^{-1/2}(K^* + U^*)U = (I - KK^*)^{-1/2}(K + U) \tag{5.4}$$

also holds. Subtracting (5.3) from (5.4) we get

$$U(I_+ - K^*K)^{-1/2} = (I_- - KK^*)^{-1/2}U \tag{5.5}$$

and using this in (5.2) we obtain

$$UK^* = KU^*. \tag{5.6}$$

Let us denote $A = KU^* \in \mathscr{L}(\mathscr{K}^-)$, equivalently $K = AU$. Using this in (5.6) it follows $A = A^*$ and since K is uniform contraction it follows that A is a uniform contraction, hence $-I_- < A < I_-$.

Conversely, let $A \in \mathscr{L}(\mathscr{K}^-)$ be such that $-I_- < A < I_-$ and set $K = AU$. Then K is a uniform contraction and

$$(I_+ - K^*K)^{1/2} = U^*(I_- - A^2)^{1/2}U, \tag{5.7}$$

$$(I_- - KK^*)^{1/2} = (I_- - A^2)^{1/2}. \tag{5.8}$$

Using these identities it follows immediately that K is a solution of (5.2). ■

5.2. Corollary. *Let the hypermaximal neutral subspaces $\mathscr{L}_1$ and $\mathscr{L}_2$ be in duality and, via theorem 2.4, assume the f.s. J satisfies $J\mathscr{L}_1=\mathscr{L}_2$. Considering the f.d. $\mathscr{K}=\mathscr{K}^+ + \mathscr{K}^-$ associated to this f.s. let U be the angular operator of $\mathscr{L}_1$. Then, the formula*

$$(5.9)\quad S=\begin{bmatrix} U^*(I_- - A^2)^{-1/2}U & -U^*A(I-A^2)^{-1/2} \\ A(I_- - A^2)^{-1/2}U & -(I_- - A^2)^{-1/2} \end{bmatrix} \quad \text{w.r.t. } \mathscr{K}=\mathscr{K}^+ + \mathscr{K}^-$$

establishes a bijective correspondence between the class of all f.s. S mapping $\mathscr{L}_1$ onto $\mathscr{L}_2$ and the class of operators $A\in\mathscr{L}(\mathscr{K}^-)$ such that $-I_- < A < I_-$.

P r o o f. This is a consequence of (4.2), proposition 5.1 and the identities (5.7) and (5.8). ■

5.3. R e m a r k. Clearly, an equivalent parametrization of the solutions of (5.2) is obtained by means of the formula

$$K=UB,$$

where $B\in\mathscr{L}(\mathscr{K}^+)$ satisfies $-I_+ < B < I_+$.

5.4. R e m a r k. Let us come back to the original setting (i.e. we do not assume $J\mathscr{L}_1=\mathscr{L}_2$). In this case we still can describe a rich set of solutions for the equation (5.1), but, in general, this is not the set of all solutions. In order to do this, let us assume $\mathscr{H}=\mathscr{K}^+=\mathscr{K}^-$ (since $\mathscr{K}^+$ and $\mathscr{K}^-$ are unitary equivalent this is no restriction). Then the angular operatore K_1, $K_2\in\mathscr{L}(\mathscr{H})$ are unitary and K_1-K_2 is invertible. It is easy to verify that for any $A\in\mathscr{L}(\mathscr{H})$, such that $0<A<I$ and $AK_i=K_iA$, $i=1, 2$ the operator

$$(5.10)\quad K=AK_1+(I-A)K_2$$

is a solution of (5.1). From what was proved in section four it follows that the f.s. obtained in the proof of theorem 2.4 corresponds to the choice $A=1/2\,I$.

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Received 11. 10. 1988

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