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One-Sided Multidimensional Approximation by Entire Functions and Trigonometric Polynomials in L_p-Metric, $0<p\leqq\infty$

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1. Introduction

One-sided approximation of functions was first considered by G. Freud [1], T. Ganelius [2]. They obtained the first nontrivial estimates of the best one-sided polynomial and spline approximation. Using the so-called averaged moduli of smoothness Bl. H. Sendov, V. A. Popov, and A. S. Andreev [4, 5, 6] obtained direct (Jackson's type) and converse (Steckin's type) theorems for the one-sided approximation in the one-dimensional case in L_p-metric, $1\leqq p\leqq\infty$. Using a multidimensional analogue of the averaged moduli of smoothness, introduced by V. A. Popov [7], direct and converse multidimensional theorems for one-sided trigonometric polynomials and entire functions approximation are proved in L_p-metric, $1\leqq p\leqq\infty$, [8], [9].

In the case L_p, $0<p<1$, the first one-dimensional result belongs to A. Shadrin [10]. This result was proved independently in [11] using another method.

The purpose of this paper is to consider the multidimensional analogues of A. Shadrin's operator [10]. Direct theorems for the best one-sided approximation in L_p-metric, $0<p\leqq\infty$ are proved. Isotopical and anisotropical cases are considered.

2. Definitions and notations

Let R^d be the d-dimensional Euclidean spaces and $\Omega\equiv\mathrm{R}^d$ or $\Omega\equiv\Pi^d=\{x\in\mathrm{R}^d: 0\leqq x_i\leqq 2\pi,\ i=1,\dots,d\}$, d-dimensional cube. For $0<p\leqq\infty$ we denote with $M_p(\Omega)$ (or M_p) the set of all measurable and bounded functions defined on R^d (in the case $\Omega=\Pi^d$ these functions are 2π-periodic with respect to each variable) for which

$$\|f\|_{L_p(\Omega)} := \|f\|_p = \left(\int_\Omega |f(x)|^p dx \right)^{1/p} < \infty, \qquad 0<p<\infty$$

and

$$\|f\|_{L_\infty(\Omega)} := \|f\|_\infty = \sup_{x\in\Omega} |f(x)|.$$

Definition 2.1.
For $f\in M_p$, $k\in\mathrm{N}$, $x\in\mathrm{R}^d$, $\delta\in\mathrm{R}^{d,+}=\{y\in\mathrm{R}^d : y_i\geqq 0, \quad i=1,\dots,d\}$ the k-th local modulus of smoothness is defined by

$$\omega_k(f,x,\delta)=\sup\{|\Delta_h^k f(t)| : t, t+kh\in \mathrm{K}_{\frac{k\delta}{2}}(x)\},$$

where

$$\text{(1)} \qquad \Delta_h^k f(t) = \sum_{m=0}^{k} (-1)^{k+m} \binom{k}{m} f(t+mh)$$

and

$$\mathrm{K}_{\frac{k\delta}{2}} = \{y\in\mathrm{R}^d : |y_i - x_i| \leqq k\delta_i/2, \qquad i=1,\dots,d\}$$

Definition 2.2. The k^{th} averaged modules of smoothness for $f\in M_p$ is defined by

$$\text{(2)} \qquad \tau_k(f,\delta)_p = \|\omega_k(f,\cdot,\ \delta)\|_{L_p(\Omega)}.$$

In the case $\delta=(\delta_1,\dots,\delta_d) : \delta_1=\delta_2=\dots=\delta_d$ the modulus (2) coincides with the well-known modulus introduced by Bl. Sendov [12] and P. Korovkin [13] in the one-dimensional case and by V. A. Popov [7] in the multidimensional case. For details see [3].

Here we consider $\tau_k(f,\delta)_p$ in the case $\delta\in\mathrm{R}^{d,+}$.

Proposition 2.1. *The following properties hold:*

(3) 1) $\omega_k(f,x,\delta)\geqq 0$, $\omega_k(f,\cdot\ \delta)\in M_\infty$;

2) If $\delta\geqq\delta'$, i.e. $\delta_i\geqq\delta_i'$, $i=1,\dots,d$, *then* $\omega_k(f,\ x,\ \delta)\geqq\omega_k(f,\ x,\ \delta')$ and $\tau_k(f,\delta)_p\geqq\tau_k(f,\delta')_p$;

3) $\omega_k(f,x+y,\delta)\leqq\omega_k(f,x,\delta+\frac{2}{k}|y|)$, *where* $|y|=(|y_1|,\dots,|y_d|)$;

4) Let $\lambda>0$, $\delta\in\mathrm{R}^{d,+}$, $0<p\leqq\infty$. *Then*

$$\tau_k(f,\lambda\delta)_p \leqq c\lceil\lambda\rceil^{k+d/\tilde{p}}\ \tau_k(f,\delta)_p,$$

where $\tilde{p}=\min(1,p)$, $\lceil\lambda\rceil=\min\{\mu\geqq\lambda : \mu$ *is an integer*$\}$ *and the constant c depends on k and d only.*

Proof. The proofs of the first three properties are standard and they can be obtained in the same way as in the one-dimensional case [3].

We give a sketch proof of the fourth property. The following estimate holds

$$\omega_k(f, x, \lceil\lambda\rceil\delta) \leqq (2\lceil\lambda\rceil)^k \sum_{\mu\in\Lambda} \omega_k(f, x_\mu, \delta),$$

where

$$\Lambda = \{\mu\in \mathbf{Z}^d : |\mu_i| \leqq \lceil\lambda\rceil, \qquad i=1,\dots,d\},$$

$$\mu\in\mathbf{Z}^d = \{\mu=(\mu_1,\dots,\mu_d) : \ \mu_i \text{ are integers, } i=1,\dots,d\},$$

$$x_\mu = x + \frac{k}{2}\langle\mu,\delta\rangle, \qquad \langle\mu,\delta\rangle = (\mu_1\delta_1,\dots,\mu_d\delta_d).$$

In the case $1\leqq p\leqq\infty$ using the Minkowski's inequality we get 4. In the case $0<p<1$ we get 4 by

$$\tau_k(f,\lceil\lambda\rceil\delta)_p \leqq \left(\int_\Omega (2\lceil\lambda\rceil)^{kp} \sum_{\mu\in\Lambda} \omega_k(f,x_\mu,\delta)^p\,\mathrm{d}x\right)^{1/p} = (2\lceil\lambda\rceil)^{k+d/p}\,\tau_k(f,\delta)_p.$$

By the fact $\tau_k(f\,\lambda\delta)_p \leqq \tau_k(f,\lceil\lambda\rceil\delta)_p$ we end the proof of 4.

Les us denote

$$HM_p(\Omega) = \{f : f\in M_p(\Omega), \qquad \omega_k(f,\cdot,\delta)\in L_p(\Omega)\}.$$

It is clear that $HM_p(\Pi^d) = M_p(\Pi^d)$, $HM_p(\mathbf{R}^d) \subsetneqq M_p(\mathbf{R}^d)$. For $n\in\mathbf{Z}^{d,+} = \{n = (n_2,\dots,n_d)\in\mathbf{Z}^d$ and $n_i\geqq 0$, $i=1,\dots,d\}$ we denote by π_n the set of all trigonometric polynomials with real coefficients of order n_i with respect to the i-th variable, $i=1,\dots,d$. For $\nu\in\mathbf{R}^{d,+}$ with $\mathscr{E}_\nu$ we denote the set of all bounded, entire functions of exponential type ν_i with respect to the i-th variable.

Definition 2.3. For $f\in M_p(\mathbf{R}^d)$ we define the best one-sided approximation of f by the functions from $\mathscr{E}_\nu$ in the L_p-metric $(0<p\leqq\infty)$ through the following expression:

$$\tilde{E}_{\mathscr{E}_\nu}(f)_p = \inf\{\|G-g\|_{L_p(\mathbf{R}^d)} : G, g\in\mathscr{E}_\nu, \qquad g\leqq f\leqq G\}.$$

Definition 2.4. For $f\in M_p(\Pi^d)$ we define the best one-sided approximation of f by the functions from π_n in the L_p-metric, $0<p\leqq\infty$ by the following expression:

$$\tilde{E}_{\pi_n}(f)_p = \inf\{\|G-g\|_{L_p(\Pi^d)} : G, g\in\pi_n, \qquad g\leqq f\leqq G\}.$$

3. Direct theorem for one-sided approximation — the isotropical case

In this section we will get an estimation for $\tilde{E}_{\mathscr{E}_v}(f)_p$ and $\tilde{E}_{\pi_n}(f)_p$ in the case $v=(v_1,\dots,v_d):v_1=\dots=v_d$ and $n=(n_1,\dots,n_d):n_1=\dots=n_d$. The estimations will be given in the terms of the averaged moduli of smoothness (Def. 2.2.) being used for $\delta=(\delta_1,\dots,\delta_d):\delta_1=\dots=\delta_d$. We substitute $v=v_1=\dots=v_d$, $n=n_1=\dots n_d$ and $\delta=\delta_1=\dots=\delta_d$ in this case.

The following theorem holds:

Theorem 3.1. *Let* $k,\ n\in\mathrm{N}$, $v>0$, $0<p\leqq\infty$ *and* $f\in HM_p(\Omega)$. *There exists a constant* $c(k,p,d)$ *such that*

a) $\tilde{E}_{\mathscr{E}_v}(f)_p\leqq c(k,p,d)\,\tau_k(f,\ 1/v)_p$

and

b) $\tilde{E}_{\pi_n}(f)_p\leqq c(k,p,d)\,\tau_k(f,\ 1/n)_p$.

Only the estimate a) will be proved because the proof of b) is similar. We shall give some comments concerning b) only.

Let

$$F_k^{\pm}(x\,;t,u)=(-1)^{k-1}\Delta_t^k f(x)\pm[\omega_k(f,x+t,\alpha_k|u|_d)+\omega_k(f,x+u,\alpha_k|t|_d)]+f(x),$$

where

$$x,t,u\in\mathrm{R}^d,\quad \alpha_k=2(1+1/k)\ \text{and for}\ v\in\mathrm{R}^d,\quad |v|_d=\max\{|v_i|,\quad i=1,\dots,d\}.$$

Lemma 3.1. *For* $x,t,u\in\mathrm{R}^d$ *the following estimates hold*

a) $F_k^-(x:t,u)\leqq f(x)\leqq F_k^+(x\,;t,u)$;

b) $F_k^+(x:t,u)-F_k^-(x\,;t,u)\leqq 4\omega_k(f,x,(\alpha_k+2/k)\max(|u|_d,|v|_d))$.

P r o o f. The estimate a) follows from the inequality

$$|\Delta_t^k f(x)|\leqq\omega_k(f,x+t,\alpha_k|u|_d)+\omega_k(f,x+u,\alpha_k|t|_d),$$

which is obtained by the fact that $x,x+kt$ belong simultaneously either to

$\mathrm{K}_{\frac{(k+1)|t|}{2}d}(x+u)$ or to $\mathrm{K}_{\frac{(k+1)|u|}{2}d}(x+t)$.

The estimate b) follows by the inequalities (see (3))

$$F_k^+(x\,;t,u)-F_k^-(x\,;t,u)=2[\omega_k(f,x+t,\alpha_k|u|_d)+\omega_k(f,x+u,\alpha_k|t|_d)]$$

$$\leqq 2[\omega_k(f,x,\alpha_k|t|_d+2|u|_d/k)+\omega_k(f,x,\alpha_k|u|_d+2|t|_d/k)]$$

$$\leqq 4\omega_k(f,x,(\alpha_k+2/k)\max(|t|_d,\ |u|_d)).$$

The following Jackson's type kernels are considered for $r\in N$, $\sigma\in R^{d,+}$, $t\in R^d$:

$$J_{r,\sigma}(t)=\prod_{i=1}^{d} j_{r,\sigma_i}(t_i),$$

where

$$j_{r,\sigma_i}(t_i)=\gamma_{r,\sigma_i}\left[\frac{\sin\sigma_i t_i/2}{\sigma_i t_i/2}\right]^{2r}, \qquad \gamma_{r,\sigma_i}^{-1}=\int_{R^1}\left[\frac{\sin\sigma_i t_i/2}{\sigma_i t_i/2}\right]^{2r} dt_i, \qquad i=1,\dots,d.$$

We shall use the following lemma (see [14]):

Lemma 3.2. *For $J_{r,\sigma}(t)$ we have*
a) $J_{r,\sigma}\in\mathscr{E}_{r\sigma}\cap L_p$ *for* $p>1/2r$;
b) j_{r,σ_i} *are even, positive functions and for* $m\in N$
$\sup\{j_{r,\sigma_i}(s): s\in[(m-1)/\sigma_i,\ m/\sigma_i]\}\leqq c(r)\sigma_i m^{-2r}$.

Definition 3.1. For $f\in HM_p(R^d)$, $k\in N$, $\sigma>0$ we define

$$g^{\pm}(f;x)=\int_{R^d}\int_{R^d} F_k^{\pm}(x;u,t)\,J_{r,\sigma}(u)\,J_{r,\sigma}(t)\,du\,dt$$

(here $\sigma_1=\sigma_2=\dots=\sigma_d=\sigma$).

The functions $g^{\pm}(f;\cdot)$ are our operators of one-sided approximation in the isotropical case.

Lemma 3.3. *For the functions $g^{\pm}$ the following properties hold:*
a) $g^{\pm}\in\mathscr{E}_{r\sigma}\cap L_\infty$;
b) *for* $x\in R^d$

$$g^{-}(f;x)\leqq f(x)\leqq g^{+}(f,x);$$

c) $g^{+}(f,x)-g^{-}(f,x)\leqq 16d\int_0^\infty \omega_k(f,x,(\alpha_k+2/k)s)j_{r,\sigma}(s)\,ds.$

P r o o f. It is clear, that $g^{\pm}\in\mathscr{E}_{r,\sigma}\cap L_\infty$, see [14, p. 135].
The assertion b) follows from Lemma 3.1 a) and the fact that

$$J_{r,\sigma}(t)\geqq 0, \qquad t\in R^d \quad \text{and} \quad \|J_{r,\sigma}\|_1=1.$$

Let us prove the assertion c). Using Fubini's theorem one may get from Lemma 3.1 b) the following

$$g^{+}(f;x)-g^{-}(f;x)=\int_{R^d}\int_{R^d}[F_k^{+}(x;t,u)-F^{-}(x;t,u)]\,J_{r,\sigma}(u)\,J_{r,\sigma}(t)\,du\,dt$$

$$\leqq 4\int_{R^d}\int_{R^d}\omega_k(f,x,(\alpha_k+2/k)\max(|u|_d,|t|_d))\,J_{r,\sigma}(u)\,J_{r,\sigma}(t)\,du\,dt$$

$$\leqq 4\left[\int_{R^d} \omega_k(f, x, (\alpha_k+2/k)|u|_d) J_{r,\sigma}(u)\,du + \int_{R^d} \omega_k(f, x, (\alpha_k+2/k)|t|_d) J_{r,\sigma}(t)\,dt\right]$$

$$\leqq 8 \sum_{i=1}^{d} \int_{R^d} \omega_k(f, x, (\alpha_k+2/k)|t_i|) J_{r,\sigma}(t)\,dt = 16d \int_0^\infty \omega_k(f, x, (\alpha_k+2/k)s) j_{r,\sigma}(s)\,ds.$$

Lemma 3.4. *Let* $0<p\leqq\infty$, r- the minimal integer exceeding $\dfrac{k\tilde{p}+1+d}{2\tilde{p}}$, $\tilde{p}=\min(1,p)$, $\sigma>0$ *and* $f\in HM_p$. *There exists a constant* $c(r,k,d,p)$ *such that*

$$\left\|\int_0^\infty \omega_k(f, \cdot, (\alpha_k+2/k)s) j_{r,\sigma}(s)\,ds\right\|_p \leqq c(r, k, d, p)\,\tau_k(f, 1/\sigma)_p.$$

Proof. Let $0<p<1$. We have by Proposition 2.1:

$$\left\|\int_0^\infty \omega_k(f, \cdot, (\alpha_k+2/k)s) j_{r,\sigma}(s)\,ds\right\|_p^p \leqq \left\|\sum_{m=1}^{\infty} \int_{\frac{m-1}{\sigma}}^{\frac{m}{\sigma}} \omega_k(f, \cdot, (\alpha_k+2/k)s) j_{r,\sigma}(s)\,ds\right\|_p^p$$

$$\leqq c^p(r)\left\|\sum_{m=1}^{\infty} m^{-2r}\omega_k\left(f, \cdot, (\alpha_k+2/k)\frac{m}{\sigma}\right)\right\|_p^p \leqq c^p(r) \sum_{m=1}^{\infty} m^{-2rp}\tau_k\left(f, (\alpha_k+2/k)\frac{m}{\sigma}\right)_p^p$$

$$\leqq c^p(r)\,\tau_k(f, 1/\sigma)_p^p \sum_{m=1}^{\infty} m^{-2rp+kp+d} \leqq c^p(r, k, d, p)\,\tau_k(f, 1/\sigma)_p^p.$$

We can change the order of summation and integration since $f\in HM_p$. Now let $1\leqq p\leqq\infty$. By Minkowski's inequality we get

$$\left\|\int_0^\infty \omega_k(f, \cdot, (\alpha_k+2/k)s) j_{r,\sigma}(s)\,ds\right\|_p \leqq \int_0^\infty \tau_k(f, (\alpha_k+2/k)s)_p j_{r,\sigma}(s)\,ds$$

$$\leqq c(r) \sum_{m=1}^{\infty} \tau_k\left(f, (\alpha_k+2/k)\frac{m}{\sigma}\right)_p m^{-2r} \leqq c(r)\,\tau_k(f, 1/\sigma)_p \sum_{m=1}^{\infty} m^{-2r+k+d}$$

$$= c(r, k, d)\,\tau_k(f, 1/\sigma)_p.$$

Proof of Theorem 3.1. We choose r- the minimal integer exceeding $\dfrac{k\tilde{p}+1+d}{2\tilde{p}}$, $\sigma=v/r$ and let $g^+(f; x)$, $g^-(f; x)$ be the functions from Definition 3.1.

By Lemmas 3.3, 3.4 and Proposition 2.1 4) we get $\tilde{E}_{\delta_v}(f)_p \leqq \|g^+-g^-\|_p \leqq c_1\,\tau_k(f, 1/\sigma)_p \leqq c\,\tau_k(f, 1/v)_p$ and this ends the proof of a).

For the proof of b) we substitute $\sigma=\left\lceil\dfrac{n}{r}\right\rceil$ and determine

$$t_n^{\pm}(f; x) = \int_{\Pi^d}\int_{\Pi^d} F_k^{\pm}(x; u, t)\, J_{r,\sigma}^*(u)\, J_{r,\sigma}^*(t)\,du\,dt,$$

where

$$J^*_{r,\sigma}(t)=\prod_{i=1}^{d} j^*_{r,\sigma}(t_i),\quad j^*_{r,\sigma}(s)=\gamma^*_{r,\sigma}\left[\frac{\sin\sigma s/2}{\sigma\sin s/2}\right]^{2r},\quad \gamma^{*\,-1}_{r,\sigma}=\int_0^{2\pi}\left[\frac{\sin\sigma s/2}{\sigma\sin s/2}\right]^{2r}ds.$$

The functions $t^{\pm}_n$ are trigonometric polynomials of degree n with respect to each variable. We have also

$$\sup\{j^*_{r,\sigma}(s): s\in[(m-1)\pi/\sigma,\ m\pi/\sigma]\leqq c(r)\,\sigma m^{-2r},\qquad m=1,2,\dots,\sigma$$

and the corresponding versions of Lemmas 3.3. and 3.4. are valid and can be proved in the same way.

4. Direct theorem for one-sided approximation — the anisotropical case

In this section the estimates for $\tilde{E}_{\mathscr{E}_v}(f)_p$ and $\tilde{E}_{\pi_n}(f)_p$ are obtained using the averaged moduli of smoothness (see Def. 2.2.). The scheme we follow is the same as in section 3 but the functions $F^{\pm}_k(x;u,t)$ have to be changed.

Let us denote $\Gamma=\{l=(l_1,\dots,l_d): l_i=0,1;\ i=1,\dots,d\}$ and for $l\in\Gamma$, $\bar{l}=(1-l_1,\dots,1-l_d)$.

For $x,t,u\in\mathrm{R}^d$ we define the following functions:

$$\tilde{F}_k^{\pm}(x;t,u)=(-1)^{k-1}\Delta^k_t f(x)$$

$$+f(x)\pm\sum_{l\in\Gamma}\omega_k(f,x+\langle l,u\rangle+\langle\bar{l},t\rangle,\alpha_k(\langle\bar{l},|u|\rangle+\langle l,|t|\rangle)),$$

where $\alpha_k=2(1+1/k)$, $\langle l,u\rangle=(l_1u_1,\dots,l_du_d)$, $|u|=(|u_1|,\dots,|u_d|)$.

Lemma 4.1. *Let $k\in\mathrm{N}$ and $x,t,u\in\mathrm{R}^d$. Then*
a) $\tilde{F}_k^-(x;t,u)\leqq f(x)\leqq\tilde{F}_k^+(x;t,u)$;
b) $\tilde{F}_k^+(x;t,u)-\tilde{F}_k^-(x;t,u)\leqq 2^{d+1}\omega_k(f,x,\alpha_k(|u|+|t|))$.

P r o o f. Let x,t and u be fixed. We determine $l^*\in\Gamma$ in the following way

$$l^*_i=\begin{cases}1 & \text{if}\quad |u_i|\leqq|t_i|\\ 0 & \text{if}\quad |u_i|>|t_i|\end{cases}\qquad i=1,\dots,d.$$

Then we assert that

$$x,x+kt\in\mathrm{K}_{(k+1)(\langle\bar{l}^*,|u|\rangle+\langle l^*,|t|\rangle)}(x+\langle l^*,u\rangle+\langle\bar{l}^*,t\rangle)$$

and consequently

$$|\Delta_t^k f(x)| \leqq \omega_k(f, x+\langle l^*, u\rangle + \langle \bar{l}^*, t\rangle,\ \alpha_k(\langle \bar{l}^*, |u|\rangle + \langle l^*, |t|\rangle))$$

and from this inequality follows a).

The assertion b) follows by Proposition 2.1.

Definition 4.1. For $f \in HM_p$ and $k, r \in \mathbb{N}$, $\sigma \in \mathbb{R}^{d,+}$ we consider the functions

$$\tilde{g}^{\pm}(f;x) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \tilde{F}_k^{\pm}(x;t,u)\, J_{r,\sigma}(t)\, J_{r,\sigma}(u)\,du\,dt.$$

The functions $\tilde{g}^{\pm} \in L_\infty$ since $\tilde{F}_k^{\pm} \in L_\infty$ and $J_{r,\sigma} \in L_1$.

Lemma 4.1. *The functions* $\tilde{g}^{\pm}$ *have the following properties:*

a) $\tilde{g}^{\pm} \in \mathscr{E}_{r,\sigma}$;

b) *for every* $x \in \mathbb{R}^d$

$$\tilde{g}^{-}(f;x) \leqq f(x) \leqq \tilde{g}^{+}(f;x).$$

Proof. According to Definition 4.1 we get

$$\tilde{g}^{\pm}(f;x) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} [(-1)^{k-1}\Delta_t^k f(x) + f(x)]\, J_{r,\sigma}(u)\, J_{r,\sigma}(t)\,du\,dt$$

$$\pm \sum_{l\in\Gamma} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \omega_k(f, x+\langle l, u\rangle + \langle \bar{l}, t\rangle,\ \alpha_k(\langle \bar{l}, |u|\rangle + \langle l, |t|\rangle))\, J_{r,\sigma}(u)\, J_{r,\sigma}(t)\,du\,dt.$$

The first term of the sum belongs to $\mathscr{E}_{r,\sigma}$, see Lemma 3.3. Let us fix $l \in \Gamma$ and let $w = \langle l, u\rangle + \langle \bar{l}, t\rangle$, $v = \langle \bar{l}, u\rangle + \langle l, t\rangle$. Applying Fubini's theorem as many times as necessary we obtain

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \omega_k(f, x+\langle l, u\rangle + \langle \bar{l}, t\rangle,\ \alpha_k(\langle \bar{l}, |u|\rangle + \langle l, |t|\rangle))\, J_{r,\sigma}(u)\, J_{r,\sigma}(t)\,du\,dt$$

$$= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \omega_k(f, x+w,\ \alpha_k|v|)\, J_{r,\sigma}(w)\, J_{r,\sigma}(v)\,dv\,dw$$

$$= \int_{\mathbb{R}^d} \psi(x+w)\, J_{r,\sigma}(w)\,dw,$$

where

$$\psi(x+w) = \int_{\mathbb{R}^d} \omega_k(f, x+w,\ \alpha_k|v|)\, J_{r,\sigma}(v)\,dv \in L_\infty(\mathbb{R}^d)$$

since $\omega_k(f, x+w, \cdot) \in L_\infty(\mathbb{R}^{d,+})$, $J_{r,\sigma} \in L_1(\mathbb{R}^d)$. Using the fact that $J_{r,\sigma}(w) \in \mathscr{E}_{r,\sigma} \cap L_1$ and the convolution properties (see [14, p. 135]) we get a). Assertion b) in fact was proved in section 3, see Lemma 3.3 b).

Theorem 4.1. *Let* $k\in\mathrm{N}$, $\nu\in\mathrm{R}^{d,+}$, $n\in\mathrm{N}^d$, $0<p\leqq\infty$ and $f\in HM_p(\Omega)$. *There exists a constant* $c(k,p,d)$ *such that*

a) $\tilde{E}_{\mathcal{E}_\nu}(f)_p\leqq c(k,p,d)\,\tau_k(f,1/\nu)_p$;

b) $\tilde{E}_{\pi_n}(f)\leqq c(k,p,d)\,\tau_k(f,1/n)_p$,

where

$$1/\nu:=(1/\nu_1,\dots,1/\nu_d),\quad 1/n:=(1/n_1,\dots,1/n_d).$$

Proof. Let us prove a). We choose $\tilde{p}=\min(1,p)$, r- the minimal integer exceeding $\dfrac{k\tilde{p}+1+d}{2\tilde{p}}$ and $\sigma=\nu/r$. By Lemmas 4.2, 4.1 we get

$$\tilde{E}_{\mathcal{E}_\nu}(f)\leqq\|\tilde{g}^+-g^-\|_p\leqq 2^{d+1}\left\|\int_{\mathrm{R}^d}\int_{\mathrm{R}^d}\omega_k(f,\cdot,\ \alpha_k(|u|+|t|))\,J_{r,\sigma}(u)\,J_{r,\sigma}(t)\,\mathrm{d}u\,\mathrm{d}t\right\|_p$$

$$\leqq 2^{3d+1}\left\|\int_{\mathrm{R}^{d,+}}\int_{\mathrm{R}^{d,+}}\omega_k(f,\cdot,\ \alpha_k(u+t))\,J_{r,\sigma}(u)\,J_{r,\sigma}(t)\,\mathrm{d}u\,\mathrm{d}t\right\|_p. \tag{4}$$

To estimate (4) we consider two cases:

1. Let $0<p<1$. We have

$$\left\|\int_{\mathrm{R}^{d,+}}\int_{\mathrm{R}^{d,+}}\omega_k(f,\cdot,\ \alpha_k(u+t))\,J_{r,\sigma}(u)\,J_{r,\sigma}(t)\,\mathrm{d}u\,\mathrm{d}t\right\|_p^p$$

$$=\left\|\sum_{\lambda\in\mathrm{N}^d}\sum_{\mu\in\mathrm{N}^d}\int_{I_{\lambda,\sigma}}\int_{I_{\mu,\sigma}}\omega_k(f,\cdot,\ \alpha_k(u+t))\,J_{r,\sigma}(u)\,J_{r,\sigma}(t)\,\mathrm{d}u\,\mathrm{d}t\right\|_p^p$$

(there $I_{\lambda,\sigma}=\{y\in\mathrm{R}^d:(\lambda_i-1)/\sigma_i\leqq y_i\leqq\lambda_i/\sigma_i,\quad i=1,\dots,d\}$)

$$\leqq c\sum_{\lambda\in\mathrm{N}^d}\sum_{\mu\in\mathrm{N}^d}\tau_k(f,\langle\lambda+\mu,1/\sigma\rangle)_p^p\prod_{i=1}^d(\lambda_i\mu_i)^{-2rp}$$

$$\leqq c_1\,\tau_k(f,1/\sigma)_p^p\sum_{\lambda\in\mathrm{N}^d}\sum_{\mu\in\mathrm{N}^d}|\lambda+\mu|_d^{kp+d}\prod_{i=1}^d(\lambda_i\mu_i)^{-2rp}$$

$$=c_1^p(k,p,d)\,\tau_k(f,1/\sigma)_p^p.$$

From (5) we get

$$\tilde{E}_{\mathcal{E}_\nu}(f)_p\leqq c_1(k,p,d)\,\tau_k(f,r/\nu)_p\leqq c(k,p,d)\,\tau_k(f,1/\nu)_p.$$

2. $1\leqq p\leqq\infty$. Using Minkowski's inequality we get

$$\left\|\int_{\mathrm{R}^{d,+}}\int_{\mathrm{R}^{d,+}}\omega_k(f,\cdot,\ \alpha_k(u+t))\,J_{r,\sigma}(u)\,J_{r,\sigma}(t)\,\mathrm{d}u\,\mathrm{d}t\right\|_p$$

$$\leqq c\,\tau_k(f,1/\sigma)_p\sum_{\lambda\in\mathrm{N}^d}\sum_{\mu\in\mathrm{N}^d}|\lambda+\mu|_d^{k+d}\prod_{i=1}^d(\lambda_i\mu_i)^{-2r}\leqq c\,\tau_k(f,1/\sigma)_p$$

and we end the proof similarly to 1.

The second part of the theorem (b) can be proved in the same way using the polynomials

$$\tilde{t}_n^{\pm}(f;x)=\int_{\pi^d}\int_{\pi^d}\tilde{F}_k^{\pm}(x;t,u)J^*_{r,\sigma}(u)J^*_{r,\sigma}(t)\,du\,dt$$

and $\sigma=\left(\left[\frac{n_1}{r_1}\right],\dots,\left[\frac{n_d}{r_d}\right]\right)$.

This ends the proof of theorem 4.1.

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