Mathematica
Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

Mathematica Balkanica

Now Series Vol. 3, 1989, Fasc. 2

B-Splines with Birkhoff Knots. Applications in the Approximations and Shape-Preserving Interpolation

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We extend some classical results from the spline functions theory to the case of polynomial splines having Birkhoff type of knots.

1. Preliminaries

We start with some basic definitions from [3], [10]. Let $X=\{x_i\}_1^m$, $x_1<\ldots<x_m$ and $E=\{e_{ij}\}_{i=1,\,j=0}^{m\,r-1}$ be an incidence matrix (E consists of 0's and 1's only). By π_r we mean the set of polynomials with real coefficients of degree at most r. It is said that the matrix E satisfies:

(i) Polya condition, if $M_k:=\sum_{j\leqq k}\sum_i e_{ij}\geqq k+1,\ k=0,\ldots,r-1$;

(ii) Strong Polya condition, if $M_k>k+1,\ k=0,\ldots,r-2$.

The matrix E is conservative if it does not contain odd supported blocks of 1's. The pair (X, E) is regular (s-regular), if E is conservative and satisfies Polya condition (resp., Strong Polya condition).

Definition 1.1. Let (X, E) be a regular pair and $|E|:=\sum_{i,j} e_{ij}$, $|E|=r+1$. The linear functional

$$D[(X, E); f]:=\sum_{e_{ij}=1} a_{ij}f^{(j)}(x_i) \tag{1.1}$$

satisfying the conditions

$$\begin{cases} D[(X, E);\varphi]=0 & \text{for} \quad \varphi(x)=x^k, \quad k=0,\ldots,r-1, \\ D[(X, E);\varphi]=1 & \text{for} \quad \varphi(x)=x^r \end{cases} \tag{1.2}$$

is said to be **divided difference** of the function f at (X, E). Denote by $\begin{bmatrix}(X, E)\\0,\dots,r\end{bmatrix}$ the matrix of coefficients of the linear system (1.2) with respect to $\{a_{ij}\}$. Since $\begin{bmatrix}(X, E)\\0,\dots,r\end{bmatrix}$ is exactly the transposed matrix for the Birkhoff interpolation problem

$$p^{(j)}(x_i)=0, \qquad e_{ij}=1, \qquad p\in\pi_r$$

and by the Atkinson-Sharma theorem [2] $\det\begin{bmatrix}(X, E)\\0,\dots,r\end{bmatrix}\neq 0$, it follows that the conditions (1.2) uniquely define the functional (1.1).

It is not difficult to prove the next properties (see [3]).

Property 1.1. $D[(X, E); f]$ coincides with the coefficient of x^r in the polynomial $p\in\pi_r$ such that $p^{(j)}(x_i)=f^{(j)}(x_i)$ for $e_{ij}=1$.

Property 1.2. If the pair (X, E) is s-regular then

$$a_{ij}\neq 0 \quad \text{if} \quad e_{ij}=1 \quad \text{and} \quad i=1 \quad \text{or} \quad i=m.$$

We call the knot (x_1, λ) of a given pair (X, E) **first**, if $e_{1\lambda}=1$ and $e_{1j}=0$ for $j>\lambda$. Similarly, the knot (x_m, μ) is said to be **last** of (X, E), if $e_{m\mu}=1$ and $e_{mj}=0$ for $j>\mu$. Both the first and the last knots of (X, E) are said to be **end knots**.

To remove a knot (x_ν, λ) from a given pair (X, E) it means to obtain a new pair (X', E') such that

$$X'=\begin{cases} X=\{x_i\}_1^m, & \text{if} \quad \Sigma_j e_{\nu j}>1,\\ \{x_1,\dots,x_{\nu-1},x_{\nu+1},\dots,x_m\}, & \text{if} \quad \Sigma_j e_{\nu j}=1,\end{cases}$$

$$E'=\{e'_{ij}\}, \qquad E'_{ij}=\begin{cases} 0, & \text{if } (i,j)=(\nu,\lambda),\\ e_{ij} & \text{otherwise.}\end{cases}$$

Definition 1.2. Let (X, E) be a regular pair with $|E|=r+1$. The function

$$B((X, E); t):=D[(X, E); (\circ-t)_+^{r-1}]/(r-1)! \tag{1.3}$$

is said to be a **polynomial B-spline of degree** $r-1$ **with knots at** (X, E).

The function (1.3) preserve the fundamental properties of the usual polynomial B-splines. The next assertion follows from Theorem 7.1.b and Theorem 7.13, [10]; see also [7].

Property 1.3. Let the pair (X, E) be s-regular and $|E|=r+1$. Then

$$B((X, E); t)=0 \quad \text{for} \quad t\notin[x_1, x_m],$$

$$B((X, E); t)>0 \quad \text{for} \quad t\in(x_1, x_m).$$

Remark 1.1. The Property 1.3 holds under weaker assumptions: the pair (X, E) to be regular and the two pairs obtained from (X, E) removing the first knot and the last knot to be regular too. Then instead of Property 1.2 we state $a_{ij} \neq 0$ only for the coefficients $\{a_{ij}\}$ corresponding to the two end knots of the pair (X, E).

Given two positive integers r, L and a pair (X, E), $x_1 < \ldots < x_m$, $E = \{e_{ij}\}_{i=1, j=0}^{m\, r-1}$, $|E| = r + L$ we define a $(r+1)$-partition of (X, E) as a sequence of pairs $\{(X_i, E_i)\}_1^L$ obtained in the following way. Let us order the elements of E in the manner $e_{1,0}, \ldots, e_{1,r-1}, \ldots, e_{m,0}, \ldots, e_{m,r-1}$ and number the 1's in this sequence from 1 to $r+L$. Let $\bar{e}_p, \bar{e}_{p+1}, \ldots, \bar{e}_q$ be the rows of E containing $r+1$ consequtive 1's starting with the i-th one. Suppose the first row $\bar{e}_p$ contains ν 1's and $\bar{e}_q$ contains μ 1's of this $(r+1)$-sample. We denote by X_i the set of knots $x_p < \ldots < x_q$ and by E_i the matrix composed from $\bar{e}_p, \ldots, \bar{e}_q$ in which all 1's in the sequences $e_{p,0}, \ldots, e_{p,r-1}$ and $e_{q,0}, \ldots, e_{q,r-1}$ except the first ν, respectively μ, are replaced by 0's.

We say that the $(r+1)$-partition $\{(X_i, E_i)\}_1^L$ of (X, E) is s-regular if all (X_i, E_i), $i = 1, \ldots, L$ are s-regular.

Having in mind Remark 1.1 we introduce other kinds of regularity to get more general results. We shall make use of the next notations. By $(X, E)_0$, $(X, E)_r$, $(\hat{X}, \hat{E})$ we mean the pairs obtained from (X, E), $|E| = r+1$ removing the first, the last, both the end knots, respectively.

Definition 1.3. The pair (X, E), $|E| = r+1$ is said to be
(i) Good regular (or G-regular), if (X, E), $(X, E)_0$, $(X, E)_r$ are all regular;
(ii) Strong Good regular (or SG-regular), if (X, E), $(X, E)_0$, $(X, E)_r$, $(\hat{X}, \hat{E})$ are all regular.

Definition 1.4. We say that the $(r+1)$-partition $\{(X_i, E_i)\}_1^L$ of (X, E) is G-regular (SG-regular) if all (X_i, E_i), $i = 1, \ldots, L$ are G-regular (resp., SG-regular).

The next assertions are proved for a s-regular $(r+1)$-partition, but they hold for G-regular one too. So, let a G-regular $(r+1)$-partition $\{(X_i, E_i)\}_i$ of a pair (X, E) be given. Then $\{B((X_i, E_i); \circ)\}_i$ form a basis in the linear space $S_{r-1}(X, E)$ of spline functions

$$(1.4) \qquad s(t) = \sum_{i=0}^{r-1} a_i t^i + \sum_{e_{i,j}=1} b_{ij} (x_i - t)_+^{r-1-j}.$$

The Variation Diminution and Extended Total Positivity of the B-splines (1.3) are proved in [3].

In Sections 2-4 we study some approximating properties of the splines from $S_{r-1}(X, E)$. A scheme for local spline approximation of functions $f \in C_{[a,b]}$

$$(1.5) \qquad f(t) \approx \Sigma_i c_i f(\tau_i) B((X_i, E_i); t) =: V_f(t)$$

is offered, extending the Schoenberg's variation diminishing spline for particular $c_i \in \mathrm{R}$ and $\tau_i \in \operatorname{supp} B((X_i, E_i); \circ)$. We prove that the splines V_f converge uniformly on [a, b] to f when $\max\{|x_{j+1} - x_j| : x_j, x_{j+1} \in X\} \to 0$ and $|E| \to \infty$. Estimations for the quantity dist $(f; S_{r-1}(X, E))$ are given. Note that the splines from $S_{r-1}(X, E)$ may be considered as splines having multiple knots with the constraints some of the coefficients of $\{(x_i - t)_+^{r-1-j}\}$ to be set zeros.

In Section 5 we consider the problem of existence, uniqueness and characterization of a smooth function solving given Birkhoff interpolation problem with minimal L_p-norm $(1<p<\infty)$ under restrictions on the r-th derivative.

2. Expansion of the unity in B-splines with Birkhoff knots

In this section we normalize the B-splines (1.3), so that they sum up to 1. To this end we prove a recurrence relation for the divided differences (1.1).

Theorem 2.1. *Let (X, E) be given SG-regular pair, $X = \{x_i\}_1^m$, $x_1 < \ldots < x_m$, $E = \{e_{ij}\}_{i=1,\, j=0}^{m\ \ r-1}$, $|E| = r+1$. Then*

$$D[(X,E); f] = (D[(X,E)_0; f] - D[(X,E)_r; f])/c, \tag{2.1}$$

where

$$c = \frac{\det\begin{bmatrix} (\hat{X}, \hat{E}) \\ 0,\ldots,r-2 \end{bmatrix} \det\begin{bmatrix} (X,E) \\ 0,\ldots,r \end{bmatrix}}{\det\begin{bmatrix} (X,E)_0 \\ 0,\ldots,r-1 \end{bmatrix} \det\begin{bmatrix} (X,E)_r \\ 0,\ldots,r-1 \end{bmatrix}} > 0. \tag{2.2}$$

Proof. Let (x_1, λ) and (x_m, μ) be the end knots of (X, E). Then

$$\begin{cases} D[(X,E)\ ; f] = a_{1\lambda} f^{(\lambda)}(x_1) + \quad \Sigma\, a_{ij} f^{(j)}(x_i) + a_{m\mu} f^{(\mu)}(x_m) \\ D[(X,E)_0; f] \ = \qquad\qquad\qquad \Sigma\, b_{ij} f^{(j)}(x_i) + b_{m\mu} f^{(\mu)}(x_m). \\ D[(X,E)_r; f] \ = c_{1\lambda} f^{(\lambda)}(x_1) + \Sigma\, c_{ij} f^{(j)}(x_i) \end{cases} \tag{2.3}$$

Everywhere in (2.3) the summation is on $\hat{e}_{ij} = 1$, $\hat{E} = \{\hat{e}_{ij}\}_{i,j}$. By the Kramer's rule it follows from (1.2) and (2.3) that

$$a_{1\lambda} = (-1)^{r+2} \det\begin{bmatrix} (X,E)_0 \\ 0,\ldots,r-1 \end{bmatrix} \Big/ \det\begin{bmatrix} (X,E) \\ 0,\ldots,r \end{bmatrix},$$

$$a_{m\mu} = \det\begin{bmatrix} (X,E)_r \\ 0,\ldots,r-1 \end{bmatrix} \Big/ \det\begin{bmatrix} (X,E) \\ 0,\ldots,r \end{bmatrix},$$

$$b_{m\mu} = \det\begin{bmatrix} (\hat{X}, \hat{E}) \\ 0,\ldots,r-2 \end{bmatrix} \Big/ \det\begin{bmatrix} (X,E)_0 \\ 0,\ldots,r-1 \end{bmatrix},$$

$$c_{1\lambda} = (-1)^{r+1} \det\begin{bmatrix} (\hat{X}, \hat{E}) \\ 0,\ldots,r-2 \end{bmatrix} \Big/ \det\begin{bmatrix} (X,E)_r \\ 0,\ldots,r-1 \end{bmatrix}.$$

Hence $b_{m\mu}/a_{m\mu} = -c_{1\lambda}/a_{1\lambda}$. Let us define c as follows

(2.4) $$c = b_{m\mu}/a_{m\mu}$$

and set

$$D[f] := D[(X, E); f] - (D[(X, E)_0; f] - D[(X, E)_r; f])/c.$$

From (2.2), (2.3) and (2.4) we have

$$D[f] = \sum_{e_{ij}=1} (a_{ij} - (b_{ij} - c_{ij})/c)\, f^{(j)}(x_i).$$

Observe now, that

(2.5) $$D[\varphi] = 0 \quad \text{for} \quad \varphi(x) = x^k, \quad k = 0, \ldots, r-2$$

and since $(\hat{X}, \hat{E})$ is regular, (2.5) has only one, the trivial solution. That is $D[f] \equiv 0$ for all functions f, which implies (2.1) provided $c \neq 0$.

It remains to prove that $c > 0$. Let $p \in \pi_r$ be such that

$$p^{(j)}(x_i) = \delta_{m\mu}, \quad e_{ij} = 1, \quad e_{ij} \in E.$$

Then $D[(X, E); p] = a_{m\mu}$ is the coefficient of x^r in $p(x)$. Remark 1.1 implies $a_{m\mu} \neq 0$. Assume that $a_{m\mu} < 0$. Then because of the relation

$$\operatorname{sign} p^{(\mu)}(x) = \operatorname{sign} a_{m\mu} < 0$$

for sufficiently large x it follows that there exists a $x_{m+1} > x_m$ with $p^{(\mu)}(x_{m+1}) = 0$. But Birkhoff interpolation problem

$$p^{(j)}(x_i) = 0, \quad e_{ij} = 1, \quad e_{ij} \in E, \quad (i, j) \neq (m, \mu),$$
$$p^{(\mu)}(x_{m+1}) = 0$$

is regular and consequently it has a unique solution $p(x) \equiv 0$, a contradiction with $p^{(\mu)}(x_m) = 1$. So $a_{m\mu} > 0$. Similarly $b_{m\mu} > 0$. The theorem is proved. ∎

Definition 2.1. Let (X, E) be given SG-regular pair with $|E| = r + 1$ and c be defined as in Theorem 2.1. Then the function

$$N((X, E); t) := cD[(X, E); (\circ - t)_+^{r-1}]$$

is said to be a normalized polynomial B-spline of degree $r-1$ with knots (X, E).

Theorem 2.2. *Suppose $\{(X_i, E_i)\}_1^{L+r}$ is a SG-regular $(r+1)$-partition of a given pair (X, E), $|E| = L + 2r$. Then*

(2.6) $\Sigma_i N_i(t) \equiv 1$ *if* $t \in \bigcup_{i=r}^{L+1} \operatorname{supp} N_i =: (a, b), \qquad N_i := N((X_i, E_i); \circ).$

Proof. Denote by c_i the normalizing coefficients corresponding to the B-splines $B((X_i, E_i); \circ)$. Fix a subinterval $(x_j, x_{j+1}) \subset (a, b)$ and a point $t \in (x_j, x_{j+1})$. For some index k we have $x_j = \min \{x_i : x_i \in X_{k+r}\}$ and $x_{j+1} = \max \{x_i : x_i \in X_{k+1}\}$. Moreover,

$$\sum_i N_i(t) = \sum_{i=k+1}^{k+r} N_i(t) = \sum_{i=k+1}^{k+r} c_i D[(X_i, E_i); (\circ - t)_+^{r-1}]$$

$$= \sum_{i=k+1}^{k+r} \left[D[(X_i, E_i)_0; (\circ - t)_+^{r-1}] - D[(X_i, E_i)_r; (\circ - t)_+^{r-1}] \right]$$

$$= D[(X_{k+r}, E_{k+r})_0; (\circ - t)_+^{r-1}] - D[(X_{k+1}, E_{k+1})_r; (\circ - t)_+^{r-1}]$$

$$= D[(X_{k+r}, E_{k+r})_0; (\circ - t)^{r-1}] - D[(X_{k+1}, E_{k+1})_r; 0] = 1 - 0 = 1$$

since $(X_i, E_i)_0 = (X_{i+1}, E_{i+1})_r$ and $D[(X_{k+r}, E_{k+r})_0; (\circ - t)^{r-1}]$ is the coefficient of x^{r-1} in the polynomial $p(x) \in \pi_{r-1}$ interpolating $(x-t)^{r-1}$ at $(X_{k+r}, E_{k+r})_0$, i.e. 1. ∎

The next corollary gives a dependence of a spline function s on (x_j, x_{j+1}) on the coefficients of the B-splines which support includes the interval (x_j, x_{j+1}).

Corollary 2.1. *According to the notations in Theorem* 2.1 *if* $s(t) = \sum_{i=k+1}^{k+r} \alpha_i N_i(t)$ *and* $t \in (x_j, x_{j+1}) = \bigcap_{i=k+1}^{k+r} \operatorname{supp} N_i$ *then*

$$\min \{\alpha_{k+1}, \dots, \alpha_{k+r}\} \le s(t) \le \max \{\alpha_{k+1}, \dots, \alpha_{k+r}\}.$$

3. Approximation of a smooth function by splines from the space $S_{r-1}(X, E)$

Let $\{(X_i, E_i)\}_1^{L+r}$ be a SG-regular $(r+1)$-partition of a given pair (X, E) with $|E| = L + 2r$. Henceforth by N_i we mean the i-th normalized B-spline $N((X_i, E_i); \circ)$. Using (2.6) we now naturally extend some estimations from [5], Chapter XII.

Suppose that ξ_i is an arbitrary point from $\operatorname{supp} N_i$, $i = 1, \dots, L + r$. Consider the spline operator

$$A : C_{[a,b]} \to S_{r-1}(X, E), \qquad [a, b] := \overline{\bigcup_{i=r}^{L+1} \operatorname{supp} N_i},$$

defined by

$$A_f(t) := \sum_{i=1}^{L} f(\xi_i) N_i(t), \qquad t \in [a, b].$$

Fix a point $\xi \in (x_j, x_{j+1}) \subset [a, b]$. Then for some integer k we have

$$A_f(\xi) = \sum_{i=k+1}^{k+r} f(\xi_i) N_i(\xi),$$

and (2.6) implies

$$f(\xi) = \sum_{i=k+1}^{k+r} f(\xi) N_i(\xi).$$

Hence

$$|f(\xi) - A_f(\xi)| \leqq \sum_{i=k+1}^{k+r} |f(\xi) - f(\xi_i)| N_i(\xi).$$

Let us introduce the notations:

$$\|f\| = \max\{|f(t)| : t \in [a, b]\}, \qquad \|X\| = \max\{|x_{i+1} - x_i| : x_i, x_{i+1} \in X\},$$

$$\|X_i\| = (\text{length of } \operatorname{supp} N_i), \qquad \|X\|_r = \max\{\|X_i\| : 1 \leqq i \leqq L + r\}.$$

Choosing ξ_i to be the midpoint of $\operatorname{supp} N_i$ we get

$$|f(\xi) - A_f(\xi)| \leqq \max\{|f(\xi) - f(\xi_i)| : k+1 \leqq i \leqq k+r\} \sum_{i=k+1}^{k+r} N_i(\xi) \leqq \omega(f; \|X\|_r/2).$$

So, there exists a constant $d_{r,E} \leqq [(r+1)/2]$, depending on r and E, such that

$$\text{(3.1)} \qquad \operatorname{dist}(f; S_{r-1}(X, E)) \leqq d_{r,E}\, \omega(f; \|X\|).$$

A theorem of Jackson's type follows from (3.1) and the next lemma.

Lemma 3.1. *Suppose that* (X, E), $|E| = 2(r+k) + L$, *has a SG-regular* $(r+1)$*-partition. Then* (X, E) *has a SG-regular* $(r+j+1)$*-partition for all* $j = 0, \ldots, k$.

Proof. We apply induction on j. For $j = 0$ the assertion is clear. Assume that the lemma holds for some natural number $j-1$ and consider an arbitrary (X_ν, E_ν) from the $(r+j+1)$-partition of (X, E). The pair (X_ν, E_ν) is regular, because, by the existence of a SG-regular $(r+1)$-partition, E_ν satisfies the Polya condition and it is conservative, since E is conservative. $(X_\nu, E_\nu)_0$ and $(X_\nu, E_\nu)_{r+j}$ coincide with some of the pairs from the $(r+j)$-partition and hence they are SG-regular. This yields that $(\hat{X}_\nu, \hat{E}_\nu)$ is regular, as obtained from $(X_\nu, E_\nu)_0$ removing the last knot. Therefore $(X_\nu, E_\nu), (X_\nu, E_\nu)_0$, $(X_\nu, E_\nu)_{r+j}$, $(\hat{X}_\nu, \hat{E}_\nu)$ are all regular. ■

Theorem 3.1. *Suppose that* (X, E) *with* $|E| = 2(r+k)+L$ *has a SG-regular* $(r+1)$*-partition. By Lemma* 3.1 (X, E) *has a SG-regular* $(r+j+1)$*-partition and* $S_{r-1}(X, E) \subset C^{\mu}_{(-\infty, \infty)}$ *for some* μ *implies* $S_{r+j-1}(X, E) \subset C^{\mu+j}_{(-\infty, \infty)}$, $j = 0, \ldots, k$. *Set* $[a, b] := \overline{\bigcup_{i=r+k}^{L+1} \operatorname{supp} N_i}$. If $f \in C^j_{[a,b]}$ *then*

$$\operatorname{dist}(f; S_{r+k-1}(X, E)) \leqq d_{r,j} \| X \|^j \omega(f^{(j)}; \| X \|), \qquad j = 0, \ldots, k,$$

where $d_{j,r,E}$ *is a constant depending on* r, j *and* E.

We omit the proof which is similar to that of Theorem XII.1, [5].

4. Extension of Schoenberg's variation-diminishing spline approximation

First we extend an important lemma due to C. de Boor and G. Fix [6] (see also [4],[5]) for the case of Birkhoff type of knots.

Lemma 4.1. *We are given a pair* (X, E), $|E| = L + r$ *and its SG-regular* $(r+1)$*-partition* $\{(X_j, E_j)\}_1^L$. *Denote by* $(\hat{X}_j, \hat{E}_j)$ *the pair obtained from* (X_j, E_j) *removing both the end knots,* $j = 1, \ldots, L$. *Consider the linear functionals*

$$\lambda_i f := \sum_{\nu=0}^{r-1} (-1)^{\nu} p_i^{(r-1-\nu)}(\xi_i) f^{(\nu)}(\xi_i)/(r-1)!, \tag{4.1}$$

where $p_i(t) = t^{r-1} +$ *(polynomial of degree at most* $r-2$*), satisfying*

$$p_i^{(\mu)}(x_k) = 0, \qquad e_{k,\mu} \in \hat{E}_j, \qquad e_{k,\mu} = 1, \tag{4.2}$$

and ξ_i *is an arbitrary point from* $\operatorname{supp} N_i$. *Then*

$$\lambda_i N_j = \delta_{ij} \qquad \text{for all } i, j. \tag{4.3}$$

Proof. We have

$$N_j(t) = c_j D[(X_j, E_j); (\circ - t)_+^{r-1}] = c_j \sum a_{k\mu} \left. \frac{d^{\mu}(s-t)_+^{r-1}}{ds^{\mu}} \right|_{s=x_k},$$

$$\lambda_i N_j = c_j \sum a_{k\mu} \lambda_i \left(\left. \frac{d^{\mu}(s-\circ)_+^{r-1}}{ds^{\mu}} \right|_{s=x_k} \right)$$

$$= c_j \sum a_{k\mu} \lambda_i \left. \left(\frac{d^{\mu}(s-\circ)_+^{r-1}}{ds^{\mu}} \right) \right|_{s=x_k} = c_j \sum a_{k\mu} \left. \frac{d^{\mu}(\lambda_i((s-\circ)_+^{r-1}))}{ds^{\mu}} \right|_{s=x_k}$$

(all summations are on $e_{k\mu} \in E_j$, $e_{k\mu} = 1$).

If $s<\xi_i$ then $\left.\dfrac{d^\nu (s-t)_+^{r-1}}{dt^\nu}\right|_{t=\xi_i}=0$, $\nu=0,\dots,r-1$, and hence $\lambda_i((s-\circ)_+^{r-1})=0$.
If $s>\xi_i$ then

$$\left.\frac{d^\nu (s-t)_+^{r-1}}{dt^\nu}\right|_{t=\xi_i}=\left.\frac{d^\nu (s-t)^{r-1}}{dt^\nu}\right|_{t=\xi_i}, \qquad \nu=0,\dots,r-1,$$

and

$$(4.4) \qquad \lambda_i((s-\circ)^{r-1})$$

$$=\sum_{\nu=0}^{r-1}(-1)^\nu p_i^{(r-1-\nu)}(\xi_i)(r-1)\dots(r-\nu)(-1)^\nu(s-\xi_i)^{r-1-\nu}/(r-1)!$$

$$=\sum_{\nu=0}^{r-1} p_i^{(r-1-\nu)}(\xi_i)(s-\xi_i)^{r-1-\nu}/(r-1-\nu)!=p_i(s),$$

since the last sum is exactly the Taylor series for the polynomial p_i. Thus,

$$\lambda_i((s-\circ)_+^{r-1})=p_i(s)(s-\xi_i)_+^0.$$

Let us set $g(s):=p_i(s)(s-\xi_i)_+^0$. Therefore

$$\lambda_i N_j=c_j\Sigma a_{k\mu}g^{(\mu)}(x_k)=c_jD[(X_j,E_j);g]=D[(X_j,E_j)_0;g]-D[(X_j,E_j)_r;g].$$

(i) Suppose $i=j$. Then $D[(X_j,E_j)_0;g]$ coincides with the coefficient of x^{r-1} in the polynomial $q\in\pi_{r-1}$, satisfying

$$(4.5)\quad \begin{cases} q^{(\mu)}(x_k)=g^{(\mu)}(x_k)=0, & e_{k\mu}\in\hat{E}_j,\quad e_{k\mu}=1,\\ q^{(\mu)}(x_k)=g^{(\mu)}(x_k)=p_i^{(\mu)}(x_k), & (x_k,\mu)\text{ is the last knot of }(X_j,E_j)_0. \end{cases}$$

Clearly, $q\equiv p_i$ because the interpolation problem (4.5) is regular, and hence $D[(X_j,E_j)_0;g]=1$.

The divided difference $D[(X_j,E_j)_r;g]$ coincides with the coefficient of x^r in the polynomial $q\in\pi_{r-1}$, satisfying

$$\begin{cases} q^{(\mu)}(x_k)=g^{(\mu)}(x_k)=0, & e_{k\mu}\in\hat{E}_j,\quad e_{k\mu}=1,\\ q^{(\mu)}(x_k)=g^{(\mu)}(x_k)=0, & \text{if }(x_k,\mu)\text{ is the first knot of }(X_j,E_j)_r \end{cases}$$

Obviously, $q\equiv 0$ and we get $D[(X_j,E_j)_r;g]=0$.
Therefore $\lambda_i N_i=1$.
(ii) Similarly, for the case $i>j$, $D[(X_j,E_j);g]=0$ since $g^{(\mu)}(x_k)=0$ for all (x_k,μ), $e_{k\mu}\in\hat{E}_i$, $e_{k\mu}=1$ and for all (x_k,μ), $x_k<\xi_i$.
(iii) For the case $i<j$ we have

$$\begin{cases} g^{(\mu)}(x_k)=0=p_i^{(\mu)}(x_k) & e_{k\mu}\in\hat{E}_i, \qquad e_{k\mu}=1 \\ g^{(\mu)}(x_k)=p_i^{(\mu)}(x_k) & \text{all} \quad x_k>\xi_i, \end{cases}$$

and since $p_i\in\pi_{r-1}$ we obtain $D[(X_j,E_j);g]=0$.

Now (i), (ii), and (iii) imply (4.3). ■

Remark 4.1. In Lemma 4.1 the values of the derivatives at the points of discontinuity are set to be the right limits of the corresponding function.

Remark 4.2. It is not difficult to see (i.e. [4]), that for $f,p_i\in\pi_{r-1}$ the functional $\lambda_i f$ does not depend on the choice of $\xi_i\in\operatorname{supp}N_i$.

From Lemma 4.1, (4.4), Remark 4.1 and Remark 4.2, choosing ξ_i to be the left bound of $\operatorname{supp}N_i$ we obtain the next corollary, extending the well-known M. S. Marsden's identity [11].

Corollary 4.1. *Using the notations of Lemma 4.1 we have*

$$(s-t)^{r-1}=\Sigma_i\, p_i(s)\,N_i(t).$$

Theorem 4.1. *Let* $\{(X_i,E_i)\}_1^{L+r}$, *be a SG-regular* $(r+1)$*-partition of a given pair* (X,E), $|E|=L+2r$ *and* $[a,b]:=\bigcup_{i=r}^{L+1}\operatorname{supp}N_i$. *Consider the approximation scheme*

$$V_f(t):=\sum_{i=1}^{L} f(\xi_i^*)\,N_i(t),\qquad t\in[a,b], \tag{4.6}$$

where ξ_i^* *is the unique zero of* $p_i^{(r-2)}$ *and* $p_i(t)$ *is defined by* (4.2). *Then*

a) V_f *preserves the polynomials from* π_1;

b) *the spline operator* V_f *is variation diminishing*;

c) *if* f *is convex or nonnegative on* $[a,b]$ *then the spline* V_f *has the same behaviour.*

Proof. Since p_i has maximal number of Birkhoff type zeros in $\operatorname{supp}N_i$ and $\hat{E}_i$ is regular it follows from Rolle's theorem that $p_i^{(r-2)}$ has a unique zero $\xi_i^*\in\operatorname{supp}N_i$. The variation diminishing property of the B-splines with Birkhoff knots is proved in [3].

Therefore

$$S^-(V_f)\le S^-(f(\xi_1^*),\dots,f(\xi_L^*))\le S^-(f), \tag{4.7}$$

where as usual $S^-(g)$ denotes the maximal number of strict sign changes of the function g in $[a,b]$, and $S^-(a_1,\dots,a_L)$ denotes the number of strict sign changes in the sequence of real numbers $a_1,\dots,a_L$.

Moreover, if $h\in\pi_1$, then $h^{(\nu)}(t)\equiv 0$, $\nu=2,\dots,r-1$ and from the uniqueness of the representation of the function $h\in S_{r-1}(X,E)$ in B-splines

$$h(t)=\sum_{i=1}^{L}\alpha_i N_i(t)$$

we get by Lemma 4.1

$$\alpha_i=h(\xi_i^*)-\frac{p_i^{(r-2)}(\xi_i^*)\,h^{(\gamma)}(\xi_i^*)}{(r-1)!}+\sum_{\nu=2}^{r-1}(-1)^{\nu}\frac{p_i^{(r-1-\nu)}(\xi_i^*)\;h^{(\gamma)}(\xi_i^*)}{(r-1)!}.$$

Hence $\alpha_i=h(\xi_i^*)$. Then (4.7) gives

$$S^-(V_f-h)=S^-(V_f-V_h)=S^-(V_{f-h})\leqq S^-(f-h). \tag{4.8}$$

If $f(t)\geqq 0$ on $[a,b]$, clearly $S^-(f)=0$ and (4.7) yields $V_f(t)\geqq 0$ on $[a,b]$. If f is convex on $[a,b]$, (4.8) valid for all $h\in\pi_1$ yields that V_f is convex on $[a,b]$ as well. ■

We now estimate the error of the approximation scheme (4.6) for the function $g(t)=t^2$. We have

$$g(t)=\sum_{i=1}^{L}\beta_i N_i(t),\qquad V_g(t)=\sum_{i=1}^{L}(\xi_i^*)^2 N_i(t),$$

$$\beta_i=(\xi_i^*)^2+2p_i^{(r-3)}(\xi_i^*)/(r-1)!$$

Therefore

$$g(t)-V_g(t)=\sum_{i=1}^{L}\frac{2p_i^{(r-3)}(\xi_i^*)}{(r-1)!}N_i(t). \tag{4.9}$$

But

$$2p_i^{(r-3)}(t)/(r-1)!=t^2-2t\,\xi_i^*+\zeta_i$$

for some ζ_i. Then

$$2p_i^{(r-3)}(\xi_i^*)/(r-1)!=-(\xi_i^*)^2+\zeta_i.$$

Since p_i satisfies maximal number zero conditions of Birkfoff type and is not identically zero the Rolle's theorem yields that $p_i^{(r-3)}(t)$ has either two different zeros $\eta_i^{(1)}$, $\eta_i^{(2)}$ or one of multiplicity 2 (in this case $\eta_i^{(1)}=\eta_i^{(2)}$) in supp N_i. Then

$$\zeta_i=\eta_i^{(1)}\eta_i^{(2)},$$

and

$$|2p_i^{(r-3)}(\xi_i^*)/(r-1)!|=|-(\xi_i^*)^2+\eta_i^{(1)}\eta_i^{(2)}|\leqq\|X_i\|^2/4. \tag{4.10}$$

From (4.9) and (4.10) it follows that

$$|g(t)-V_g(t)| \leqq \sum_{i=1}^{L} \| X_i \|^2 N_i(t)/4 \leqq \| X \|_r^2/4 \leqq d_{r,E} \| X \|^2, \quad t\in[a,b].$$

Lemma 4.2. *For* $g(t)=t^2$

$$\| g-Vg \| \leqq d_{r,E} \| X \|^2,$$

where the constant $d_{r,E} \leqq r^2/4$ *depends on* r, E *but not on* X.

So we have:

(i) by Theorem 4.1 the spline operator V_f is positive;
(ii) by Theorem 2.2 $V_1=1$;
(iii) by Theorem 4.1 $V_h=h$ for all $h\in\pi_1$;
(iv) by Lemma 4.2 $\| g-V_g \| \to 0$ when $\| X \| \to 0$ for $g(t)=t^2$.

Applying the Korovkin theorem about the convergence of a sequence of positive operators we obtain

Theorem 4.2. *Using the notations of this section,*

$$\| f-V_f \| \to 0 \text{ when } \| X \| \to 0$$

for every continuous on $[a,b]$ *function* f.

5. Optimal shape-preserving Birkhoff interpolation

We are given $X=\{x_i\}_1^m$, $x_1<\ldots<x_m$, an incidence matrix $E=\{e_{ij}\}_{i=1,\, j=0}^{m\; r-1}$, $|E|=r+L$ and arbitrary real numbers $Y=\{y_{ij}: e_{ij}\in E,\ e_{ij}=1\}$. Suppose the pair (X, E) has a G-regular $(r+1)$-partition $\{(X_\nu, E_\nu)\}_1^L$. Denote by Y_ν the subset of Y $Y_\nu := \{y_{ij}: (x_i, j)$ is a knot of $(X_\nu, E_\nu)\}$.

Definition 5.1. The data (X, Y, E) are said to be r-strictly convex if

$$\Delta_\nu > 0, \qquad \nu=1,\ldots,L,$$

where $\Delta_\nu := D[(X_\nu, E_\nu);\gamma]$, γ is any function satisfying

$$\gamma^{(j)}(x_i)=y_{ij} \quad \text{for all } i,\ j.$$

Let us fix the above data (X, Y, E), a natural number r and a real number p, $1<p<\infty$. Now we define the class of functions

$$F_p^r(X, Y, E) := \{\gamma\in W^r_{p[x_1, x_m]}: \gamma^{(r)} \geqq 0, \qquad \gamma^{(j)}(x_i)=y_{ij},\ e_{ij}\in E,\ e_{i,j}=1\}.$$

It is known that the set $F_p^r(X, Y, E)$ may be empty if $r=3$ and E having 1's only in its first column. In the case $F_p^r(X, Y, E)\neq\emptyset$ and $1<p<\infty$ we prove that there exists a unique function f from this class with

$$\|f^{(r)}\|_p = \inf\{\|\gamma^{(r)}\|_p : \gamma\in F_p^r(X,Y,E)\}, \quad \|g\|_p := \left(\int_{x_i}^{x_m} |g(t)|^p \,\mathrm{d}t\right)^{1/p},$$

and characterize the solution to the extremal problem. This result extends the cases of Lagrange and Hermite interpolation considered in [1], [8], [9] and others.

Lemma 5.1. *$F_p^r(X,Y,E)\neq\emptyset$ if and only if there exists a function $g\in L_p[x_1,x_m]$, $g\geqq 0$, such that $\int_{x_1}^{x_m} g(t)B_\nu(t)\,\mathrm{d}t=\Delta_\nu$, $\nu=1,\dots,L$.*

Proof. Suppose $\gamma\in F_p^r(X,Y,E)$. Then from the Peano theorem

$$\gamma(x)=p(x)+\int_{x_1}^{x_m}\frac{(x-t)_+^{r-1}}{(r-1)!}\gamma^{(r)}(t)\,\mathrm{d}t, \qquad p\in\pi_{r-1}.$$

Thus

$$\Delta_\nu = D[(X_\nu,E_\nu);\gamma]=\int_{x_1}^{x_m} B_\nu(t)\gamma^{(r)}(t)\,\mathrm{d}t.$$

That is, we may choose $g=\gamma^{(r)}$.

Let us set $\gamma(x):=p(x)+\int_{x_1}^{x_m}\frac{(x-t)_+^{r-1}}{(r-1)!}g(t)\,\mathrm{d}t$, where the polynomial $p\in\pi_{r-1}$ is determined by the interpolation conditions at $(X_1,Y_1)_r$:

$$(5.1)\qquad \begin{cases} p^{(j)}(x_i)=y_{ij}-\int_{x_1}^{x_m}\frac{(x-t)_+^{r-1-j}}{(r-1-j)!}g(t)\,\mathrm{d}t, \\ e_{ij}\in E_1, \quad (x_i,j)\neq \text{the last knot of } (X_1,E_1). \end{cases}$$

Since (X_1,E_1) is G-regular, then the interpolation problem (5.1) is poised and we define uniquely polynomial p. Then from

$$\Sigma\, a_{ij}^{(1)}\gamma^{(j)}(x_i)=D[(X_1,E_1);\gamma]=\int_{x_1}^{x_m} B_1(t)g(t)\,\mathrm{d}t=\Delta_1=\Sigma\, a_{ij}^{(1)}y_{ij}$$

(summations are on the indices i,j with $e_{ij}\in E_1$, $e_{ij}=1$) and from the fact that the coefficient in the divided difference $D[(X_1,E_1);\circ]$ corresponding to the last knot of (X_1,E_1) is not zero (see Remark 1.1), we get $\gamma^{(j)}(x_i)=y_{ij}$ for the last knot (x_i,j) of (X_1,E_1). Similarly using G-regularity of the $(r+1)$-partition $\{(X_\nu,E_\nu)\}_1^L$ we obtain $\gamma^{(j)}(x_i)=y_{ij}$ for all knots (x_i,j). I.e. $\gamma\in F_p^r(X,Y,E)$. The lemma is proved. ■

Suppose $1<q<\infty$. Define the map $\Phi:\mathrm{R}^L\to\mathrm{R}$ by

$$\Phi(\bar a):=\int_{x_1}^{x_m}\left(\sum_{\nu=1}^{L} a_\nu B_\nu(t)\right)_+^q \mathrm{d}t, \qquad \bar a=(a_1,\dots,a_L)\in\mathrm{R}^L.$$

Lemma 5.2. *If $\bar{a}, \bar{b} \in \mathbb{R}^L$ and $a_\nu \leqq b_\nu$, $\nu = 1, \ldots, L$,* then $\Phi(\bar{a}) \leqq \Phi(\bar{b})$.

Proof. Since $B_\nu \geqq 0$ we have $\Sigma_\nu a_\nu B_\nu \leqq \Sigma_\nu b_\nu B_\nu$. Then $(\Sigma_\nu a_\nu B_\nu(t))_+^q \leqq (\Sigma_\nu b_\nu B_\nu(t))_+^q$ and hence $\Phi(\bar{a}) \leqq \Phi(\bar{b})$. ■

Theorem 5.1. *Let (X, E) be a given pair, $X = \{x_i\}_1^m$, $x_1 < \ldots < x_m$, $E = \{e_{ij}\}_{i=1, j=0}^{m\ r-1}$, $|E| = r + L$ and $\{(X_\nu, E_\nu)\}_1^L$ be a G-regular $(r+1)$-partition of (X, E). Suppose $Y = \{y_{ij} : e_{ij} \in E,\ e_{ij} = 1\}$ are real numbers, such that the data (X, Y, E) are r-strictly convex. Then the problems*

(A) *find a function γ from $F_p^r(X, Y, E)$, $1 < p < \infty$;*
(B) *find $\bar{a}^* := (a_1^*, \ldots, a_L^*) \in \mathbb{R}^L$, satisfying*

$$\int_{x_1}^{x_m} B_\nu(t) \left(\sum_{i=1}^{L} a_i^* B_i(t) \right)_+^{q-1} dt = \Delta_\nu, \qquad \nu = 1, \ldots, L, \qquad 1 < q < \infty$$

are both solvable or both unsolvable.

Proof. The proof uses Lemma 5.1 and Lemma 5.2 and is not essentially different from the proof of Theorem 1, [9]. One can see [1]. ■

Suppose that $F_p^r(X, Y, E) \neq \emptyset$ and consider the extremal problem
(C) find a function $f \in F_p^r(X, Y, E)$, such that

$$\|f^{(r)}\|_p = \inf\{\|\gamma^{(r)}\|_p : \gamma \in F_p^r(X, Y, E)\}.$$

Combining Theorem 5.1 and the next lemma proved in [8] we obtain the main result in this section Theorem 5.2.

Lemma 5.3. *Let $1 < q < \infty$ be fixed, S be finite-dimensional subspace of L_q, Λ be a linear functional on S, G be the set of all functions $g \in L_p$, $1/p + 1/q = 1$, such that $g \geqq 0$ and the functional $\langle \circ, g \rangle$ coincides with Λ on S. If there exists $h \in S$ such that $g_0 := h_+^{q-1} \in G$ then $\|g_0\|_p < \inf\{\|g\|_p : g \in G\}$.*

Theorem 5.2. *Let $p, r, (X, Y, E)$ be as in Theorem 5.1 and $F_p^r(X, Y, E) \neq \emptyset$. Then the problem (C) has a unique solution f, such that $f^{(r)} = (\Sigma_i a_i^* B_i)_+^{q-1}$, where $\{a_i^*\}$ is the solution to the problem (B).*

Proof. The uniqueness follows from the strict convexity of the L_p-norm, $1 < p < \infty$. By Theorem 5.1 there exists $\bar{a}^* \in \mathbb{R}^L$ such that

$$\int_{x_1}^{x_m} B_\nu(t) (\Sigma_i a_i^* B_i(t))_+^{q-1} dt = \Delta_\nu, \qquad \nu = 1, \ldots, L.$$

Then for $h := \Sigma_i a_i^* B_i$ we can apply Lemma 5.3. In our case $S = S_{r-1}(X, E) = \{\Sigma_i \alpha_i B_i : \bar{\alpha} \in \mathbb{R}^L\}$, the functional Λ is given by $\Lambda(\Sigma_i \alpha_i B_i) = \Sigma_i \alpha_i \Delta_i$, $\bar{\alpha} \in \mathbb{R}^L$, $\Delta_i = D[(X_i, E_i); Y]$, and

$$G = \{g \in L_p : g \geqq 0, \langle B_i, g \rangle = \int_{x_1}^{x_m} B_i(t) g(t) dt = \Delta_i\}.$$

Therefore there exists a solution f to the problem (C), such that

$f^{(r)} = (\Sigma_i a_i^* B_i)_+^{q-1}$. ■

Acknowledgements. The author is grateful to Professor Borislav Bojanov for his valuable comments and help during the preparation of this paper.

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Received 21. 10. 1988