Mathematica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

Mathematica Balkanica

Now Series Vol. 3, 1989, Fasc. 3-4

A Classification of Rings with Zero Divisors

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Two kinds of additive subgroups in associative rings with zero divisors are introduced and studied.

Introduction

In order to classify associative rings with zero divisors we introduce two new concepts. These are two kinds of additive subgroups. We study some properties of the above rings and mainly in the rings Z/mZ, Z is the set of integers. At the end of this paper we give an example of the ring of $n \times n$ square matrices.

Definition 1. Let R be an associative ring. We shall call strong V-group (s. V-g) every additive subgroup $M \neq \{0\}$ of R satisfying the following condition:

$$rm = mr = 0, \quad \forall m \in M \quad \text{iff} \quad r \in M.$$

Definition 2. Let R be an associative ring. W shall call a weak V-group (w. V-g) every additive subgroup $M \neq \{0\}$ of R satisfying the following conditions:
(a) $\forall m \in M - \{0\}$, m is left and right zero divisor,
(b) if $r \in R$ such that $rm = mr = 0$, $\forall m \in M$, then $r \in M$.
Obviously if M is s. V-g, then M is also w. V-g.

Theorem 1. *If M_1, M_2 are two s. V-g of a ring R and if $M_1 \subset M_2$, then $M_1 = M_2$.*

Proof. Let $r \in M_2$ and $r \notin M_1$. $r \in M_2 \Rightarrow rm = mr = 0$, $\forall m \in M_2 \Rightarrow rm = mr = 0$, $\forall m \in M_1 \Rightarrow r \in M_1$, contradiction. So in any ring R every s. V-g has no subgroup which is s. V-g.

Theorem 2. *If $m = pq$, where p, q are prime numbers, $p \neq q$, the Z/mZ, has no w. V-g.*

Proof. First let $\bar{\lambda} = \lambda + mZ$, $\forall \lambda \in Z$. Z/mZ has two additive subgroups: $M_1 = \{\bar{0}, \bar{p}, \ldots, \overline{p(q-1)}\}$ and $M_2 = \{\bar{0}, \bar{q}, \ldots, \overline{q(p-1)}\}$. We have $\bar{q} M_1 = \{\bar{0}\}$ and $\bar{p} M_2 = \{\bar{0}\}$ but $\bar{q} \notin M_1$, $\bar{p} \notin M_2$, so no-one of M_1 and M_2 is a w. V-g.

Theorem 3. *Let Z/mZ be an additive p-group, $m = p^a$, and $a > 1$; then $M = \{\bar{0}, \bar{p}, \overline{2p}, \ldots, \overline{p(p^{a-1}-1)}\}$ is a w. V-g.*

Proof. a) For $\bar{n}=\overline{\varkappa p}$, $\forall \varkappa \in \{0, 1, \ldots, p^{a-1}-1\}$, from $(n, m) \neq 1$ we obtain that $\bar{n}$ is 0 or zero divisor.

b) Let $\bar{\mu} \notin M$; then $p \not| \mu$ so for the element $\bar{p} \in M$ we have $\bar{\mu}$, $\bar{p} \neq \bar{0}$, i.e. $\mu p \not\equiv 0 \pmod{p^a}$. That means that there is no element $\bar{\mu} \notin$M such that $\bar{\mu}.\bar{m}=\bar{m}\ .\bar{\mu}=\bar{0}$, $\forall \bar{m} \in M$.

Remark. In the above theorem if $a=2$, then $M=\{\bar{0}, \bar{p}, \ldots, \overline{p(p-1)}\}$ is a s. V-g, because $\bar{m}\ .\bar{m}'=\bar{m}'.\ \bar{m}=\bar{0}$, $\forall \bar{m}, \bar{m}' \in M$ and there is no $\bar{\mu} \notin M$ such that $\bar{\mu}\ .\bar{m}=\bar{m}\ .\bar{\mu}=\bar{0}$, $\forall \bar{m} \in M$.

Theorem 4. *If $m=p^a\sigma$, where $a>1$ and $p \not| \sigma$, then $M=\{\bar{0}, \bar{p}, \overline{2p}, \ldots, \overline{p,(p^{a-1}\sigma-1)}\}$ is a w. V-g of Z/mZ.*

Proof. a) Let $\bar{n}=\overline{\varkappa p}$, where $\varkappa \in \{0, 1, \ldots, p^{a-1}\sigma-1\}$, because of $(n, m) \neq 1$ we obtain that $\bar{n}$ is 0 or zero divisor.

b) Now let $\bar{\mu} \notin M$ so $p \not| \mu$, then for $\bar{n}=\bar{p}$ supposing $\bar{\mu}\ .\bar{p}=\bar{0}$, i.e. if $\mu p \equiv 0 \pmod{p^a\sigma}$ then p/μ, contradiction. Thus $\forall \bar{\mu} \notin M$ we get $\bar{\mu}\ .\bar{p} \neq \bar{0}$ Q.E.D.

Theorem 5. *Let R be an associative ring. If R has a s. V-g, then R[x] has also a s. V-g.*

Proof. (We use the technique of [1, p. 97].)

Let M be a s. V-g of R and $f(x)$, $g(x) \in M[x]$, where

$$f(x)=a_0+a_1x+\ldots+a_mx^m, \quad g(x)=b_0+b_1x+\ \ldots\ +b_nx^n$$

and the leading coefficients a_m, b_n are different from zero. Then $f(x).g(x)\ =c_0+c_1x+\ \ldots\ +c_{m+n}x^{m+n}$, where

$$c_i=\sum_{j=0}^{i} a_jb_{i-j}=a_0b_i+a_1b_{i-1}+\ \ldots\ +a_ib_0$$

(taking $a_\varkappa=0$ or $b_\varkappa=0$, if $a_\varkappa$ or $b_\varkappa$ is missing). Obviously $f(x).g(x)=0$, $\forall f(x)$, $g(x) \in M[x]$. Now let $h(x)=h_0+h_1x+\ldots h_\varkappa x^\varkappa \in R[x]$, and

$$h(x).f(x)=f(x).h(x)=0, \quad \forall\ f(x) \in M[x].$$

For $f(x)=a_0$ we get $\forall\ a_0 \in M$

$$f(x).h(x)=a_0h_0+a_0h_1x+\ldots+a_0h_\varkappa x^\varkappa=0 \Rightarrow$$

$$a_0h_0=a_0h_1=\ \ldots\ =a_0h_\varkappa=0 \text{ (likewise } h_0a_0=h_1a_0=\ \ldots\ =h_\varkappa a_0=0)$$

so, because M is a s. V-g, we obtain that h_0, $h_1, \ldots, h_\varkappa \in M$ and $h(x) \in M[x]$; therefore $M[x]$ is a s. V-g Q.E.D.

Remark. Trivially we extend the above result to $R[x_1, \ldots, x_n]$.

We conclude with an example using the same notation and theorems as in [2].

Example. Let R be the ring of $n \times n$ square matrices over a field K with characteristic $\neq 2$. We define an *xy*-symmetric matrix $A=(a_{ij})$ as one which satisfies the following

$$\begin{array}{ll}\text{(I)} & a_{ij}=a_{n+1-i,j} \\ \text{(II)} & a_{ij}=-a_{i,n+1-j}\end{array}\Bigg\}\ \forall\, i,\ j=1,\dots,\ n.$$

We define similarly the yx-symmetric matrices. The set $S_{xy}=\{A/A$ is an xy-symmetric matrix over $K\}$ is a s. V-g of R.

Proof. In [2] one can see that

$A^{|T}=A$ means that A is a symmetric matrix with respect to the vertical median,

$A^{-T}=-A$ means that A is a skew symmetric matrix with respect to the horizontal median.

So a matrix A is xy-symmetric iff

$$\text{(III)} \qquad A=A^{|T}=-A^{-T}$$

or equivalently

$$A=A\,.\,1'=-1'\,.\,A,$$

where $1'=1'(n)$ is the transpose of the unit $n\times n$ matrix with respect to the vertical median.

In order to prove that S_{xy} is a s. V-g of R first we observe that S_{xy} is an additive group and for every A and B of S_{xy} we get

$$AB=-A\,.\,1'\,.\,1'\,.\,B=-AB, \text{ because } 1'\,.\,1'=1,$$

so $2AB=0$ and $AB=0$. Similarly $BA=0$, i.e. $AB=BA=0$.

It remains to see that if $AD=DA=0$ for every A of S_{xy} then $D=(d_{ij})\in S_{xy}$. Suppose $n=2p$. From $AD=0$ we get

$$0=\sum_{s=1}^{p} a_{is}d_{sj}+\sum_{s=p+1}^{2p} a_{is}d_{sj}=\sum_{s=1}^{p} a_{is}d_{sj}+\sum_{s=p+1}^{2p}(-a_{i,2p+1-s})d_{sj}=$$

$$a_{i1}(d_{1j}-d_{2p,j})+a_{i2}(d_{2j}-d_{2p-1,j})+\dots+a_{ip}(d_{pj}-d_{p+1,j})$$

for all $a_{ij}\in K$. So $d_{ij}=d_{2p,j},\dots,\ d_{pj}=d_{p+1,j}$ hence D satisfies condition (I).

Likewise $DA=0$, $\forall\, A\in S_{xy}$ implies that D satisfies condition (II).

For $n=2p+1$ we work in a similar way observing that $a_{i,p+1}=0$ for $i=1,\dots,$ n.

It follows therefore that $D\in S_{xy}$.

References

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Received 30.05.1983