Mathematica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

Mathematica
Balkanica

Now Series Vol. 3, 1989, Fasc. 3-4

Numerical Solutions to the Gurtin-MacCamy Equation

T. Kostova, M. Marcheva*

Presented by V. Popov

A numerical method with rate of convergence $O(\Delta t)$ is proposed to find an approximate solution of the so-called von Foerster-Gurtin-MacCamy system which describes the dynamics of a population with respect to its age-structure.

1. Introduction

In the last decades mathematical methods are coming strong in biology. Ecology and more precisely – population theory is considered to be a classical field of their application.

The first who applied mathematical equations to the theory of communities is thought to be Malthus with his well-known law of exponential growth. The modern successors of that simplest model are much finer and correct descriptions of reality and provide problems which are rather complicated and of great interest from mathematical point of view.

These peculiarities are also possessed by the model offered in 1974 by M. E. Gurtin and R. C. MacCamy [1]. It describes the dynamics of a closed population taking account of its age-structure. Moreover, the birth and death moduli depend on the total population size $P(t)$. The following equations supplemented by the initial condition are usually referred to as von Foerster-Gurtin-MacCamy system:

$$(1.1)\qquad \begin{cases} \dfrac{\partial u}{\partial a}+\dfrac{\partial u}{\partial t}=-\,d[a,\int\limits_0^\infty u(a,t)da]\,.\,u(a,t) \\ u(0,t)=\int\limits_0^\infty b[a,\int\limits_0^\infty u(a,t)da]\,.\,u(a,t)da\,, \\ u(a,0)=\alpha(a) \end{cases}$$

where $u(a, t)$ is the size of the population of age a at time t;

$\alpha(a)$ is the initial age distribution of the population;

$b(a, P)$ is called the birth modulus (average number of offsprings, produced by an individual of age a);

$d(a, P)$ is called the death modulus (the death-rate at age a per unit population of age a).

The question of existence, uniqueness and stability of the solution of this non-linear hyperbolic partial integro-differential equation was investigated by

M. E. Gurtin and R. C. MacCamy [1]. Though the exact solution is not known, almost no attempts have been made to find an approximate solution. In T. V. Kostova [2] are offered some analytical approximations and a numerical method to approximate the solution of a simpler case of the system (1.1). In the present paper we consider a difference scheme method for finding a numerical solution to the original system with some additional minimal restrictions which are stated down in what follows. We show that the numerical approximations are bounded and that the scheme is convergent with rate $O(\Delta t)$.

2. Statement of the problem

We are concerned with the numerical solution of Gurtin-MacCamy system (1.1). We should consider the problem in some preliminarily fixed finite interval of time

$$0 \leqq t \leqq T. \tag{2.1}$$

Let the following conditions be satisfied:

I. 1. d, α, b are nonnegative functions;

2. d, b are smooth;

3. $\alpha \in C$ and it is compactly supported in the interval $[0, S]$, i. e. $\alpha(a)=0$ for $a \notin [0, S]$;

4. $B_{max} = \sup\limits_{\substack{\alpha>0 \\ y \geqq 0}} b(a, y) < \infty$;

5. The initial conditions are compatible.

II. The solution $u(a, t)$ of (1.1) is twice continuously differentiable in the considered rectangular:

$$\mathrm{R} = \{(a, t) : 0 \leqq t \leqq T, \quad 0 \leqq a \leqq A\}. \tag{2.2}$$

It is easy to establish by integrating the equations (1.1) along the characteristic lines $a\text{-}t=\text{const}$, that $u(a, t)$, differs from zero only in the region

$$\mathrm{W} = \{(a, t) : 0 \leqq t \leqq T, \quad a - t \leqq S\}.$$

Let us denote: $A = S + T$. For simplicity we shall consider the problem in the rectangular $\mathrm{R} \supset \mathrm{W}$.

We intend to construct a numerical method to find an approximate solution of the problem

$$\begin{cases} \dfrac{\partial u}{\partial a} + \dfrac{\partial u}{\partial t} = -d[a, \int\limits_0^A u(a, t)\mathrm{d}a] \,.\, u(a, t) \\ u(0, t) = \int\limits_0^A b[a, \int\limits_0^A (a, t)\mathrm{d}a] \,.\, u(a, t)\mathrm{d}a \\ u(a, 0) = \alpha(a), \qquad\qquad \text{for} \quad (a, t) \in \mathrm{R}. \end{cases} \tag{2.3}$$

3. Some properties of the solution in the discrete case

Let us consider the uniform mesh

(3.1) $$\Omega_{\Delta t}=\{(a_k,\ t_l);\quad k=0,\ 1,\ 2,\dots,\ K\quad l=0,\ 1,\ 2,\dots,\ L\},$$

where $a_k=k.\Delta t,\ t_l=l.\Delta t,\ \Delta t$ is the step in both directions and

$$(K+1).\Delta t>A\geqq K.\Delta t,\quad (L+1).\Delta t>T\geqq L.\Delta t.$$

Note that $K\geqq L$ (as $A\geqq T$).

We denote $u_{k,l}=u(k.\Delta t,\ l.\Delta T)$. Having in mind that

$$\frac{\partial u_{k,l}}{\partial a}=\frac{u_{k\cdot l}-u_{k-1,l}}{\Delta t}+O(\Delta t)$$

we can easily prove that

$$\left.\frac{\partial u}{\partial a}+\frac{\partial u}{\partial t}\right|_{k,l}=\frac{u_{k,l}-u_{k-1,l}+u_{k-1,l}-u_{k-1,l-1}}{\Delta t}+O(\Delta t)$$

$$=\frac{u_{k,l}-u_{k-1,l-1}}{\Delta t}+O(\Delta t).$$

We approximate the integral using the trapezoidal rule

$$\int\limits_0^A u(a,\ t_l)\mathrm{d}a=\int\limits_0^{K\cdot\Delta t} u(a,\ t_l)\mathrm{d}a+O(\Delta t)=I_{\Delta t}u_{\cdot,l}+O(\Delta t),$$

where

(3.2) $$I_{\Delta t}u_{\cdot,l}=\frac{\Delta t}{2}\cdot\{u_{0,l}+u_{K,l}+2\sum_{j=1}^{K-1}u_{j,l}\}.$$

Since

$$\int\limits_0^A u(a,\ t_l)\mathrm{d}a=\int\limits_0^A[u(a,\ t_{l-1})+O(\Delta t)]\mathrm{d}a=\int\limits_0^A u(a,\ t_{l-1})\mathrm{d}a+O(\Delta t)$$

we can approximate the integral in the following manner:

(3.3) $$\int\limits_0^A u(a,\ t_l)\mathrm{d}a=I_{\Delta t}u_{\cdot,l-1}+O(\Delta t).$$

Thus for each point of the mesh $\Omega_{\Delta t}$ it is fulfilled:

$$u_{k,l}=u_{k-1,l-1}[1-\Delta t.d(a_{k-1},\ I_{\Delta t}u_{\cdot,l-1})]+O(\Delta t^2),$$
$$k=1,\ 2,\dots,\ K;\ l=1,\ 2,\dots,\ L$$

$$(3.4)\quad \begin{cases} u_{0,l}=I_{\Delta t}[b(.,\ I_{\Delta t}u_{\cdot,l-1}).u_{\cdot,l-1}]+O(\Delta t) \\ \qquad\qquad\qquad\qquad\qquad l=1,2,\dots,L \\ u_{k,0}=\alpha_k=\alpha(k.\Delta t) \qquad k=0,1,\dots,K. \end{cases}$$

Simultaneously we consider the difference scheme

$$(3.5)\quad \begin{cases} w_{k,l}=w_{k-1,l-1}[1-\Delta t.d(a_{k-1},\ I_{\Delta t}w_{\cdot,l-1})], \\ \qquad\qquad\qquad\qquad\qquad k=1,2,\dots,K;\ l=1,2,\dots,L \\ w_{0,l}=I_{\Delta t}[b(.,\ I_{\Delta t}w_{\cdot,l-1}).w_{\cdot,l-1}], \\ \qquad\qquad\qquad\qquad\qquad l=1,2,\dots,L \\ w_{k,0}=\alpha_k=\alpha(k.\Delta t), \qquad k=0,1,\dots,K. \end{cases}$$

It is very important to prove some boundedness properties of the solution of system (3.5). This is done in the following

Theorem 1. *Let* $\Delta t\to 0$, $L\to\infty$, $K\to\infty$, *so that* $\Delta t.L\leqq T<\Delta t.(L+1)$, $\Delta t.K\leqq A<\Delta t.(K+1)$. *Then there exist constants* δ *and* C *such that whenever* $\Delta t<\delta$:

1. $w_{k,l}\geqq 0$ *for all* $k,\ l$: $k\leqq K,\quad l\leqq L$;
2. $w_{k,l}\leqq C,\quad k\leqq K,\quad l\leqq L.$

We shall prove first the following simple Lemma:

Lemma 1. *Let for a fixed* l *and for all* $k=0,1,\dots,K$ *be satisfied* $w_{k,l-1}\geqq 0$. *Then*:

$$(3.6)\qquad w_{0,l}\leqq B_{\max}.I_{\Delta t}w_{\cdot,l-1}.$$

Proof. It follows from (3.5_2) that

$$w_{0,l}=\frac{\Delta t}{2}.\{b(0,\ I_{\Delta t}w_{\cdot,l-1}).w_{0,l-1}+b(K,\ I_{\Delta t}w_{\cdot,l-1}).w_{K,l-1}$$

$$+2.\sum_{j=1}^{K-1} b(j,\ I_{\Delta t}w_{\cdot,l-1}).w_{j,l-1}\}$$

$$\leqq\frac{\Delta t}{2}.\{B_{\max}.w_{0,l-1}+B_{\max}.w_{K,l-1}+2.\sum_{j=1}^{K-1} B_{\max}.w_{j,l-1}\}=B_{\max}.I_{\Delta t}w_{\cdot,l-1}.$$

Proof of Theorem 1. We shall prove both statements of Theorem 1 jointly by induction on 1 and at each step simultaneously for all $k=1,2,\dots,K$. Let us denote

$$\alpha_{\max}=\max_{0\leqq a\leqq S}\alpha(a)$$

(3.7) $$\varphi_{k,l}=1-\Delta t\,.\,d(a_k,\ I_{\Delta t}w_{\cdot,l}).$$

Then (3.5_1) appears as

(3.8) $$w_{k,l}=w_{k-1,l-1}\cdot\varphi_{k-1,l-1}\,.$$

Let $R=e^{\frac{3}{2}\cdot B_{\max}\cdot T}.A.\alpha_{\max}+(e^{\frac{3}{2}\cdot B_{\max}\cdot T}-1).\alpha_{\max}$
and

(3.9) $$\delta_1=\frac{1}{2\,.\max\limits_{\substack{0\leqq a\leqq A\\0\leqq y\leqq R}} d(a,\ y)}.$$

Let $\Delta t<\delta_1$ and $\Delta t<2$. We denote

$$R_j=(\frac{3}{2}.\Delta t.B_{\max}+1).I_{\Delta t}w_{\cdot,j-1}+\frac{3}{4}.\Delta t^2.B_{\max}\cdot\alpha_{\max},\quad j=2,\dots,\ L.$$

We consider the following statements:

$$S_1(j):w_{k,j}\geqq 0\qquad \forall k=0,\ 1,\dots,\ K;$$
$$S_2(j):I_{\Delta t}w_{\cdot,j}\leqq R_j;$$
$$S_3(j):w_{k,j}\leqq\alpha_{\max}\qquad \text{if } k\geqq j.$$

Let $2<l<L$. *We shall show that if* $S_1(l-1)$, $S_3(l-1)$, $S_3(l-2)$, and $S_2(j)$, $j=2,\dots,\ l-1$ are true, then $S_1(l)$, $S_2(l)$, and $S_3(l)$ are also true.

First of all, the validity of $S_2(j)$, $j=2,\dots,l-1$ leads to the following chain of inequalities:

$$I_{\Delta t}w_{\cdot,l-1}\leqq R_{l-1}\leqq(\frac{3}{2}.\Delta t.B_{\max}+1).R_{l-2}+\frac{3}{4}.\Delta t^2.B_{\max}\cdot\alpha_{\max}$$
$$\leqq(\frac{3}{2}.\Delta t.B_{\max}+1).((\frac{3}{2}.\Delta t.B_{\max}+1).I_{\Delta t}w_{\cdot,l-3}+\frac{3}{4}.\Delta t^2.B_{\max}\cdot\alpha_{\max})$$
$$+\frac{3}{4}.\Delta t^2.B_{\max}\cdot\alpha_{\max}\leqq\dots\leqq(\frac{3}{2}.\Delta t.B_{\max}+1)^{l-2}.I_{\Delta t}w_{\cdot,1}$$
$$+((\frac{3}{2}.\Delta t.B_{\max}+1)^{l-3}+\dots+1).\frac{3}{4}.\Delta t^2.B_{\max}\cdot\alpha_{\max}=(\frac{3}{2}.\Delta t.B_{\max}+1)^{l-2}.I_{\Delta t}w_{\cdot,1}$$
$$+\frac{(\frac{3}{2}.\Delta t.B_{\max}+1)^{l-2}-1}{\frac{3}{2}.\Delta t.B_{\max}}.\frac{3}{4}.\Delta t^2.B_{\max}\cdot\alpha_{\max}\leqq(\frac{3}{2}.\Delta t.B_{\max}+1)^{l-2}.I_{\Delta t}w_{\cdot,1}$$
$$+((\frac{3}{2}.\Delta t.B_{\max}+1)^l-1).\alpha_{\max};$$

(3.10)

$$I_{\Delta t} w_{\cdot,1} = \frac{\Delta t}{2} \cdot \{w_{0,1} + w_{K,1} + 2 \,.\, \sum_{j=1}^{K-1} w_{j,1}\} \leqq \frac{\Delta t}{2} \cdot \{B_{\max} \cdot I_{\Delta t} w_{\cdot,0} + w_{K-1,0} + 2 \,.\, \sum_{j=0}^{K-2} w_{j,0}\}$$

$$\leqq \frac{\Delta t}{2} \cdot \{B_{\max} \,.\, A \,.\, \alpha_{\max} + 2 \,.\, K \,.\, \alpha_{\max}\} \leqq (\frac{\Delta t}{2} \,.\, B_{\max} + 1) \,.\, A \,.\, \alpha_{\max}$$

$$\leqq (\frac{3}{2} \,.\, \Delta t \,.\, B_{\max} + 1) \,.\, A \,.\, \alpha_{\max}.$$

Then (3.10) gives $I_{\Delta t} w_{\cdot,l-1} \leqq R$.

Therefore $\varphi_{k,l-1} \in [\frac{1}{2}, 1]$ and from (3.8) it follows that

(3.11)
$$w_{k,l} \geqq 0, \qquad \forall k = 1, 2, \ldots, K,$$
$$w_{k,l} \leqq w_{k-1,l-1} \leqq \ldots \leqq w_{k-l,0} < \alpha_{\max} \quad \text{for} \quad k \geqq l.$$

From (3.5) it follows:

$$w_{0,l} = I_{\Delta t} b(.,\ I_{\Delta t} w_{\cdot,l-1}) \geqq 0.$$

Therefore $S_1(1)$ and $S_3(1)$ are true. Now we shall use $S_1(l-1)$ and $S_3(l-2)$ to prove the validity of $S_2(l)$

$$I_{\Delta t} w_{\cdot,l-2} = \frac{\Delta t}{2} \cdot \{w_{0,l-2} + w_{K,l-2} + 2 \,.\, \sum_{j=1}^{K-1} w_{j,l-2}\}$$

$$\leqq \frac{\Delta t}{2} \cdot \{2 \,.\, w_{1,l-1} + \alpha_{\max} + 4 \,.\, \sum_{j=2}^{K} w_{j,l-1}\}$$

$$\leqq \frac{\Delta t}{2} \cdot \{\alpha_{\max} + 2 \,.\, w_{K,l-1} + 2 \,.\, w_{0,l-1} + 2 \,.\, w_{K,l-1} + 4 \,.\, \sum_{j=1}^{K-1} w_{j,l-1}\}$$

(3.12)
$$\leqq \frac{\Delta t}{2} \,.\, 3 \,.\, \alpha_{\max} + 2 \,.\, I_{\Delta t} w_{\cdot,l-1}.$$

Therefore

$$I_{\Delta t} w_{\cdot,l} \leqq \frac{\Delta t}{2} \cdot \{w_{0,l} + w_{K,l} + 2 \cdot \sum_{j=1}^{K-1} w_{j,l}\}$$

$$\leqq \frac{\Delta t}{2} \cdot \{B_{\max} \cdot I_{\Delta t} w_{\cdot,l-1} + w_{K-1,l-1} + 2 \cdot \sum_{j=0}^{K-2} w_{j,l-1}\}$$

$$\leqq \frac{\Delta t}{2} \cdot \{B_{\max} \cdot I_{\Delta t} w_{\cdot,l-1} + w_{0,l-1}\} + I_{\Delta t} w_{\cdot,l-1}$$

$$\leqq \frac{\Delta t}{2} \cdot \{B_{\max} \cdot I_{\Delta t} w_{\cdot,l-1} + B_{\max} \cdot I_{\Delta t} w_{\cdot,l-2}\} + I_{\Delta t} w_{\cdot,l-1}$$

$$\leqq \frac{\Delta t}{2} \cdot \{B_{\max} \cdot I_{\Delta t} w_{\cdot,l-1} + 2 \cdot B_{\max} \cdot I_{\Delta t} w_{\cdot,l-1} + \frac{3}{2} \cdot \Delta t \cdot B_{\max} \cdot \alpha_{\max}\} + I_{\Delta t} w_{\cdot,l-1}$$

$$\leqq (\frac{3}{2} \cdot \Delta t \cdot B_{\max} + 1) \cdot I_{\Delta t} w_{\cdot,l-1} + \frac{3}{4} \cdot \Delta t^2 \cdot B_{\max} \cdot \alpha_{\max} = R_l < R.$$

We have to check now the validity of $S_1(2)$, $S_3(1)$, $S_3(2)$, and $S_2(2)$. **First**, note that

$$I_{\Delta t} w_{\cdot,0} \leqq A \cdot \alpha_{\max} < R$$

$$I_{\Delta t} w_{\cdot,1} = \frac{\Delta t}{2} \cdot \{w_{0,1} + w_{K,1} + 2 \cdot \sum_{j=1}^{K-1} w_{j,1}\}$$

$$\leqq \frac{\Delta t}{2} \cdot \{B_{\max} \cdot I_{\Delta t} w_{\cdot,0} + w_{K-1,0} + 2 \cdot \sum_{j=0}^{K-2} w_{j,0}\}$$

$$\leqq \frac{\Delta t}{2} \cdot \{B_{\max} \cdot A \cdot \alpha_{\max} + 2 \cdot K \cdot \alpha_{\max}\} \leqq (\frac{\Delta t}{2} \cdot B_{\max} + 1) \cdot A \cdot \alpha_{\max}$$

$$\leqq (\frac{3}{2} \cdot \Delta t \cdot B_{\max} + 1) \cdot A \cdot \alpha_{\max} < R.$$

Therefore $\frac{1}{2} \leqq \varphi_{k-s,2-s} \leqq 1$ for $s=1$, 2. These inequalities directly imply

1. $0 \leqq w_{k,1} = \varphi_{k-1,0} w_{k-1,0} \leqq \alpha_{\max}$, $k \geqq 1$,
   $w_{0,1} = I_{\Delta t} b(\cdot, I_{\Delta t} w_{\cdot,0}) \cdot w_{\cdot,0} \geqq 0$;

2. $0 \leqq w_{k,2} = w_{k-2,0} \cdot \varphi_{k-2,0} \cdot \varphi_{k-1,1} \leqq \alpha_{\max}, \; k \geqq 2,$

$w_{1,2} = w_{0,1} \cdot \varphi_{0,1} \geqq 0,$

$w_{0,2} = I_{\Delta t} b(., \; I_{\Delta t} w._{,1}) . w._{,1} \geqq 0.$

Thus we proved that $S_1(2)$, $S_3(1)$, and $S_3(2)$ are true.

Second, it is obvious that $S_3(0)$ and $S_1(1)$ are also true. Further, proceeding exactly as before (in the case of $I_{\Delta t} w._{,l}$, $l \geqq 3$), we can easily verify that

$$I_{\Delta t} w._{,2} \leqq R_2 \quad \text{(i.e. } S_2(2) \text{ is also true).}$$

Thus we have proved that $S_1(j)$, $S_2(j)$, and $S_3(j)$ hold true for all $j=3, \dots, L$ when $\Delta t < \min(\delta_1, 2) = \delta$. Therefore, as is seen from the chain of inequalities (3.10)

$$I_{\Delta t} w._{,j} < R, \quad j=3, \dots, L.$$

To conclude the proof of Theorem 1, we have to establish that $w_{k,j}$, $k<j$ are bounded. Really

$$w_{k,j} = \varphi_{k-1,j-1} \dots \varphi_{0,j-k} \leqq I_{\Delta t} w._{,j-k-1} < R.$$

In view of this inequality and (3.11), we choose

$$C = \max(R, \; \alpha_{\max})$$

and the proof of Theorem 1 is completed.

4. Proof of the convergence of the method

Let us denote $\varepsilon_{k,l} = u_{k,l} - w_{k,l}$, $\forall k=0, \dots, K; \; l=0, \dots, L.$

Because of the boundedness of $|u_{k,l}|$ and $|w_{k,l}|$ there exists a constant C_ε (independent of k, l) such that

$$|\varepsilon_{k,l}| \leqq C_\varepsilon. \tag{4.1}$$

We seek more rigorous estimate for $|\varepsilon_{k,l}|$.

After subtracting the system (3.4) from the system (3.5) we consider the equations for the error

$$\varepsilon_{k,l} = u_{k,l} - w_{k,l} = u_{k-1,l-1} - \Delta t \,.\, u_{k-1,l-1} \,.\, d(a_{k-1}, \; I_{\Delta t} u._{,l-1})$$

$$+ O(\Delta t^2) - w_{k-1,l-1} + \Delta t \,.\, w_{k-1,l-1} \,.\, d(a_{k-1}, \; I_{\Delta t} w._{,l-1})$$

$$=\varepsilon_{k-1,l-1}-\Delta t\,.\,u_{k-1,l-1}\,.\,d(a_{k-1},\ I_{\Delta t}u_{\cdot,l-1})$$

$$+\Delta t\,.\,w_{k-1,l-1}\,.\,d(a_{k-1},\ I_{\Delta t}u_{\cdot,l-1})+O(\Delta t^2)$$
$$-\Delta t\,.\,w_{k-1,l-1}\,.\,[d(a_{k-1},\ I_{\Delta t}u_{\cdot,l-1})-d(a_{k-1},\ I_{\Delta t}w_{\cdot,l-1})]$$

$$=\varepsilon_{k-1,l-1}(1-\Delta t\,.\,d(a_{k-1},\ I_{\Delta t}u_{\cdot,l-1}))$$

$$-\Delta t\,.\,w_{k-1,l-1}\frac{\partial d(a_{k-1},\,y)}{\partial y}_{/y=\xi_{l-1}}\,.\,I_{\Delta t}\varepsilon_{\cdot,l-1}+O(\Delta t^2).$$

Then

$$\varepsilon_{k,l}=\varepsilon_{k-1,l-1}[1-\Delta t\,.\,d(a_{k-1},\ I_{\Delta t}u_{\cdot,l-1})]$$

$$-\Delta t\,.\,w_{k-1,l-1}\,.\,I_{\Delta t}\varepsilon_{\cdot,l-1}\,.\,\frac{\partial d(a_{k-1},y)}{\partial y}_{/y=\xi_{l-1}}+0(\Delta t^2),$$

where $\xi_{l-1}\in(I_{\Delta t}w_{\cdot,l-1}\vee I_{\Delta t}u_{\cdot,l-1})$. By $(a\vee b)$ we mean $(\min(a,\ b),\ \max(a,\ b))$

$$u_{0,l}-w_{0,l}=I_{\Delta t}[b(.,\ I_{\Delta t}u_{\cdot,l-1})\,.\,u_{\cdot,l-1}]-I_{\Delta t}[b(.,\ I_{\Delta t}w_{\cdot,l-1})\,.\,w_{\cdot,l-1}]+O(\Delta t)$$

$$=I_{\Delta t}[b(.,\ I_{\Delta t}u_{\cdot,l-1})\,.\,u_{\cdot,l-1}]-I_{\Delta t}[b(.,\ I_{\Delta t}u_{\cdot,l-1})\,.\,w_{\cdot,l-1}]$$

$$+I_{\Delta t}[b(.,\ I_{\Delta t}u_{\cdot,l-1})\,.\,w_{\cdot,l-1}]-I_{\Delta t}[b(.,\ I_{\Delta t}w_{\cdot,l-1})\,.\,w_{\cdot,l-1}]+O(\Delta t)$$

$$=I_{\Delta t}[b(.,\ I_{\Delta t}u_{\cdot,l-1})\,.\,\varepsilon_{\cdot,l-1}]+O(\Delta t)+I_{\Delta t}[(b(.,\ I_{\Delta t}u_{\cdot,l-1})-b(.,\ I_{\Delta t}w_{\cdot,l-1}))\,.\,w_{\cdot,l-1}]$$

$$=I_{\Delta t}[b(.,\ I_{\Delta t}u_{\cdot,l-1})\,.\,\varepsilon_{\cdot,l-1}]+O(\Delta t)+I_{\Delta t}[\frac{\partial b(.,y)}{\partial y}_{/y=\mathring{\eta}_{l-1}}\,.\,w_{\cdot,l-1}]\,.\,I_{\Delta t}\varepsilon_{\cdot,l-1},$$

where $\mathring{\eta}_{l-1}\in(I_{\Delta t}u_{\cdot,l-1}\vee I_{\Delta t}w_{\cdot,l-1})$. So

$$\varepsilon_{0,l}=I_{\Delta t}[b(.,\ I_{\Delta t}u_{\cdot,l-1})\,.\,\varepsilon_{\cdot,l-1}]+O(\Delta t)$$

$$+I_{\Delta_t}[\frac{\partial b(.,y)}{\partial y}_{/y=\mathring{\eta}_{l-1}}\,.\,w_{\cdot,l-1}]\,.\,I_{\Delta t}\varepsilon_{\cdot,l-1}\,.$$

The last difference appears as

$$\varepsilon_{k,0}=0.$$

Thus we arrive at the following system for the error

$$(4.2)\quad \begin{cases} \varepsilon_{k,l}=\varepsilon_{k-1,l-1}[1-\Delta t\,.\,d(a_{k-1},\ I_{\Delta t}u_{\cdot,l-1})]+O(\Delta t^2) \\ \qquad\qquad -\Delta t\,.\,w_{k-1,l-1}\,.\,I_{\Delta t}\,\varepsilon_{\cdot,l-1}\,.\,\dfrac{\partial d(a_{k-1},y)}{\partial y}{}_{/y=\xi_{l-1}}\,; \\ \varepsilon_{0,l}=I_{\Delta t}[b(.,I_{\Delta t}u_{\cdot,l-1})\,.\,\varepsilon_{\cdot,l-1}]+O(\Delta t) \\ \qquad\qquad +I_{\Delta t}[\dfrac{\partial b(.,y)}{\partial y}{}_{/y=\eta^{\circ}_{l-1}}\,.\,w_{\cdot,l-1}]\,.\,I_{\Delta t}\varepsilon_{\cdot,l-1} \\ \varepsilon_{k,}=0. \end{cases}$$

Since this system is too complicated, and in the same time, our goal is to restrict $|\varepsilon_{k,l}|$, we shall use some inequalities to make (4.2) more concise.

Let us denote

$$\psi_{k,l}=1-\Delta t\,.\,d(a_k,\ I_{\Delta t}u_{\circ,l}).$$

Since

$$\int_0^A u(a,\ t_l)\mathrm{d}a=I_{\Delta t}u_{\cdot,l}+O(\Delta t^2)$$

and, on the other hand,

$$\int_0^A u(a,\ t_l)\mathrm{d}a\leqq M\,.\,A,\quad \text{where}\quad M=\max_{\substack{0\leqq a\leqq A\\ 0\leqq t\leqq T}} u(a,\ t)$$

then there exists a constant Const, such that

$$I_{\Delta t}u_{\cdot,l}\leqq \text{Const}.$$

Thus if we require that $\Delta t\leqq \dfrac{1}{2\,.\,\max\limits_{\substack{0\leqq a\leqq A\\ 0\leqq y\leqq \text{Const}}} \mathrm{d}(a,y)}$ then

$$(4.3)\qquad \frac{1}{2}\leqq\psi_{k,l}\leqq 1.$$

Beside that we denote

$$\frac{\partial d(a_k,\ y)}{\partial y}{}_{/y=\xi_l}=\theta_{k,l}\,.$$

Next we condude from the assumptions for d and the fact that $I_{\Delta t}w_{\cdot,l}$ and $I_{\Delta t}u_{\cdot,l}$ are bounded, that

$$\left|\frac{\partial d(a,\,y)}{\partial y}\Big/_{y=\xi_l}\right| \leqq \max_{\substack{0\leqq a\leqq A\\ 0\leqq y\leqq \max}} \left|\frac{\partial u(a,\,y)}{\partial y}\right|_{(\mathrm{Const}\ .\ R)} = (\text{denotation}) = D$$

(4.4)

$$\Rightarrow |\theta_{k,l}| \leqq D \quad \forall k=1,\,2,\ldots,\,K \quad l=1,\,2,\ldots,\,L$$

$$\left|\frac{\partial b(a_k,\,y)}{\partial y}\Big/_{y=\eta_l^k}\right| \leqq \max_{\substack{0\leqq a\leqq A\\ 0\leqq y\leqq \max}} \left|\frac{\partial b(a,\,y)}{\partial y}\right|_{(\mathrm{Const}\ .\ R)} = (\text{denotation}) = B.$$

With this in mind we restrict $|\varepsilon_{0,l}|$:

$$|\varepsilon_{0,l}| \leqq \mathrm{I}_{\Delta t}[b(.,\; I_{\Delta t}u_{.,l}).|\varepsilon_{.,l-1}|] + O(\Delta t) + I_{\Delta t}\Big[\frac{\partial b(.,\,y)}{\partial y}\Big/_{y=\eta_{l-1}}.\,w_{.,l-1}\Big].I_{\Delta t}|\varepsilon_{.,l-1}|$$

$$\leqq B_{\max}.I_{\Delta t}|\varepsilon_{.,l-1}| + B.I_{\Delta t}w_{.,l-1}.I_{\Delta t}|\varepsilon_{.,l-1}| + O(\Delta t)$$

$$\leqq (B_{\max}.B.R).I_{\Delta t}|\varepsilon_{.,l-1}| + O(\Delta t) = E.I_{\Delta t}|\varepsilon_{.,l-1}| + O(\Delta t),$$

where $E = B_{\max} + B.R$. Then the system (4.2) has the form

$$(4.5)\quad \begin{cases} \varepsilon_{k,l} = \varepsilon_{k-1,l-1}.\psi_{k-1,l-1} - \Delta t.w_{k-1,l-1}.\theta_{k-1,l-1}.I_{\Delta t}\varepsilon_{.,l-1} + O(\Delta t^2) \\ |\varepsilon_{0,l}| \leqq E.I_{\Delta t}|\varepsilon_{.,l-1}| + O(\Delta t) \\ \varepsilon_{k,0} = 0, \qquad k=1,\,2,\ldots,\,K \quad l=1,\,2,\ldots,\,L. \end{cases}$$

Let us consider the equality (4.5_1).

1. If $\theta_{k-1,l-1}$ and $I_{\Delta t}\varepsilon_{.,l-1}$ are with opposite signs then the inequalities (4.3), (4.4) and (4.1) imply that

$$|\varepsilon_{k,l}| \geqq \frac{1}{2}.|\varepsilon_{k-1,l-1}| + O(\Delta t).$$

2. If $\theta_{k-1,l-1}$ and $I_{\Delta t}\varepsilon_{.,l-1}$ are with one and the same sign then $(-\Delta t.w_{k-1,l-1}.\theta_{k-1,l-1}.I_{\Delta t}\varepsilon_{.,l-1})$ can be disregarded

$$|\varepsilon_{k,l}| \geqq \frac{1}{2}.|\varepsilon_{k-1,l-1}| + O(\Delta t^2).$$

In both cases we conclude that

$$|\varepsilon_{k-1,l-1}| \leqq 2.|\varepsilon_{k,l}| + O(\Delta t).$$

From this inequality it follows in the same way as it is done in (3.12) that

$$I_{\Delta t}|\varepsilon_{\cdot,l-1}|\leqq 2\,.\,I_{\Delta t}|\varepsilon_{\cdot,l}|+O(\Delta t).$$

Having in mind this the inequality (4.4_2) can be written also in the form

$$(4.4_2)^* \qquad |\varepsilon_{0,l}|\leqq 2\,.\,E\,.\,I_{\Delta t}|\varepsilon_{\cdot,l}|+O(\Delta t).$$

Now we can prove the following theorem:

Theorem 2. *The solution at an arbitrary fixed point* $(a_0, t_0)\in\{(a, t): 0\leqq a\leqq A, 0\leqq t\leqq T\}$ *of the problem* (3.5) *using a suitable mesh with a step* Δt *converges when* $\Delta t\to 0$ *with rate of convergence* $O(\Delta t)$ *to the exact solution of the population problem* (3.4) *taken at the same point.*

Proof. Let k and l are fixed such that $a_0=\Delta t\,.\,k$, $t_0=\Delta t\,.\,l$.
I. We consider the case when $k\geqq l$.

$$|\varepsilon_{k,l}|=|\varepsilon_{k-1,l-1}\,.\,\psi_{k-1,l-1}-\Delta t\,.\,w_{k-1,l-1}\,.\,\theta_{k-1,l-1}\,.\,I_{\Delta t}\varepsilon_{\circ,l-1}+O(\Delta t^2)|$$

$$\leqq[|\varepsilon_{k-2,l-2}|\,.\,\psi_{k-2,l-2}+\Delta t\,.\,w_{k-2,l-2}\,.\,|\theta_{k-2,l-2}|\,.\,I_{\Delta t}|\varepsilon_{\circ,l-2}|$$

$$+O(\Delta t^2)]|\,.\,\psi_{k-1,l-1}+\Delta t\,.\,w_{k-1,l-1}\,.\,|\theta_{k-1,l-1}|\,.\,I_{\Delta t}|\varepsilon_{\circ,l-1}|+O(\Delta t^2)\leqq\ldots.$$

$$\leqq|\varepsilon_{k-l,0}|\,.\,\psi_{k-l,0}\ldots\psi_{k-1,l-1}+\Delta t\,.\,\{w_{k-1,l-1}\,.\,|\theta_{k-1,l-1}|\,.\,I_{\Delta t}|\varepsilon_{\cdot,l-1}|$$

$$+\ldots+w_{k-l,0}\,.\,|\theta_{k-l,0}|\,.\,I_{\Delta t}|\varepsilon_{\cdot,0}|\,.\,\psi_{k-l+1,1}\ldots\psi_{k-1,l-1}\}$$

$$+O(\Delta t^2)\,.\,(1+\psi_{k-1,l-1}+\psi_{k-1,l-1}\,.\,\psi_{k-2,l-2}+\ldots$$

$$(4.6) \qquad \ldots+\psi_{k-1,l-1}\ldots\psi_{k-l,0}).$$

According to Theorem 1 all $w_{k,l}$ are bounded. The inequalities (4.3) and (4.4) yield the same for all $\psi_{k,l}$ and $\theta_{k,l}$. Thus out of all multipliers in the right-hand side of (4.6) only $I_{\Delta t}|\varepsilon_{\cdot,l}|$ has not been evaluated. So we consider

$$I_{\Delta t}|\varepsilon_{\cdot,l}|=\frac{\Delta t}{2}\,.\,\{|\varepsilon_{0,l}|+|\varepsilon_{K,l}|+2\,.\sum_{j=1}^{K-1}|\varepsilon_{j,l}|\}\leqq\frac{\Delta t}{2}\,.\,\{E\,.\,I_{\Delta t}|\varepsilon_{\cdot,l-1}|+O(\Delta t)$$

$$+|\varepsilon_{K-1,l-1}\,.\,\psi_{K-1,l-1}-\Delta t\,.\,w_{K-1,l-1}\,.\,\theta_{K-1,l-1}\,.\,I_{\Delta t}\varepsilon_{\cdot,l-1}|$$

$$+2\,.\sum_{j=1}^{K-1}|\varepsilon_{j-1,l-1}\,.\,\psi_{j-1,l-1}-\Delta t\,.\,w_{j-1,l-1}\,.\,\theta_{j-1,l-1}\,.\,I_{\Delta t}\varepsilon_{\cdot,l-1}|\}$$

$$\leqq \frac{3}{2}.\Delta t.E.I_{\Delta t}|\varepsilon_{\cdot,l-1}|+I_{\Delta t}|\varepsilon_{\cdot,l-1}|+O(\Delta t^2)+\frac{\Delta t^2}{2}.[|w_{K-1,l-1}.\theta_{K-1,l-1}|$$

$$+2.\sum_{j=1}^{K-1}|w_{j-1,l-1}.\theta_{j-1,l-1}|].I_{\Delta t}|\varepsilon_{\cdot,l-1}|$$

$$\leqq[\frac{\Delta t^2}{2}.C.D.2.K+\frac{3}{2}.\Delta t.E+1].I_{\Delta t}|\varepsilon_{\cdot,l-1}|+O(\Delta t^2)$$

$$\leqq[(C.D.A+\frac{3}{2}.E).\Delta t+1].I_{\Delta t}|\varepsilon_{\cdot,l-1}|+O(\Delta t^2)\leqq\dots$$

We denote $(C.D.A+\frac{3}{2}.E)=P=\text{constant}$

$$\leqq(P.\Delta t+1)^l.I_{\Delta t}|\varepsilon_{\cdot,0}|+[(P.\Delta t+1)^{l-1}+\dots+1].O(\Delta t^2)\leqq.$$

Having in mind that $(P.\Delta t+1)^l$ is bounded by a constant we obtain

$$\leqq\frac{(P.\Delta t+1)^l-1}{P.\Delta t}.O(\Delta t^2)\leqq O(\Delta t)$$

$$\Rightarrow I_{\Delta t}|\varepsilon_{\cdot,l}|\leqq O(\Delta t). \tag{4.7}$$

The above inequality and the inequality (4.6) imply that

$$|\varepsilon_{k,l}|\leqq O(\Delta t^2).[|w_{k-1,l-1}.\theta_{k-1,l-1}|+\dots$$

$$+|w_{k-l+1,1}.\theta_{k-l+1,1}.\psi_{k-l+1,1}\dots\psi_{k-1,l-1}|]$$

$$+O(\Delta t^2).[1+\psi_{k-1,l-1}+\dots+\psi_{k-1,l-1}\dots\psi_{k-l+1,1}]\leqq(l-1).O(\Delta t^2).C.D\leqq O(\Delta t)$$

$$\Rightarrow|\varepsilon_{k,l}|\leqq O(\Delta t),\quad \forall k,\ l:k\geqq l. \tag{4.8}$$

II. We consider the case when $k<l$. Then

$$\varepsilon_{k,l}=\varepsilon_{0,l-k}.\psi_{k-1,l-1}\dots\psi_{0,l-k}-\Delta t.\{w_{k-1,l-1}.\theta_{k-1,l-1}.I_{\Delta t}\varepsilon_{\cdot,l-1}+\dots$$

$$+w_{1,l-k+1}.\theta_{1,l-k+1}.I_{\Delta t}\varepsilon_{\cdot,l-k}.\psi_{k-1,l-1}\dots\psi_{2,l-k+2}\}$$

$$+O(\Delta t^2).[1+\psi_{k-1,l-1}+\dots+\psi_{k-1,l-1}\dots\psi_{1,l-k+1}]. \tag{4.9}$$

The inequality (4.7) holds again because its verification does not depend on the relation between k and l. Therefore

$$|\varepsilon_{0,l-k}| \leqq E \, . \, I_{\Delta t} |\varepsilon_{\cdot,l-k-1}| + O(\Delta t) \leqq O(\Delta t).$$

The equality (4.9) and the last inequality imply that

$$|\varepsilon_{k,l}| \leqq |\varepsilon_{0,l-k}| \cdot |\psi_{k-1,l-1}| \ldots |\psi_{0,l-k}| + \Delta t \, . \, \{|w_{k-1,l-1} \cdot \theta_{k-1,l-1}| \, . \, O(\Delta t) + \ldots$$

$$+ w_{1,l-k+1} \cdot |\theta_{1,l-k+1}| \, . \, O(\Delta t) \, . \, |\psi_{k-1,l-1}| \ldots |\psi_{2,l-k+2}|\}$$

$$+ O(\Delta t^2) \, . \, [1 + |\psi_{k-1,l-1}| + \ldots + |\psi_{k-1,l-1}| \ldots |\psi_{1,l-k+1}|]$$

$$\leqq O(\Delta t) + l \, . \, O(\Delta t^2) + l \, . \, O(\Delta t^2) \leqq O(\Delta t) \Rightarrow |\varepsilon_{k,l}| \leqq O(\Delta t) \qquad \forall k, \; l : k < l. \tag{4.10}$$

The derived estimates in (4.8) and (4.10) show that

$$|\varepsilon_{k,l}| \xrightarrow[\Delta t \to 0]{} 0 \text{ with rate of convergence } O(\Delta t).$$

This completes the proof of Theorem 2.

References

1. M. E. Gurtin, R. C. MacCamy. Non-linear age-dependent population dynamics. *Archs Ration. Mech. Analysis*, **54**, 1974, 281-300.
2. T. V. Kostova. Numerical solutions of a hyperbolic differential-integral equations. *Comput. Math. Applic.*, **15**, No 6-8, 1988, 427-436.

Institute of Mathematics
Bulgarian Academy of Sciences
P.O.B 373
1090 Sofia
BULGARIA

Received 01. 12. 1988