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Acoustic Scattering Theory for a Multi-Layered Scatterer in Low-Frequency

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An incident acoustic plane wave is scattered by a multi-layered scatterer. Integral representations for the total, exterior field, as well as for the normalized scattering amplitude are given. Using low-frequency techniques the wave problem is reduced to an iterative sequence of elliptic boundary value problems, which can be solved successively, using methods of potential theory. Low-frequency expansions are given for the total exterior field and the normalized scattering amplitude. We evaluate the leading term of the normalized scattering amplitude and the scattering cross-section in the low-frequency region in the case of a multi-layered scatterer with a rigid or soft core.

1. Introduction

We consider the three-dimensional problem of Helmholtz equation corresponding to the scattering of an acoustic wave by a multi-layered scatterer.

J. Rayleigh's [9] idea of approximating a scattering problem by the corresponding static problem when the wavelength is large compared to the characteristic dimension of the scatterer has been extensively investigated. In electromagnetics, A. Stevenson has developed a procedure for obtaining the general term of the electromagnetic field in low-frequency expansion [10]. The main contribution to electromagnetic and acoustic scattering in low-frequencies was made by R. Kleinman. In his paper [7] he formulated the Dirichlet boundary value problem (soft scatterer). E. Ar and R. Kleinman considered the Neumann boundary value problem for the Helmholtz equation (rigid scatterer) [1]. Some important theorems for scattering are proved by V. Twersky [11]. G. Dassios has investigated the scattering of acoustic wave by a penetrable body [3] and by a penetrable body with an impenetrable core [4]. He applied the method to a general ellipsoid with a confocal ellipsoidal core. Low-frequencies expansions have already been used in elasticity [5].

In this work an integrated and systematic theory for scattering of acoustic wave by a multi-layered scatterer is developed. A multi-layered scatterer consists of many closed penetrable surfaces S_j, $j=1, 2, \dots, q-1$ and an impenetrable (core) S_q. Every surface S_a is inside S_b for $a>b$. For $q=2$ we have the scattering problem for a penetrable body with an impenetrable core [4]. So, due to its generality, this work includes all previously studied problems as special cases.

We give an integral representation for the total (incident plus scattered) exterior field. Based on the asymptotic form of the integral representation for the total field, we express the normalized scattering amplitude in a closed form.

All the wave fields of the problem as well as the fundamental solution are expanded in power series of the wave number. Using these expansions the wave problem is reduced to a sequence of potential problems which can be solved recursively by means of appropriate harmonic functions. The particular potential problems that determine all the coefficients are stated explicitely. Integral representations for every coefficients, as well as their asymptotic form, far away from the scatterer, are found. The far field form of the integral representation of the total exterior field provides particular solutions of the corresponding Poisson equations.

We prove that when the core of the multi-layered scatterer is rigid the leading low-frequency approximation of the normalized scattering amplitude is of order k_1^3 and of scattering cross-section of order k_1^4. On the other hand, for a soft core, we prove that the leading terms of scattering amplitude and cross-section are of order k^1 and k^0, respectively.

2. Formulation of the problem

We consider the space R^3 to be divided by means of closed and smooth surfaces S_j, $j=1, 2,\ldots, q$ into $q+1$ regions $V_1, V_2,\ldots, V_{q+1}$ with the property $\partial V_j \cap \partial V_{j+1}=S_j$, $j=1, 2,\ldots, q$. Every surface S_{j+1} is inside S_j for every $j=1, 2,\ldots, q-1$. The set of all these regions V_j and surfaces S_j, $j=1, 2,\ldots, q$ constitutes the "multi-layered scatterer".

Let ρ_j be the mass densities and γ_j be the compressibilities in the regions V_j, $j=1, 2,\ldots, q$. The space V_{q+1} is the impenetrable core. We assume that the origin of coordinates is in the V_{q+1}. We also assume that a plane acoustic wave $\psi^{\mathrm{in}}(\boldsymbol{r})$ is incident upon the multi-layered scatterer. Let ψ be the scattered wave and $\psi_j(\boldsymbol{r})$ the total acoustic wave for the spaces V_j, $j=1, 2,\ldots, q$. The fields $\psi^{\mathrm{in}}(\boldsymbol{r})$, $\psi_j(\boldsymbol{r})$, $\psi(\boldsymbol{r})$ must satisfy Helmholtz's equations

$$(\nabla^2+k_j^2)\Phi(\boldsymbol{r})=0, \quad \boldsymbol{r}\in V_j, \tag{1}$$

where $k_j=\dfrac{2\pi}{\lambda_j}$ are the wavenumbers in the regions V_j.

The wavenumbers are connected with the relations

$$k_j^2=\frac{\gamma_1}{\rho_1}.\frac{\rho_j}{\gamma_j}k_1^2, \quad j=1, 2,\ldots, q, \tag{2}$$

On the surfaces S_j of the multi-layered scatterer the transition conditions

$$\psi_j(\boldsymbol{r})=\psi_{j+1}(\boldsymbol{r}), \quad \boldsymbol{r}\in S_j \tag{3}$$

$$\partial_n\psi_j(\boldsymbol{r})=\frac{\rho_j}{\rho_{j+1}}\partial_n\psi_{j+1}(\boldsymbol{r}), \quad \boldsymbol{r}\in S_j \tag{4}$$

should hold, for every $j=1, 2,\ldots, q-1$, where $\partial_n=\hat{\boldsymbol{n}}\,.\nabla$ is the outward normal derivate on S_j and $\hat{\boldsymbol{n}}$ the outward normal on S_j.

On the boundary S_q of the core, the field ψ_q satisfies

$$\varepsilon\psi_q(\boldsymbol{r})+(1-\varepsilon)\partial_n\psi_q(\boldsymbol{r})=0, \quad \boldsymbol{r}\in S_q, \tag{5}$$

where $\varepsilon\in[0, 1]$. In particular, for $\varepsilon=1$, (5) becomes a Dirichlet boundary condition, for $\varepsilon=0$ a Neumann boundary condition and $0<\varepsilon<1$ a boundary condition of Robin's type (impedance boundary condition).

The incident wave has the form

$$\psi^{\text{in}}(\boldsymbol{r})=e^{ik_1\hat{\boldsymbol{k}}\,.\,\boldsymbol{r}}, \tag{6}$$

where $\hat{\boldsymbol{k}}$ is the unit vector in the direction of propagation and $\boldsymbol{r}$ is the observation vector.

The scattered field must satisfy the radiation condition

$$\lim_{r\to\infty}\int_{S_r}|\partial_r\psi(\boldsymbol{r})-ik_1\psi(\boldsymbol{r})|^2\,\mathrm{d}s(\boldsymbol{r})=0 \tag{7}$$

S_r being a sphere of radius r and ∂_r the outward normal derivative on S_r.

3. Integral representations

In this section we will construct an integral representation of the solution in V_1 that contains explicitly the physical dependence as well as the geometrical dependence of our scattering problem.

The scattered wave $\psi(\boldsymbol{r})$ satisfies in V_1 the well-known Helmholtz's integral representation [3]

$$\psi(\boldsymbol{r})=\frac{1}{4\pi}\int_{S_1}\left[\psi(\boldsymbol{r}')\partial_n\frac{\mathrm{e}^{ik_1|\boldsymbol{r}-\boldsymbol{r}'|}}{|\boldsymbol{r}-\boldsymbol{r}'|}-\frac{\mathrm{e}^{ik_1|\boldsymbol{r}-\boldsymbol{r}'|}}{|\boldsymbol{r}-\boldsymbol{r}'|}\partial_n\psi(\boldsymbol{r}')\right]\mathrm{d}s(\boldsymbol{r}') \tag{8}$$

where the radiation condition given by Eq. (7) is included.

The total field $\psi_1(\boldsymbol{r})$, $\boldsymbol{r}\in V_1$ is the superpotion of the incident and the scattered field

$$\psi_1(\boldsymbol{r})=\mathrm{e}^{ik_1\hat{\boldsymbol{k}}\,.\,\boldsymbol{r}}+\psi(\boldsymbol{r}), \quad \boldsymbol{r}\in V_1. \tag{9}$$

Since the incident wave $\exp(ik_1\hat{\boldsymbol{k}}\,.\,\boldsymbol{r})$ belongs to the kerner of the differential operator of the problem we have that

$$\int_{S_1} [e^{ik_1 \hat{k} \cdot r} \partial_n \frac{e^{ik_1 |r-r'|}}{|r-r'|} - \frac{e^{ik_1 |r-r'|}}{|r-r'|} \partial_n e^{ik_1 \hat{k} \cdot r}] \mathrm{d}s(r') = 0. \tag{10}$$

From (8), (9) and (10) we conclude that

$$\psi_1(r) = e^{ik_1 \hat{k} \cdot r} + \frac{1}{4\pi} \int_{S_1} [\psi_1(r') \partial_n \frac{e^{ik_1 |r-r'|}}{|r-r'|} - \frac{e^{ik_1 |r-r'|}}{|r-r'|} \partial_n \psi_1(r')] \mathrm{d}s(r'). \tag{11}$$

Introducing the boundary conditions given by (3) and (4) we have

$$\begin{aligned} \psi_1(r) = e^{ik_1 \hat{k} \cdot r} &+ \frac{1}{4\pi} \int_{S_1} \psi_2(r') \partial_n \frac{e^{ik_1 |r-r'|}}{|r-r'|} \mathrm{d}s(r') \\ &- \frac{1}{4\pi} \frac{\rho_1}{\rho_2} \int_{S_1} \frac{e^{ik_1 |r-r'|}}{|r-r'|} \partial_n \psi_2(r') \mathrm{d}s(r'). \end{aligned} \tag{12}$$

Applying Green's first identity successively on $\psi_j(r)$, $(\exp(ik_1|r-r'|))/|r-r'|$ in V_j with $\partial V_j = S_{j-1} - S_j$, $j=2, 3, \ldots, q$, using that $\psi_j(r)$, $(\exp(ik_1|r-r'|))/|r-r'|$ are solutions of Eqs (1) in V_j and V_1, respectively, and introducing the boundary conditions given by (3), (4) we have the relations

$$\begin{aligned} \int_{S_1} \psi_2(r') \partial_n \frac{e^{ik_1 |r-r'|}}{|r-r'|} \mathrm{d}s(r') = &\int_{S_q} \psi_q(r') \partial_n \frac{e^{ik_1 |r-r'|}}{|r-r'|} \mathrm{d}s(r') \\ &- k_1^2 \sum_{j=2}^{q} \int_{V_j} \psi_j(r') \frac{e^{ik_1 |r-r'|}}{|r-r'|} \mathrm{d}v(r') \\ &+ \sum_{j=2}^{q} \int_{V_j} \nabla \psi_j(r') . \nabla_{r'} \frac{e^{ik_1 |r-r'|}}{|r-r'|} \mathrm{d}v(r'), \end{aligned} \tag{13}$$

$$\begin{aligned} \frac{\rho_1}{\rho_2} \int_{S_1} \frac{e^{ik_1 |r-r'|}}{|r-r'|} \partial_n \psi_2(r') \mathrm{d}v(r') = &\frac{\rho_1}{\rho_q} \int_{S_q} \frac{e^{ik_1 |r-r'|}}{|r-r'|} \partial_n \psi_q(r') \mathrm{d}s(r') \\ - k_1^2 \sum_{j=2}^{q} \frac{\gamma_1}{\gamma_j} \int_{V_j} \frac{e^{ik_1 |r-r'|}}{|r-r'|} \psi_j(r') \mathrm{d}v(r') &+ \sum_{j=2}^{q} \frac{\rho_1}{\rho_j} \int_{V_j} \nabla \psi_j(r') . \nabla_{r'} \frac{e^{ik_1 |r-r'|}}{|r-r'|} \mathrm{d}v(r'). \end{aligned} \tag{14}$$

Substituting (13), (14), (5) into (12) we have the integral representation of the solution $\psi_1(r)$ in V_1

$$\psi_1(\boldsymbol{r})=e^{ik_1\hat{\boldsymbol{k}}\cdot\boldsymbol{r}}-\frac{\varepsilon}{4\pi}\frac{\rho_1}{\rho_q}\int_{S_q}\frac{e^{ik_1|\boldsymbol{r}-\boldsymbol{r}'|}}{|\boldsymbol{r}-\boldsymbol{r}'|}\partial_n\psi_q(\boldsymbol{r}')ds(\boldsymbol{r}')$$

$$(15)\qquad +\frac{1}{4\pi}\int_{S_q}\left[\varepsilon\frac{\rho_1}{\rho_q}+\left(\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}-ik_1\right)\frac{(\boldsymbol{r}-\boldsymbol{r}')\cdot\hat{\boldsymbol{n}}}{|\boldsymbol{r}-\boldsymbol{r}'|}\right]\psi_q(\boldsymbol{r}')\frac{e^{ik_1|\boldsymbol{r}-\boldsymbol{r}'|}}{|\boldsymbol{r}-\boldsymbol{r}'|}ds(\boldsymbol{r}')$$

$$+\frac{1}{4\pi}k_1^2\sum_{j=2}^{q}\left(\frac{\gamma_1}{\gamma_j}-1\right)\int_{V_j}\psi_j(\boldsymbol{r}')\frac{e^{ik_1|\boldsymbol{r}-\boldsymbol{r}'|}}{|\boldsymbol{r}-\boldsymbol{r}'|}dv(\boldsymbol{r}')$$

$$+\frac{1}{4\pi}\sum_{j=2}^{q}\left(1-\frac{\rho_1}{\rho_j}\right)\int_{V_j}\left(\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}-ik_1\right)\frac{e^{ik_1|\boldsymbol{r}-\boldsymbol{r}'|}}{|\boldsymbol{r}-\boldsymbol{r}'|}\frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|}\cdot\nabla\psi_j(\boldsymbol{r}')dv(\boldsymbol{r}').$$

So, we have succeeded in expressing the field $\psi_1(\boldsymbol{r})$ in terms of two surface integrals on S_q (of the core) and a sum of volume integrals over the regions V_j of the fields ψ_j, $j=2, 3,\ldots, q$.

4. The scattering amplitude and cross-section

In the present section we obtain asymptotic expansions for the scattered field in the radiation zone. We need to elaborate some computational formulae in appropriate forms.

Using the asymptotic relations

$$(16)\qquad |\boldsymbol{r}-\boldsymbol{r}'|=r-\hat{\boldsymbol{r}}\cdot\boldsymbol{r}'+0\left(\frac{1}{r}\right),\quad r\to\infty$$

$$(17)\qquad \frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|}=\hat{\boldsymbol{r}}+0\left(\frac{1}{r}\right),\quad r\to\infty$$

$$(18)\qquad \frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}=\frac{1}{r}+0\left(\frac{1}{r^2}\right),\quad r\to\infty$$

we finally obtain the following asymptotic forms

$$(19)\qquad \frac{e^{ik_1|\boldsymbol{r}-\boldsymbol{r}'|}}{|\boldsymbol{r}-\boldsymbol{r}'|}=ik_1\,e^{-ik_1\hat{\boldsymbol{r}}\cdot\boldsymbol{r}'}\frac{e^{ik_1r}}{ik_1r}+0\left(\frac{1}{r^2}\right),\quad r\to\infty$$

$$(20)\qquad \nabla_{r'}\frac{e^{ik_1|\boldsymbol{r}-\boldsymbol{r}'|}}{|\boldsymbol{r}-\boldsymbol{r}'|}=k_1^2\hat{\boldsymbol{r}}e^{-ik_1\hat{\boldsymbol{r}}\cdot\boldsymbol{r}'}\frac{e^{ik_1r}}{ik_1r}+0\left(\frac{1}{r^2}\right),\quad r\to\infty.$$

From (9), (15), (19) and (20) we have

$$\psi(\boldsymbol{r})=g(\hat{\boldsymbol{r}},\ \hat{\boldsymbol{k}})\ h_0^1(k_1 r)+0\left(\frac{1}{r^2}\right),\quad r\to\infty, \tag{21}$$

where

$$h_0^1(k_1 r)=\frac{e^{ik_1 r}}{ik_1 r} \tag{22}$$

is the zeroth order spherical Hankel function of the first kind and

$$\begin{aligned} g(\hat{\boldsymbol{r}},\ \hat{\boldsymbol{k}})= \frac{1}{4\pi}\Big\{ & ik_1\varepsilon\frac{\rho_1}{\rho_q}\int_{S_q}[\psi_q(\boldsymbol{r}')-\partial_n\psi_q(\boldsymbol{r}')]\mathrm{e}^{-ik_1\hat{\boldsymbol{r}}\cdot\boldsymbol{r}'}\mathrm{d}s(\boldsymbol{r}') \\ & +k_1^2\int_{S_q}\hat{\boldsymbol{r}}\cdot\hat{\boldsymbol{n}}\psi_q(\boldsymbol{r}')\mathrm{e}^{-ik_1\hat{\boldsymbol{r}}\cdot\boldsymbol{r}'}\mathrm{d}s(\boldsymbol{r}')+ik_1^3\sum_{j=2}^{q}\left(\frac{\gamma_1}{\gamma_j}-1\right)\int_{V_j}\psi_j(\boldsymbol{r}')\mathrm{e}^{-ik_1\hat{\boldsymbol{r}}\cdot\boldsymbol{r}'}\mathrm{d}v(\boldsymbol{r}') \\ & +k_1^2\sum_{j=2}^{q}\left(1-\frac{\rho_1}{\rho_j}\right)\int_{V_j}\hat{\boldsymbol{r}}\cdot\nabla\psi_j(\boldsymbol{r}')\mathrm{e}^{-ik_1\hat{\boldsymbol{r}}\cdot\boldsymbol{r}'}\mathrm{d}v(\boldsymbol{r}')\Big\},\quad r\to\infty \end{aligned} \tag{23}$$

is the normalized scattering amplitude.

The scattering amplitude g satisfies [11] the reciprocity theorem

$$g(\hat{\boldsymbol{r}},\ \hat{\boldsymbol{k}})=g(-\hat{\boldsymbol{k}},\ -\hat{\boldsymbol{r}}) \tag{24}$$

and the scattering theorem

$$g(\hat{\boldsymbol{r}},\ \hat{\boldsymbol{k}})+g^*(\hat{\boldsymbol{k}},\ \hat{\boldsymbol{r}})=\frac{-1}{2\pi}\int g(\hat{\boldsymbol{p}},\ \hat{\boldsymbol{k}})\ g^*(\hat{\boldsymbol{p}},\ \hat{\boldsymbol{r}})\ \mathrm{d}\Omega(\hat{\boldsymbol{p}}), \tag{25}$$

where the asterisk indicates the complex conjugate and the integration is taken over all angles.

We now define the scattering cross-section as the ratio of the time average rate (over a period) at which energy is scattered by the body, to the corresponding time average rate at which the energy of the incident wave crosses a unit area normal to the direction of propagation. The scattering cross-section has the dimensions of area and is a measure of the disturbance caused to the incident wave by the scatterer.

The scattering cross-section σ_s satisfies [4] the relation

$$\sigma_s=\frac{1}{k_1^2}\int|g(\hat{\boldsymbol{r}},\ \hat{\boldsymbol{k}})\,|^2\mathrm{d}\Omega(\hat{\boldsymbol{r}}) \tag{26}$$

and the optical theorem [11].

5. Low-frequency expansions

The incident plane wave $\psi^{\text{in}}(\boldsymbol{r})=\exp(ik_1\hat{\boldsymbol{k}}\cdot\boldsymbol{r})$ is analytic at $k_1=0$ and we assume that the fields $\psi_j(\boldsymbol{r})$, $j=1, 2,\ldots, q$ are also analytic at $k_1=0$. Therefore the fields $\psi_j(\boldsymbol{r})$ can be expanded in convergent power series of k_1

$$\psi_j(\boldsymbol{r})=\sum_{n=0}^{\infty}\frac{(ik_1)^n}{n!}\Phi_n^{(j)}(\boldsymbol{r}),\quad \boldsymbol{r}\in V_j,\quad j=1, 2,\ldots, q. \tag{27}$$

Substituting (27) in (1) and equating equal powers of k_1, we obtain the following differential equations

$$\nabla^2\Phi_n^{(j)}(\boldsymbol{r})=n(n-1)\frac{\gamma_1}{\rho_1}\frac{\rho_j}{\gamma_j}\Phi_{n-2}^{(j)}(\boldsymbol{r})\left|\begin{array}{l}\boldsymbol{r}\in V_j\\ j=1, 2,\ldots, q\\ n=0, 1, 2,\ldots .\end{array}\right. \tag{28}$$

The boundary conditions given by Eqs (3), (4), (5) are transformed into the boundary conditions

$$\Phi_n^{(j)}(\boldsymbol{r})=\Phi_n^{(j+1)}(\boldsymbol{r}),\quad \boldsymbol{r}\in S_j \tag{29}$$

$$\partial_n\Phi_n^{(j)}(\boldsymbol{r})=\frac{\rho_j}{\rho_{j+1}}\partial_n\Phi_n^{(j+1)}(\boldsymbol{r}),\quad \boldsymbol{r}\in S_j \tag{30}$$

$$\varepsilon\Phi_n^{(q)}(\boldsymbol{r})+(1-\varepsilon)\,\partial_n\Phi_n^{(q)}(\boldsymbol{r})=0,\quad \boldsymbol{r}\in S_q \tag{31}$$

for every $j=1, 2,\ldots, (q-1)$ and $n=0, 1, 2,\ldots$

Similarly, substituting (27) and the expansions

$$e^{ik_1\hat{\boldsymbol{k}}\cdot\boldsymbol{r}}=\sum_{n=0}^{\infty}\frac{(ik_1)^n}{n!}(\hat{\boldsymbol{k}}\cdot\boldsymbol{r})^n \tag{32}$$

$$\frac{e^{ik_1|\boldsymbol{r}-\boldsymbol{r}'|}}{|\boldsymbol{r}-\boldsymbol{r}'|}=\sum_{n=0}^{\infty}\frac{(ik_1)^n}{n!}|\boldsymbol{r}-\boldsymbol{r}'|^{n-1} \tag{33}$$

$$\nabla_{\boldsymbol{r}'}\frac{e^{ik_1|\boldsymbol{r}-\boldsymbol{r}'|}}{|\boldsymbol{r}-\boldsymbol{r}'|}=\left(\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}-ik_1\right)(\boldsymbol{r}-\boldsymbol{r}')\sum_{n=0}^{\infty}\frac{(ik_1)^n}{n!}|\boldsymbol{r}-\boldsymbol{r}'|^{n-2} \tag{34}$$

into (15) we obtain the following integral relations among the coefficients $\Phi_0^{(j)}$, $\Phi_1^{(j)},\ldots, \Phi_n^{(j)}$

$$\Phi_n^{(1)}(\boldsymbol{r})=(\hat{\boldsymbol{k}}\,.\,\boldsymbol{r})^n-\frac{\varepsilon}{4\pi}\frac{\rho_1}{\rho_q}\sum_{m=0}^{n}\binom{n}{m}\int_{S_q}|\boldsymbol{r}-\boldsymbol{r}'|^{n-m-1}\,\partial_n\Phi_m^{(q)}(\boldsymbol{r}')ds(\boldsymbol{r}')$$

$$+\frac{1}{4\pi}\sum_{m=0}^{n}\binom{n}{m}\int_{S_q}\left[\frac{\varepsilon\rho_1}{\rho_q}|\boldsymbol{r}-\boldsymbol{r}'|^2-(n-m-1)(\boldsymbol{r}-\boldsymbol{r}')\,.\,\hat{\boldsymbol{n}}\right]|\boldsymbol{r}-\boldsymbol{r}'|^{n-m-3}\Phi_m^{(q)}(\boldsymbol{r}')ds(\boldsymbol{r}')$$

(35)
$$-\frac{1}{4\pi}\sum_{j=2}^{q}\left(\frac{\gamma_1}{\gamma_j}-1\right)\sum_{m=0}^{n}\binom{n}{m}m(m-1)\int_{V_j}|\boldsymbol{r}-\boldsymbol{r}'|^{n-m-3}\Phi_m^{(j)}(\boldsymbol{r}')dv(\boldsymbol{r}')$$

$$-\frac{1}{4\pi}\sum_{j=2}^{q}\left(1-\frac{\rho_1}{\rho_j}\right)\sum_{m=0}^{n}\binom{n}{m}(n-m-1)\int_{V_j}|\boldsymbol{r}-\boldsymbol{r}'|^{n-m-3}(\boldsymbol{r}-\boldsymbol{r}')\,.\,\nabla\Phi_m^{(j)}(\boldsymbol{r}')dv(\boldsymbol{r}').$$

The asymptotic representation for n-th order coefficient can be derived from the integral relation (35) if we omit the n-th term in the right-hand side, which is of order $\frac{1}{r}$. So we have

$$U_n^{(1)}(\boldsymbol{r})=(\hat{\boldsymbol{k}}\,.\,\boldsymbol{r})^n-\frac{\varepsilon}{4\pi}\frac{\rho_1}{\rho_q}\sum_{m=0}^{n-1}\binom{n}{m}\int_{S_q}|\boldsymbol{r}-\boldsymbol{r}'|^{n-m-1}\partial_n\Phi_m^{(q)}(\boldsymbol{r}')ds(\boldsymbol{r}')$$

$$+\frac{\varepsilon}{4\pi}\frac{\rho_1}{\rho_q}\sum_{m=0}^{n-1}\binom{n}{m}\int_{S_q}|\boldsymbol{r}-\boldsymbol{r}'|^{n-m-1}\Phi_m^{(q)}(\boldsymbol{r}')ds(\boldsymbol{r}')$$

(36)
$$-\frac{1}{4\pi}\sum_{m=0}^{n-2}\binom{n}{m}(n-m-1)\int_{S_q}\hat{\boldsymbol{r}}\,.\,\hat{\boldsymbol{n}}\,|\boldsymbol{r}-\boldsymbol{r}'|^{n-m-2}\,\Phi_m^{(q)}(\boldsymbol{r}')ds(\boldsymbol{r}')$$

$$-\frac{1}{4\pi}\sum_{j=2}^{q}\left(\frac{\gamma_1}{\gamma_j}-1\right)\sum_{m=0}^{n-3}\binom{n}{m}m(m-1)\int_{V_j}|\boldsymbol{r}-\boldsymbol{r}'|^{n-m-3}\,\Phi_n^{(j)}(\boldsymbol{r}')dv(\boldsymbol{r}')$$

$$-\frac{1}{4\pi}\sum_{j=2}^{q}\left(1-\frac{\rho_1}{\rho_j}\right)\sum_{m=0}^{n-2}\binom{n}{m}(n-m-1)\int_{V_j}|\boldsymbol{r}-\boldsymbol{r}'|^{n-m-2}\,\hat{\boldsymbol{r}}\,.\,\nabla\Phi_m^{(j)}(\boldsymbol{r}')dv(\boldsymbol{r}').$$

By substitution of $U_n^{(1)}(\boldsymbol{r})$ given by (36) in Eq. (28) we obtain a particular solution of (28). Hence, $\Phi_n^{(1)}$ can be decomposed into the sum of two terms

(37)
$$\Phi_n^{(1)}=U_n^{(1)}+W_n^{(1)},$$

where $U_n^{(1)}$ is given by (36) and $W_n^{(1)}$ is a harmonic function in V_1, regular at infinity (i.e. $W_n^{(1)}(\boldsymbol{r})=0(\frac{1}{r})$, $r\to\infty$) and such that the boundary conditions (3), (4) are satisfied. We also note, that (35) provides $\Phi_n^{(1)}$ in terms of $\Phi_0^{(j)}$, $\Phi_1^{(j)},\ldots,$ $\Phi_n^{(j)}$, while (36) gives $U_n^{(1)}$ (i.e. asymptotic form of $\Phi_n^{(1)}$) in terms of $\Phi_0^{(j)}, \Phi_1^{(j)}, \ldots, \Phi_{n-1}^{(j)}$, $j=2, 3, \ldots, q$.

The boundary value problem for Helmholtz's equation is thus reduced to a sequence of boundary value problems for Laplace's (for $n=0$, 1) and Poisson's (for $n \geqq 2$) equations.

6. The far-field behaviour at low-frequencies

Substitution of (27) and of

$$e^{-ik_1 \hat{r}\cdot r'} = \sum_{n=0}^{\infty} (-1)^n \frac{(ik_1)^n}{n!} (\hat{r}\cdot r')^n$$

into (23) yields

$$\begin{aligned} g(\hat{r}, \hat{k}) = \frac{ik_1}{4\pi} \sum_{n=0}^{\infty} \frac{(ik_1)^n}{n!} \sum_{m=0}^{n} \binom{n}{m} (-1)^m \Bigg[& \varepsilon \frac{\rho_1}{\rho_q} \int_{S_q} (\Phi_{n-m}^{(q)}(r') - \partial_n \Phi_{n-m}^{(q)}(r'))(r \cdot r')^m \mathrm{d}s(r') \\ & - ik_1 \int_{S_q} (\hat{r}\cdot r')^m (\hat{r}\cdot \hat{n})\, \Phi_{n-m}^{(q)}(r')\, \mathrm{d}s(r') \\ & + k_1^2 \sum_{j=2}^{q} \left(\frac{\gamma_1}{\gamma_j} - 1\right) \int_{V_j} (\hat{r}\cdot r')^m \Phi_{n-m}^{(j)}(r') \mathrm{d}v(r') \\ & - ik_1 \sum_{j=2}^{q} \left(1 - \frac{\rho_1}{\rho_j}\right) \int_{V_j} (\hat{r}\cdot r')^m \hat{r}\cdot \nabla \Phi_{n-m}^{(j)}(r') \mathrm{d}v(r') \Bigg]. \end{aligned} \tag{38}$$

For a multi-layered scatterer with a soft core ($\varepsilon=1$) the normalized scattering amplitude is $0(k_1)$. The leading term approximation, as $k_1 \to 0$, is

$$g(\hat{r}, \hat{k}) = -\frac{ik_1}{4\pi} \frac{\rho_1}{\rho_q} \int_{S_q} \partial_n \Phi_0^{(q)}(r') \mathrm{d}s(r'). \tag{39}$$

If the multi-layered scatterer has inversion symmetry ($r \in S$ implies $-r \in S$), using the scattering theorem, we conclude that the normalized scattering amplitude, in the case of a rigid core ($\varepsilon=0$) is $0(k_1^3)$. The leading term approximation, as $k_1 \to 0$, is

$$\begin{aligned} g(\hat{r}, \hat{k}) = \frac{(ik_1)^3}{4\pi} \Bigg[& \int_{S_q} (\hat{r}\cdot r')(\hat{r}\cdot \hat{n})\, \Phi_0^{(q)}(r')\mathrm{d}s(r') - \int_{S_q} (\hat{r}\cdot \hat{n})\, \Phi_1^{(q)}(r')\mathrm{d}s(r') \\ & - \sum_{j=2}^{q} \left(\frac{\gamma_1}{\gamma_j} - 1\right) \int_{V_j} \Phi_0^{(j)}(r')\mathrm{d}v(r') - \sum_{j=2}^{q} \left(1 - \frac{\rho_1}{\rho_j}\right) \int_{V_j} \hat{r}\cdot \nabla \Phi_1^{(j)}(r')\mathrm{d}v(r') \\ & + \sum_{j=2}^{q} \left(1 - \frac{\rho_1}{\rho_j}\right) \int_{V_j} (\hat{r}\cdot r')\hat{r}\cdot \nabla \Phi_0^{(j)}(r')\mathrm{d}v(r') \Bigg]. \end{aligned} \tag{40}$$

In order to evaluate the leading low-frequency approximation for the scattering cross-section, we substitute the expansions (39), (40) into the relation (26). We observe that the leading low-frequency approximation of the scattering cross-section for the multi-layered scatterer with a soft core is independent of the wave number, while in the case of the rigid core it is of order k_1^4. In other words, the total energy scattered by a multi-layered scatterer with a rigid core is four orders of magnitude in the wave number more than the corresponding total energy scattered by a multi-layered scatterer with a soft core. These results are in agreement with Rayleigh's law of scattering for sound waves, which says that in the low-frequency limit the total energy scattered by a rigid scatterer is proportional to the fourth inverse power of the wavelength.

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