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Separability of Topological Spaces by Continuous Maps

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Introduction

For every pair Y, R of topological spaces the equivalence relation $x \sim y$ if and only if $f(x) = f(y)$, for every continuous map f of Y into R, divides Y into equivalence classes. Let us consider the quotient space of the corresponding partition of Y. Spaces for which this quotient space is a singleton (that is every continuous map into R is constant), have been considered by many authors (see, for example, the references in [1]).

In the present paper we examine the case where the quotient space is not a singleton. Actually, we examine the subspace of the quotient space which consists of the classes which are not singletons. Let us denote this space by $M_0(Y, R)$ (see Theorem below). A corollary of this Theorem is that for every Hausdorff (resp., Urysohn, resp., regular) space M_0, there exists a Hausdorff (resp., Urysohn, resp., regular) space Y such that the space $M_0(Y, R)$ is homeomorphic to M_0.

Certainly, every element of $M_0(Y, R)$ considered as a subspace of Y, is divided by the same relation into new equivalence classes. Continuing in this manner, to every natural number n there corresponds a set of subsets of Y which are the equivalence classes of the elements of the set corresponding to the natural number $n-1$. This correspondence can be extended over the infinite ordinal numbers, considering that the elements of the set corresponding to a limit ordinal are the equivalence classes of the intersections of the elements of the sets corresponding to less ordinal numbers.

The above described process stops yielding new subsets of Y if the equivalence classes can not be further divided. This means that the equivalence class either is a singleton or all its continuous maps into R are constant. It is easy to prove that if a space does not contain subspaces with this property then the process stops yielding new classes if all final classes are singletons.

Obviously, the set of all classes arising out of the above process can be partially ordered with respect to "contained". For every class, the set of all equivalence classes of "next step" can be considered as a subset of the corresponding quotient space. Thus, to every topological space, there, finally, corresponds a "gradually" ordered set with "step topologies" characterizing the "type of R-separability" of the space.

The question which arises is under what conditions a "gradually" ordered set with "step topologies" is the "type of R-separability" of the topological space. Theorem below answers this question.

Definitions

Let X, R be topological spaces. Two points x, y of X are called: 1) R-separable, if there exists a continuous map f of X into R such that $f(x) \neq f(y)$; 2) R-equivalent (denoted by $x \sim y$) if for every continuous map f of X into R, $f(x)=f(y)$.

The space X (as well as its topology) is called: 1) R-separable, if for every distinct point x, y of X, there exists a continuous map f of X into R such that $f(x) \neq f(y)$; 2) R-monolithic, if every continuous map of X into R is constant. Obviously, the relation $x \sim y$ is an equivalence relation in X dividing X into equivalence classes which are closed subsets of X.

An ordered set $M \neq \emptyset$ is called gradually ordered if to every ordinal number α less than an ordinal number γ, there corresponds a non-empty subset M_α of M such that: 1) $M = \bigcup_{\alpha<\gamma} M_\alpha$, 2) $M_\alpha \cap M_\beta = \emptyset$, $\alpha \neq \beta$, $\alpha<\gamma$, $\beta<\gamma$, 3) if $x \in M_\alpha$, $\beta<\alpha<\gamma$, then there exists a unique point $y \in M_\beta$ such that $y \leqq x$, 4) if $x \neq y$, $y<x$ and $y \in M_\beta$, then $x \in M_\alpha$, where $\beta+1 \leqq \alpha$.

The set M_α is called the α-cut of M. Obviously, every α-cut of M is uniquely determined. Also, every totally ordered subset l of M is well-ordered and hence its ordinal type is an ordinal number which will be denoted by $\delta(l)$.

A point $a \in M$ is called maximal, if the relation $a \leqq b$, $b \in M$ implies $a=b$. The set of maximal points of M will be denoted by $N(M)$.

For every $a \in M$ we set $l(a)=\{b \in M : b \leqq a\}$. Obviously, if $a \in N(M)$, then the set $l(a)$ is a maximal totally ordered subset of M and the ordinal number $\delta(l(a))$ is a non-limit ordinal. The set of all maximal totally ordered subsets of M will be denoted by $L(M)$.

We denote by L_M^α the set of all totally ordered subsets l of M such that $\delta(l)=\alpha$ and $l \cap M_\beta$ is a singleton for every $\beta<\alpha$. We set $\{l_\beta\}=l \cap M_\beta$. Obviously, $l \cap M_\beta = \emptyset$ if $\alpha \leqq \beta$. Set $L_M = \bigcup_\alpha L_M^\alpha$. For every $l \in L_M^\alpha$ (resp., $a \in M_\alpha$), we denote by $(l, +)$ (resp., $(a, +)$) the set of all $c \in M_\alpha$ (resp., $c \in M_{\alpha+1}$) such that $b<c$, for every $b \in l$ (resp., $a<c$).

If $\alpha=\beta+1$, then $l \in L_M^\alpha$ if and only if $l(l_\beta)=l$.

We say that a gradually ordered set M has step topologies, if on the sets M_0 and $(l, +)$, for every $l \in L_M$ a topology is given.

Two ordered sets M, N are said to be similar if there exists a map $f: M \to N$ which is one-to-one and onto and which has the property that for every x, $y \in M$, $x \leqq y$ if and only if $f(x) \leqq f(y)$. The map f is called a similarity of M into N. If M, N are similar gradually ordered sets with step topologies, then we say that similarity preserves topologies, if $f|_{M_0}$ and $f|_{(l,+)}$, for every $l \in M$, is a homeomorphism. (Obviously, $f(M_\alpha)=N_\alpha$ and if $l \in L_M^\alpha$ then $f(l) \in L_N^\alpha$ and $f((l, +))=(f(l), +)$.)

Let Y, R be topological spaces. We denote by $K(Y, R)$ the set of all equivalence classes of Y which are proper subsets of Y and not singletons. Hence if Y is R-monolithic or R-separable, then $K(Y, R)=\emptyset$. In the sequel, if a space is R-monolithic or R-separable then it will be considered non-empty and not a singleton.

We now define, for every pair of Hausdorff spaces a gradually ordered set $M(Y, R)$ with step topologies.

For every ordinal number α, we, first, define by induction, a set $M_\alpha(Y, R)$ as follows: We set $M_0(Y, R)=K(Y, R)$. If $\alpha=\beta+1$, then a subset F of Y belongs to $M_\alpha(Y, R)$ if and only if there exists an element F' of $M_\beta(Y, R)$, such that $F\in K(F', R)$. And, if α is a limit ordinal, then the set F belongs to $M_\alpha(Y, R)$ if and only if, for every $\beta<\alpha$, there exists $F_\beta\in M_\beta(Y, R)$ such that $F\in K(\bigcap_{\beta<\alpha} F_\beta, R)$.

The following properties of the sets $M_\alpha(Y, R)$ are easily verified by induction:

1) Every element of $M_\alpha(Y,R)$ is non-empty and not a singleton;

2) If $F_1, F_2\in M_\alpha(Y,R)$, then $F_1\cap F_2=\emptyset$;

3) If $F_1\in M_\alpha(Y,R)$ and $\beta<\alpha$, then there exists a unique element F_2 of $M_\beta(Y,R)$ such that $F_1\subseteq F_2$ and $F_1\neq F_2$;

4) There exists an ordinal number γ, such that $M_\alpha(Y,R)=\emptyset$, for every $\gamma\leqq\alpha$ and $M_\alpha(Y,R)\neq\emptyset$, for every $\alpha<\gamma$.

We now set $M(Y,R)=\bigcup_{\alpha<\gamma} M_\alpha(Y,R)$ and on the set $M(Y,R)$ we define $F_1\leqq F_2$ if and only if $F_2\subseteq F_1$. From the above properties of the sets $M_\alpha(Y,R)$, it follows that the set $M(Y,R)$ is gradually ordered and its α-cut is the set $M_\alpha(Y,R)$.

On the sets M_0 and $(l, +)$, where $l\in L_{M(Y,R)}$, we define the following topologies $\tau(M_0)$ and $\tau(l, +)$, respectively. The topology $\tau(M_0)$ on M_0 is the relative topology with respect to the quotient space $D(M_0)$, where $D(M_0)$ is the partition of Y whose elements are the elements of M_0 and the singletons $\{a\}$, for every a belonging to Y and not belonging to any element of M_0.

The topology $\tau(l, +)$ on $(l, +)$ is defined by replacing in the above definition of $\tau(M_0)$, the space Y by $\bigcap_{F\in l} F$ and the set M_0 by $(l, +)$.

Obviously, the topologies $\tau(M_0)$ and $\tau(l, +)$ are R-separable.

Thus, the set $M(Y,R)$ becomes gradually ordered with step topologies.

Let $l=\{F_\alpha : \alpha<\delta(l)\}$ be a maximal totally ordered subset of $M(Y,R)$. Set $Y(l)=\bigcap_{\alpha<\delta(l)} F_\alpha$. Note that the set $Y(l)$ has the following properties: If the ordinal number $\delta(l)$ is a limit ordinal, then either the set $Y(l)$ is empty or a singleton, or the space $Y(l)$ is R-monolithic or R-separable. If $\delta(l)$ is a non-limit ordinal, then the space $Y(l)$ is R-monolithic or R-separable (the set $Y(l)$ is non-empty and not a singleton).

Theorem. *Let R be a non-empty Hausdorff space not consisting of a single point and M be a gradually ordered set with step R-separable Hausdorff (resp., Urysohn, resp., regular) topologies. Suppose that for every maximal ordered subset l of M there corresponds a space $M(l)$ such that if $\delta(l)$ is a non-limit ordinal number, then the space $M(l)$ is R-monolithic or R-separable Hausdorff (resp., Urysohn, resp., regular) space, and if $\delta(l)$ is a limit ordinal then either the set $M(l)$ is empty or* *a singleton, or the space $M(l)$ is R-monolithic or R-separable Hausdorff (resp., Urysohn, resp., regular) space. Then, there exists a Hausdorff (resp., Urysohn, resp., regular) space Y and a similarity $f: M \to M(Y, R)$ preserving the topologies such that the space $Y(f(l))$ is homeomorphic to the space $M(l)$.*

Proof. For the construction of the space Y we first consider a space X, and the spaces $I^1(X, \Lambda_0)$, $I^2(X, \Lambda_1) \ldots I^n(X, \Lambda_{n-1}) \ldots$ constructed in [1], choosing appropriately the sets $\Lambda_0, \Lambda_1, \ldots, \Lambda_n, \ldots$ Then we prove that $I(X)$ is the required space. (For notation and definitions, see [1].)

Let M be a gradually ordered set with step topologies. Let $N(M)$ be the set of maximal points of M and $L(M)$ be the set of maximal totally ordered subsets of M. We consider the spaces $M(l)$, $l \in L(M)$ and the set $M \setminus N(M)$ to be pairwise disjoint. We identify each point a of the set $N(M)$ with a point of the space $M(l(a))$ which is also denoted by a. Observe that the set $M(l(a))$ is neither empty nor a singleton.

We set

$$X = (M \setminus N(M)) \bigcup \bigcup_{l \in L(M)} M(l)$$

and on the set X we define the following topology:

A subset U of X is open if and only if $U \cap M(l)$, $l \in L(M)$ is open in $M(l)$, $U \cap M_0$ is open in M_0 and $U \cap (l, +)$, $l \in L_M$ is open in $(l, +)$.

We denote by $B(X)$ a basis of the space X.

Obviously, the space X is Hausdorff (resp., Urysohn, resp., regular).

In the set X we also define an order, extending the order in M, as follows: $a \leqq b$, if and only if one of the following cases is true: 1) $a = b$, 2) $a, b \in M$ and $a \leqq b$ 3) $a \in M$, $a \in l$, for $l \in L(M)$ and $b \in M(l)$.

Let T be a regular space each pair of points of which is separated by a continuous map of T into R, except for a unique pair, say (p^-, p^+). (See, for example, the space $T_1(R)$ in [1].) Obviously, for the space $T, K(T, R) = \{p^-, p^+\}$.

Set $J = T \setminus \{p^-, p^+\}$. We denote by $B(J)$ a basis of the space J and by $B^0(p^-)$ and $B^0(p^+)$ a basis of open neighbourhoods of p^- and p^+, respectively, in the space T. Also, by $B(p^-)$ (resp., $B(p^+)$) we denote the set of all sets of the form $V \setminus \{p^-\}$ (resp., $V \setminus \{p^+\}$), where $V \in B^0(p^-)$ (resp., $V \in B^0(p^+)$).

Construct now the spaces $I^1(X, \Lambda_0)$, $I^2(X, \Lambda_1), \ldots, I^n(X, \Lambda_{n-1}), \ldots$ and $I(X)$. For their construction, it suffices to define the sets $\Lambda_0, \Lambda_1, \ldots, \Lambda_n, \ldots$ This can be done by induction as follows: A pair (a, b) belongs to the set Λ_0 if and only if $a \in M, b \in X$, $a \neq b$, and $a \leqq b$. A pair (a, b) belongs to Λ_n, $n = 1, 2, \ldots$, if and only if $a \in M$ and $b \in J^\lambda$, $\lambda = (c, d) \in \Lambda_{n-1}$, $c \neq a$, $a \leqq c$.

For every $a \in M$ we denote by $X(a, \leqq)$ the set of all $b \in X$ for which $a \leqq b$. Observe that if $a \in N(M)$, then the subspace $X(a, \leqq)$ of X coincides with the subspace $M(l(a))$.

Denote by $\Lambda_n(a, \leqq)$, $n = 0, 1, 2, \ldots$ the set of all $\lambda = (c, d) \in \Lambda_n$ such that $a \leqq c$. Obviously, if $\lambda = (c, d) \in \Lambda_0(a, \leqq)$, then $c, d \in X(a, \leqq)$, that is, the space $I^1(X(a, \leqq), \Lambda_0(a, \leqq))$ is well-defined. Let U be an open subset of $X(a, \leqq)$ and V be an open subset of X such that $V \cap X(a, \leqq) = U$. Then for every $H \in B(p^-)$ and $G \in B(p^+)$, $O^1(U, H, G) = O^1(V, H, G) \cap I^1(X(a, \leqq), \Lambda_0(a, \leqq))$, where $O^1(U, H, G)$ and

$O^1(V_1, H, G)$ are subsets of the spaces $I^1(X(a, \leqq), \Lambda_0(a, \leqq))$ and $I^1(X, \Lambda_0)$, respectively. Hence, the space $I^1(X(a, \leqq), \Lambda_0(a, \leqq))$ can be considered as a subspace of $I^1(X, \Lambda_0)$.

It can be easily proved, by induction, that if $\lambda = (c, d) \in \Lambda_{n-1}(a, \leqq)$, n=2, 3,..., then $c, d \in I^{n-1}(X(a, \leqq), \Lambda_{n-2}(a, \leqq))$, that is, the space $I^n(X(a, \leqq), \Lambda_{n-1}(a, \leqq))$ is well-defined. As above this space can be considered as a subspace of $I^n(X, \Lambda^{n-1})$. Therefore, the space $I(X(a, \leqq))$ can be considered as a subspace of $I(X)$.

We denote by $\Lambda(a)$ (resp., $\Lambda_n(a)$) the set of all pairs $\lambda \in \bigcup_{n=0}^{\infty} \Lambda_n$ (resp., $\lambda \in \Lambda_n(a, \leqq)$) such that $\lambda = (a, b)$. Also, by $E(a)$ (resp., $E_n(a)$) we denote the set of all points $b \in I(X)$ for which there exists $\lambda \in \Lambda(a)$ (resp., $\lambda \in \Lambda_n(a)$) such that $\lambda = (a, b)$.

Proposition 1. *If* $a \leqq b$, *then*

$$I(X(b, \leqq)) \subseteq I(X(a, \leqq)).$$

Proposition 2. *If* $a, b \in M_\alpha$, $a \neq b$, *then*

$$I(X(a, \leqq)) \cap I(X(b, \leqq)) = \emptyset.$$

Proposition 3. $I(X) = \bigcup_{a \in M_0} I(X(a, \leqq))$.

Proposition 4. *If* $a \in M \setminus N(M)$ (*resp.*, $a \in N(M)$), *then* $I(X(a, \leqq)) = \{a\} \cup \bigcup_{b \in (a,+)} I(X(b, \leqq)) \cup \bigcup_{\lambda \in \Lambda(a)} J^\lambda$ (resp., $I(X(a, \leqq)) = X(a, \leqq) \cup \bigcup_{\lambda \in \Lambda(a)} J^\lambda$).

Propositions 1, 2, and 3 are easily proved by the following relations (1), (2), and (3), respectively,

(1) $$X(b, \leqq) \subseteq X(a, \leqq), \quad \Lambda_n(b, \leqq) \subseteq \Lambda_n(a, \leqq),$$

(2) $$X(a, \leqq) \cap X(b, \leqq) = \emptyset, \quad \Lambda_n(a, \leqq) \cap \Lambda_n(b, \leqq) = \emptyset,$$

(3) $$\bigcup_{a \in M_0} X(a, \leqq) = X, \quad \bigcup_{a \in M_0} \Lambda_n(a, \leqq) = \Lambda_n.$$

The relations are proved by induction.
Proposition 4 follows from the relations

$$X(a, \leqq) = \{a\} \cup \bigcup_{b \in (a,+)} X(b, \leqq)$$

(resp., $X(a, \leqq) = M(l(a))$)

$$\Lambda_n(a, \leqq) = \Lambda_n(a) \cup \bigcup_{b \in (a,+)} \Lambda_n(b, \leqq), \quad n = 0, 1, 2, \ldots$$

(resp., $\Lambda_0(a, \leqq) = \Lambda(a)$ and $\Lambda_n(a, \leqq) = \emptyset$, $n = 1, 2, \ldots$).

Proposition 5. If $\lambda \in \Lambda(a)$, then the set J^λ is open in $I(X(a, \leqq))$.

Proof. Let $\lambda \in \Lambda_n(a)$, $n=0, 1, 2, \ldots$ Then the set J^λ is open in the space $I^{n+1}(X(a, \leqq), \Lambda_n(a, \leqq))$. By definition of the set $\Lambda_{n+1}(a, \leqq)$ it follows that if $\lambda'=(c,d)\in\Lambda_{n+1}(a, \leqq)$, then $c, d \notin J^\lambda$ that is, $\Lambda^-(J^\lambda)\cup\Lambda^+(J^\lambda)\cup\Lambda(J^\lambda)=\emptyset$. Hence, $O^1(J^\lambda, H, G)=J^\lambda$, for every $H\in B(p^-)$ and $G\in B(p^+)$.

By induction it is proved that $O^k(J^\lambda, H, G)=J^\lambda$, $k=2, 3, \ldots$, and hence $O(J^\lambda, H, G)=J^\lambda$, that is, the set J^λ is open in $I(X(a, \leqq))$. (See, [1] for the notation.)

Proposition 6. *Let U be an open subset of M_0. Then the set $I(X(U, \leqq)) = \bigcup_{a\in U} I(X(a, \leqq))$ is open in $I(X)$.*

Proof. We set $X(U)=\bigcup_{a\in U} X(a, \leqq)$ and we observe that this set is an open subset of X. Further, by the construction of $I(X)$, it follows that for every $H\in B(p^-)$, $G\in B(p^+)$ the set $O(X(U), H, G)$ coincides with $I(X(U, \leqq))$. Hence, $I(X(U, \leqq))$ is open in $I(X)$.

Proposition 7. *The point $a\in M\backslash N(M)$ is isolated in the space $\{a\}\cup\bigcup_{b\in(a,+)} I(X(b, \leqq))$.*

Proof. The point a is isolated in the space $X(a, \leqq)$. Since

$$\Big(\bigcup_{\lambda\in\Lambda(a)} J^\lambda\Big)\bigcap\Big(\bigcup_{b\in(a,+)} I(X(b, \leqq))\Big)=\emptyset,$$

it follows that if $U=\{a\}$, then the set $O(U, H, G)$ is open in the space $I(X(a, \leqq))$ and $O(U, H, G)\subseteq X(a, \leqq)\cup\bigcup_{\lambda\in\Lambda(a)} J^\lambda$. Therefore,

$$O(U, H, G)\bigcap\Big(\{a\}\cup\bigcup_{b\in(a,+)} I(X(b, \leqq))\Big)=\{a\}$$

and hence the set $\{a\}$ is open in the space $\{a\}\cup\bigcup_{b\in(a,+)} I(X(b, \leqq))$.

Proposition 8. *If $a\in M\backslash N(M)$ (resp., $a\in N(M)$), then*

$$E(a)=\bigcup_{b\in(a,+)} I(X(b, \leqq)) \tag{4}$$

$$(\textit{resp., } E(a)=M(l(a))\backslash\{a\}). \tag{5}$$

Proof. It is obvious that

$$E_0(a)=\bigcup_{b\in(a,+)} X(b, \leqq) \tag{6}$$

$$(\text{resp., } E_0(a)=X(a, \leqq)\backslash\{a\}). \tag{7}$$

and that

$$E_1(a)=\bigcup_{b\in(a,+)} (I^1(X(b, \leqq), \Lambda_0(b, \leqq))\backslash X(b, \leqq)). \tag{8}$$

By induction it can be easily proved that

$$(9)\quad E_n(a)=\bigcup_{b\in(a,+)}(I^n(X(b,\leqq),\Lambda_{n-1}(a,\leqq))\setminus I^{n-1}(X(b,\leqq),\Lambda_{n-2}(a,\leqq)))$$

(resp., $E_n(a)=\emptyset$, $n=2,3,\dots$).
Then relation (4) (resp., (5)) follows by (6), (8) and (9) (resp., by (7) and (10)).
Let α be a limit ordinal number and $l\in L^{\alpha}_{M(Y,R)}$.

Proposition 9.

$$(10)\qquad \bigcap_{\beta<\alpha} I(X(l_\beta,\leqq))=\bigcup_{b\in(l,+)} I(X(b,\leqq)).$$

Proof. It is obvious that

$$(11)\qquad \bigcap_{\beta<\alpha} X(l_\beta,\leqq)=\bigcup_{b\in(l,+)} X(b,\leqq).$$

Also,

$$(12)\qquad \bigcap_{\beta<\alpha} \Lambda_n(l_\beta,\leqq)=\bigcup_{b\in(l,+)} \Lambda_n(b,\leqq),\ n=0,1,2,\dots.$$

Then, relation (10) follows from (11) and (12).

Proposition 10. Let $a\in M\setminus N(M)$ (resp., $a\in N(M)$). Then the set $F=\{a\}\cup\bigcup_{b\in(a,+)} I(X(b,\leqq))$ (resp., $F=M(l(a))$) is the unique element of the set $K(I(X(a,\leqq)),R)$.

Proof. Let $x\in\bigcup_{b\in(a,+)} I(X(b,\leqq))$ (resp., $x\in M(l(a))\setminus\{a\}$) and h be an arbitrary continuous map of $I(X(a,\leqq))$ into R. By proposition 8, it follows that there exists an integer $n\geqq 0$ and $\lambda\in\Lambda_n$ (resp., $\lambda\in\Lambda_0$) such that $\lambda=(a,x)$. Let g be the restriction of h in J^λ. Since the map $g\circ\bar{i}^\lambda$ into R is continuous, it follows that $g\circ\bar{i}^\lambda(p^-)=g\circ\bar{i}^\lambda(p^+)$ or $g(a)=g(x)$, that is, $h(a)=h(x)$ and hence every point of the set F is R-equivalent to the point a, i.e., the set F is contained in an equivalence class of the space $I(X(a,\leqq))$ (for notation see [1]). It remains to prove that if $x\in I(X(a,\leqq))\setminus F$, then the singleton $\{x\}$ is an equivalence class of the space $I(X(a,\leqq))$. Let $x\in I(X(a,\leqq))\setminus F$. By Proposition 4 it follows that there exists $\lambda\in\Lambda(a)$ such that $x\in J^\lambda$.

Let h be a continuous map of T into R. We denote by $\bar{h}$ the map of $I(X(a,\leqq))$ into R, defined as follows: If $y\in\overline{J^\lambda}^{I(X(a,\leqq))}$, then $\bar{h}(y)=h((\bar{i}^\lambda)^{-1}(y))$ and if $y\in I(X(a,\leqq))\setminus\overline{J^\lambda}^{I(X(a,\leqq))}$, then $\bar{h}(y)=h(p^-)=h(p^+)$. By Proposition 5, the set J^λ is an open subset of $I(X(a,\leqq))$ and hence the map $\bar{h}$ is continuous.

Now, let y be a point of $I(X(a,\leqq))$, $y\neq x$. We consider a continuous map h of T into R such that if $y\in\overline{J^\lambda}^{I(X(a,\leqq))}$ then $h((\bar{i}^\lambda)^{-1}(x))\neq h((\bar{i}^\lambda)^{-1}(y))$ and if $y\notin J^\lambda$, then

$h(\bar{i}^{\lambda})^{-1}(x) \neq h(p^{-})$. Then $\bar{h}(x) \neq \bar{h}(y)$ and hence the points x, y are not R-equivalent, that is, the singleton $\{x\}$ is an equivalence class of $I(X(a, \leqq))$.

Proposition 11. *If $a \in M_0$ and $x \in I(X(a, \leqq)) \setminus F$, then $\{x\}$ is an equivalence class of $I(X)$.*

Proof. Let $x \in I(X(a, \leqq)) \setminus F$, $y \in I(X)$, $y \neq x$. There exists $\lambda \in \Lambda(a)$ such that $x \in J^{\lambda}$. Also, there exists a map $h : \bar{J}^{\lambda} \to R$ such that if $y \in \bar{J}^{\lambda}$, then $h(x) \neq h(y)$ and if $y \notin \bar{J}^{\lambda}$, then $h(x) \neq h(p^{+})$. The map h extends to a map $\bar{h} : I(X) \to R$ considering that, if $z \in I(X) \setminus \bar{J}^{\lambda}$, then $\bar{h}(z) = h(p^{+})$. As in Proposition 10, $\bar{h}$ is continuous and, in all cases, $\bar{h}(x) \neq \bar{h}(y)$. Therefore, $\{x\}$ is an equivalence class of $I(X)$.

Proposition 12. *If f is a continuous map of M_0 into R, then the map $g : I(X) \to R$ defined by $g(x) = f(a)$, for every $a \in M_0$, $x \in I(X(a, \leqq))$ is continuous.*

Proof. Let U be an open set of R. Then $g^{-1}(U) = \bigcup_{a \in f^{-1}(U)} I(X(a, \leqq))$. Since $f^{-1}(U)$ is open in M_0, by Proposition 6, it follows that the set $\bigcup_{a \in f^{-1}(U)} I(X(a, \leqq))$ is open in $I(X)$. Hence g is continuous.

We now set $Y = I(X)$ and we prove that Y is the required space. Note that by its construction the space Y is Hausdorff (resp. Urysohn, resp. regular) (see [1]).

The elements of the set $M_0(Y, R)$ are the elements of the set $K(Y, R)$. Since the topology on M_0 is R-separable, by Proposition 12, it follows that every element of $K(I(X), R)$ is contained in $I(X(a, \leqq))$, for some $a \in M_0$. From Propositions 10 and 11 it follows that a subset F of $I(X)$ belongs to $K(Y, R)$ if and only if it has the form $\{a\} \cup \bigcup_{b \in (a, +)} I(X(b, \leqq))$ if $a \in M_0 \setminus N(M)$ or the form $M(l(a))$, if $a \in M_0 \cap N(M)$.

We now consider the map $f_0 : M_0 \to M_0(Y, R)$ setting $f_0(a) = F = \{a\} \cup \bigcup_{b \in (a, +)} I(X(b, \leqq))$. By the above it follows that f_0 is the restriction of the projection of $I(X)$ onto the corresponding quotient space. Hence f_0 is continuous. In order to prove that M_0 and $M_0(Y, R)$ are homeomorphic, it suffices to prove that f_0^{-1} is continuous. Let U be an open subset of M_0. The set $I(X(U, \leqq))$ is open in $I(X)$, (Proposition 6) and it is a union of some elements of the quotient space. Hence, the set V of all elements of quotient space which are contained in $I(X(U, \leqq))$ is open. Obviously, $V \cap K(Y, R) = f_0(U)$. Therefore, $f_0(U)$ is open and hence f_0^{-1} is continuous.

We now suppose that for every $\beta < \alpha$ there exists an one-to-one and onto map $f_\beta : M_\beta \to M_\beta(Y, R)$, such that if $l \in L_M^\beta$, then the map $f_\beta|_{(l, +)}$ is a homeomorphism. Further, we suppose that $f_\beta(a) = \{a\} \cup \bigcup_{b \in (a, +)} I(X(b, \leqq))$, if $a \in M_\beta \setminus N(M)$ and $f_\beta(a) = M(l(a))$, if $a \in M_\beta \cap N(M)$. Let α be a non-limit ordinal number, hence $\alpha = \beta + 1$. We shall first prove that every element of the set $M_\alpha(Y, R)$ has the form $\{a\} \cup \bigcup_{b \in (a, +)} I(X(b, \leqq))$, if $a \in M_\alpha \setminus N(M)$ or the form $M(l(a))$, if $a \in M_\alpha \cap N(M)$.

By definition of the set $M_\alpha(Y, R)$, a set F belongs to $M_\alpha(Y, R)$ if and only if there exists an element $F' \in M_\beta(Y, R)$ such that $F \in K(F', R)$. Let $F \in M_\alpha(Y, R)$. The corresponding element $F' \in M_\beta(Y, R)$, will have the form $\{b\} \cup \bigcup_{a \in (b, +)} I(X(a, \leqq))$ if

$b \in M_\beta \setminus N(M)$, or the form $F' = M(l(b))$, if $b \in M_\beta \cap N(M)$. The set F' cannot have the form $M(l(b))$ because $K(M(l(b)), R) = \emptyset$. By Proposition 7, the point b is isolated, hence $F \in K(\bigcup_{a \in (b,+)} I(X(a, \leqq)), R)$. Let $M' = \{c \in M : b < c\}$. Observe that M' can be considered as a gradually ordered set with step topologies and that $(M')_0 = (b, +)$. The space X', which is constructed from M' in the same manner the space X constructed from M, coincides with the subset $X(b, \leqq) \setminus \{b\}$ of X. Thus, $I(X') = \bigcup_{a \in (b,+)} I(X(a, \leqq))$. By Propositions 10, 11 and 12, it follows that

$$F = \{a\} \bigcup \bigcup_{c \in (a,+)} I(X(c, \leqq)), \text{ if } a \in M_\alpha \setminus N(M) \text{ or } F = M(l(a)), \text{ if } a \in M_\alpha \cap N(M).$$

By Proposition 12, it follows that there exists a homeomorphism $f_b : (b, +) \to K(\bigcup_{a \in (b,+)} I(X(a, \leqq)), R) = K(\{b\} \bigcup \bigcup_{a \in (b,+)} I(X(a, \leqq)), R) = K(F', R)$. Obviously, $M_\alpha = \bigcup_{a \in M_\beta} (b, +)$. Define the map $f_\alpha : M_\alpha \to M_\alpha(Y, R)$ considering that it coincides with f_b on the set $(b, +)$. It is clear that f_α is one-to-one and onto.

If α is a limit ordinal number and F is an element of $M_\alpha(Y, R)$, then, by definition of $M_\alpha(Y, R)$, for every $\beta < \alpha$ there exists an element F_β of the set $M_\beta(Y, R)$ such that $F \in K(\bigcap_{\beta < \alpha} F_\beta, R)$. The elements of the set $M_\beta(Y, R)$ are pairwise disjoint and every element of $M_{\beta+1}(Y, R)$ is contained in a unique element of $M_\beta(Y, R)$. Since $F_{\beta+1} \cap F_\beta \neq \emptyset$, it follows that $F_{\beta+1} \in K(F_\beta, R)$. Therefore, by assumption, every set F_β has the form $\{a_\beta\} \bigcup \bigcup_{b \in (a_\beta, +)} I(X(b, \leqq))$, $a_\beta \in M_\beta \setminus N(M)$ (and not the form $M(l(a_\beta))$, $a_\beta \in M_\beta \cap N(M)$, for then $K(F_\beta, R) = \emptyset$). Since the point a_β is isolated in F_β (Proposition 7) it follows that $F_{\beta+1} \subseteq \bigcup_{b \in (a_\beta, +)} I(X(b, \leqq))$. Hence, $a_\beta \leqq a_{\beta+1}$. Let $l = \{a_\beta \in M : \beta < \alpha\}$. Obviously, $\bigcap_{\beta < \alpha} F_\beta = \bigcap_{\beta < \alpha} I(X(a_\beta, \leqq))$. By Proposition 9, it follows that $\bigcap_{\beta < \alpha} I(X(a_\beta, \leqq)) = \bigcup_{b \in (l,+)} I(X(b, \leqq))$. As in case where α is a non-limit ordinal, for the set F there exists an element $a \in (l, +)$ such that $F \in K(I(X(a, \leqq)), R)$. Hence by Proposition 10, either $F = \{a\} \bigcup \bigcup_{c \in (a,+)} I(X(c, \leqq))$, if $a \in M_\alpha \setminus N(M)$ or $F = M(l(a))$, if $a \in M_\alpha \cap N(M)$.

Similarly, there exists a homeomorphism $f_l : (l, +) \to K(\bigcup_{b \in (l,+)} I(X(b, \leqq)), R)$. Obviously, $M_\alpha = \bigcup_{l \in L_M^\alpha} (l, +)$. Define the map $f_\alpha : M_\alpha \to M_\alpha(Y, R)$ considering that f_α coincides with f_l on the set $(l, +)$. It is clear that f_α is one-to-one and onto.

Thus, for every α we constructed an one-to-one map $f_\alpha : M_\alpha \to M_\alpha(Y, R)$. The required map f is defined by $f(a) = f_\alpha(a)$, for every $a \in M_\alpha$.

Now, let l be a maximal totally ordered subset of M. If $\delta(l)$ is a limit ordinal number, then obviously $\bigcap_{\beta < \delta(l)} X(l_\beta, \leqq) = M(l)$. Also, it can be easily proved that $\bigcap_{\beta < \delta(l)} I(X(l_\beta, \leqq)) = M(l)$. By the construction of f it follows that $\bigcap_{\beta < \delta(l)} f(l_\beta) = \bigcap_{\beta < \delta(l)} I(X(l_\beta, \leqq))$. Therefore, $\bigcap_{\beta < \delta(l)} f(l_\beta) = M(l)$, that is, the space $Y(f(l))$

is homeomorphic to $M(l)$. If $\delta(l)$ is not a limit ordinal, hence $\delta(l)=\alpha+1$ it follows that $\bigcap_{\beta<\delta(l)} f(l_\beta)=f(l_\alpha)$. The point l_α is a maximal point of M and hence $Y(f(l))=f(l_\alpha)=M(l)$, that is, the space $Y(f(l))$ is homeomorphic to the space $M(l)$. The proof of the Theorem is completed.

References

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