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Low-Frequency Scattering by an Ellipsoidal Dielectric with a Confocal Ellipsoidal Perfect Conductor Core

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In this paper we consider the problem of scattering of an electromagnetic wave by an ellipsoidal dielectric scatterer which contains a perfect conductor confocal ellipsoidal core. We consider the problem in the low-frequency region. Explicit closed-form solutions for the zeroth and first-order approximations are provided in terms of the physical and geometric characteristics of the scatterer, as well as the direction cosines of the incidence and observation points. The leading low-frequency term for the normalized spherical scattering amplitude and the scattering cross-section are also given explicitly. Degenerate cases, the spheroids, the sphere, the needle and the disc are considered as special cases. As degenerate case of the above problem, the problem of the perfect conductor and the dielectric is also considered.

1. Introduction

In [5] we gave a systematic analysis of the electromagnetic scattering problem at low-frequencies. The present work refers to the application of our general method to a triaxial ellipsoidal dielectric scatterer which contains a perfect conductor confocal ellipsoidal core. It turned out that the lack of rotational symmetry for the scatterer makes the problem very difficult to solve in closed analytical form. Moreover, the existence of the core which imposes new boundary conditions on its surface adds new difficulties to the problem. So, a new calculational technique had to be introduced in order to find the first two low-frequency approximations in terms of ellipsoidal harmonics, Lamé functions, and standard elliptic integrals.

A. F. Stevenson [9, 10] was the first to study the electromagnetic scattering problem. He has examined the scattering when the scatterer is an ellipsoidal dielectric. All the results of the present work are in agreement with Stevenson's results and can be derived as degenerate cases of our problem.

A main contribution to electromagnetic scattering in low frequencies is made by R. Kleinman. In [1, 2] are given results for scatterers of spheroidal shape when we have a Dirichlet or Neumann problem. The scattering of acoustic and elastic waves by scatterers containing a core are examined in [3, 6].

In Sec. 2 we formulate the problem we discuss in this paper. For completeness, we include all the necessary formulae, far-field expressions, integral representations, fundamental solutions, scattering amplitude and scattering cross-section which are proved in [5].

In Sec. 3 we introduce the ellipsoidal harmonic functions in order to reflect the geometrical peculiarities of the scatterer. We give all the definitions and the useful relations among the ellipsoidal harmonics, Lamé functions and the elliptic integrals.

In Sec. 4 we apply a technique in order to solve the zeroth-order coefficient problem and the first-order coefficient problem.

The normalized spherical scattering amplitude and the scattering cross-section are evaluated in Sec. 5.

Finally, in Sec. 6 we discuss the special cases that correspond to degenerate ellipsoids and the degenerate cases which are obtained if there is a particular relation between the material constants or if there is geometrical degeneration.

2. Formulation of the problem

Let us assume that the triaxial ellipsoid

$$\sum_{i=1}^{3} \frac{x^2}{a_i^2} \leqq 1, \quad 0<a_3<a_2<a_1<+\infty \tag{1}$$

is a dielectric with dielectric constant ε_2 and permeability μ_2 which lies in an infinite homogeneous isotropic medium V_1 with dielectric constant ε_1 and permeability μ_1. Let us also assume that a triaxial ellipsoid

$$\sum_{i=1}^{3} \frac{x_i^2}{\beta_i^2} \leq 1, \quad 0<\beta_3<\beta_2<\beta_1<+\infty \tag{2}$$

which is a perfect conductor lies entirely within the first ellipsoid and is confocal with it. Let us define V_2 the space between the surfaces of the two ellipsoids.

A harmonic time dependence $\exp\{-i\omega t\}$ where ω is the angular frequency is suppressed throughout this work.

An "incident" plane electric wave E^{in} propagates in the medium V_1 along the propagation vector $\hat{k}$. Let the corresponding magnetic wave is H^{in}. The two waves have the form

$$\begin{aligned} \boldsymbol{E}^{\mathrm{in}}(\boldsymbol{r}) &= \hat{\boldsymbol{b}} e^{ik_1 \hat{\boldsymbol{k}} \cdot \boldsymbol{r}} \\ \boldsymbol{H}^{\mathrm{in}}(\boldsymbol{r}) &= (\hat{\boldsymbol{k}} \times \hat{\boldsymbol{b}}) \left(\frac{\varepsilon_1}{\mu_1}\right)^{1/2} e^{ik_1 \hat{\boldsymbol{k}} \cdot \boldsymbol{r}}, \end{aligned} \tag{3}$$

where $\hat{\boldsymbol{b}}$ is the unit polarization vector for the electric field $\hat{\boldsymbol{b}} \cdot \hat{\boldsymbol{k}} = 0$ and k_1 is the propagation constant for V_1.

The ellipsoid (1) called the "scatterer" disturbs the propagation of the incident wave. The ellipsoid (2) will be called the core of the scatterer. If $\boldsymbol{E}(r)$, $\boldsymbol{H}(r)$ are the scattered electric and magnetic waves respectively and $\boldsymbol{E}_i(r)$, $\boldsymbol{H}_i(r)$ the total fields for the spaces V_i, $i=1, 2$, then (due to linearity) the total waves are given by the sum of the incident plus the scattered field. All the above fields satisfy the equations

$$\nabla \times \nabla \times \boldsymbol{w}(\boldsymbol{r}) - k_i^2 \boldsymbol{w}(\boldsymbol{r}) = \boldsymbol{0}, \quad \boldsymbol{r} \in V_i, \quad i = 1, 2 \tag{4}$$
$$\nabla . \boldsymbol{w}(\boldsymbol{r}) = 0,$$

where

$$k_i^2 = \omega^2 \mu_i \varepsilon_i. \tag{5}$$

The boundary conditions for the electric field on the surface of the dielectric S_1 are given by the equations

$$\hat{\boldsymbol{n}} \times \boldsymbol{E}_1(\boldsymbol{r}') = \hat{\boldsymbol{n}} \times \boldsymbol{E}_2(\boldsymbol{r}')$$
$$, \quad \boldsymbol{r}' \in S_1. \tag{6}$$
$$\hat{\boldsymbol{n}} \times (\nabla \times \boldsymbol{E}_1(\boldsymbol{r}')) = \frac{\mu_1}{\mu_2} \hat{\boldsymbol{n}} \times (\nabla \times \boldsymbol{E}_2(\boldsymbol{r}'))$$

For the magnetic field the boundary conditions on S_1 can be derived by Eqs (6) substituting $\boldsymbol{E}_i$ with $\boldsymbol{H}_i$ and μ_i with ε_i. On the surface of the perfect conductor S_0 the following equations must be satisfied

$$\hat{\boldsymbol{n}} \times \boldsymbol{E}_2(\boldsymbol{r}') = \boldsymbol{0}$$
$$, \quad \boldsymbol{r}' \in S_0. \tag{7}$$
$$\hat{\boldsymbol{n}} \times \nabla \times \boldsymbol{H}_2(\boldsymbol{r}') = \boldsymbol{0}$$

On the surface S_1 of the dielectric the boundary condition

$$\int_S \hat{\boldsymbol{n}} . \boldsymbol{E}(\boldsymbol{r}') \, dS(\boldsymbol{r}') = 0, \tag{8}$$

must also be satisfied [5].

The scattered fields $\boldsymbol{E}(\boldsymbol{r})$, $\boldsymbol{H}(\boldsymbol{r})$ satisfy the radiation condition, due to Sommerfeld

$$\lim_{r \to \infty} \boldsymbol{r} \times \left\{ \nabla \times \begin{pmatrix} \boldsymbol{E}(\boldsymbol{r}) \\ \boldsymbol{H}(\boldsymbol{r}) \end{pmatrix} \right\} + i k_1 r \begin{pmatrix} \boldsymbol{E}(\boldsymbol{r}) \\ \boldsymbol{H}(\boldsymbol{r}) \end{pmatrix} = \boldsymbol{0} \tag{9}$$

uniformly over all directions.

In what follows we will examine the electric field only, because as we can prove the determination of the electric field suffices for the evaluation of the total electromagnetic field. The total electric field admits the following integer representatiton [5]

$$\begin{aligned} E_1(r) = E^{in}(r) &+ \frac{1}{4\pi}\frac{\mu_1}{\mu_2}\int_{S_0} \nabla \times E_2(r') . (\hat{n} \times \tilde{\Gamma}(r,r'))\, dS(r') \\ &- \frac{1}{4\pi}\int_{V_2} \{(\frac{\varepsilon_2}{\varepsilon_1} - 1) k_1^2 E_2(r') . \tilde{\Gamma}(r,r') \\ &+ (1 - \frac{\mu_1}{\mu_2}) \nabla \times E_2(r') . \nabla_{r'} \times \tilde{\Gamma}(r,r')\}\, dU(r'), \end{aligned} \tag{10}$$

where the fundamental dyadic has the analytical form

$$\begin{aligned} \tilde{\Gamma}(r,r') = \frac{e^{ik|r-r'|}}{k^2|r-r'|^3}\{k^2(r-r') \otimes (r-r') \\ + (1 - ik|r-r'|)(\tilde{I} - 3\frac{(r-r') \otimes (r-r')}{|r-r'|^2})\} - \frac{e^{ik}|r-r'|}{|r-r'|}\tilde{I}. \end{aligned} \tag{11}$$

We have proved in [5] that the normalized spherical scattering amplitude is given by the

$$\begin{aligned} g(\hat{r},\hat{k}) = \frac{1}{4\pi}\{-ik_1\frac{\mu_1}{\mu_2}\int_{S_0} \nabla \times E_2(r') . (\hat{n} \times (\tilde{I} - \hat{r} \otimes \hat{r}))\, e^{-ik_1\hat{r}.r'} dS(r') \\ + (\frac{\varepsilon_2}{\varepsilon_1} - 1)\, ik_1^3 \int_{V_2} E_2(r') . (\tilde{I} - \hat{r} \otimes \hat{r})\, e^{-ik_1\hat{r}.r'} dU(r') \\ + (1 - \frac{\mu_1}{\mu_2}) k_1^2 \int_{V_2} \nabla \times E_2(r') . (\tilde{I} \times r)\, e^{-ik_1 r.r'} dU(r')\}. \end{aligned} \tag{12}$$

The scattering cross-section, which is actually a measure of the interaction of the scattered and the incident wave, is defined as the ratio of the time average rate at which energy is scattered by the body, to the corresponding time average rate at which the energy of the incident wave crosses a unit area normal to the direction of propagation. The scattering cross-section is equal to

$$\sigma = \frac{1}{k_1^2}\int_{|\hat{r}|=1} |g(\hat{r},\hat{k})|^2 d\Omega(\hat{r}). \tag{13}$$

In [5] we have shown that if we consider the expansions

$$E_i(\mathbf{r}) = \sum_{n=0}^{\infty} \frac{(ik_1)^n}{n!} \Phi_n^{(i)}(r), \quad r \in V_i, \quad i = 1, 2, \tag{14}$$

the following sequence of partial differential equations is obtained

$$\begin{aligned} &\nabla \times \nabla \times \boldsymbol{\Phi}_n^{(i)}(\boldsymbol{r}) + n(n-1)\, m_i\, \boldsymbol{\Phi}_{n-2}^{(i)}(\boldsymbol{r}) = \boldsymbol{0}, \\ &\nabla \cdot \boldsymbol{\Phi}_n^{(i)}(\boldsymbol{r}) = 0, \quad n = 0, 1, 2, \ldots, \quad \boldsymbol{r} \in V_i, \quad i = 1, 2, \end{aligned} \tag{15}$$

where

$$m_i = \begin{cases} 1, & \text{for} \quad i = 1 \\ \dfrac{\mu_2 \varepsilon_2}{\mu_1 \varepsilon_1}, & \text{for} \quad i = 2. \end{cases} \tag{16}$$

The boundary conditions are transformed into the boundary conditions

$$\left.\begin{aligned} &\hat{\boldsymbol{n}} \times \boldsymbol{\Phi}_n^{(1)}(\boldsymbol{r}') = \hat{\boldsymbol{n}} \times \boldsymbol{\Phi}_n^{(2)}(\boldsymbol{r}') \\ &\hat{\boldsymbol{n}} \times (\nabla \times \boldsymbol{\Phi}_n^{(1)}(\boldsymbol{r}')) = \frac{\mu_1}{\mu_2}\, \hat{\boldsymbol{n}} \times (\nabla \times \boldsymbol{\Phi}_n^{(2)}(\boldsymbol{r}')) \end{aligned}\right\} , \quad \boldsymbol{r}' \in S_1 \tag{17}$$

on the surface of the dielectric, and

$$\hat{\boldsymbol{n}} \times \boldsymbol{\Phi}_n^{(2)}(\boldsymbol{r}') = \boldsymbol{0}, \quad \boldsymbol{r}' \in S_0 \tag{18}$$

on the surface of the perfect conductor.

The coefficient $\boldsymbol{\Phi}_n^{(1)}(\boldsymbol{r})$, which forms the n-th order low-frequency approximation of our scattering problem has the following integral representation [5]

$$\begin{aligned} \boldsymbol{\Phi}_n^{(1)}(\boldsymbol{r}) = {} & \hat{\boldsymbol{b}}\, (\hat{\boldsymbol{k}} \cdot \boldsymbol{r})^n + \frac{1}{4\pi} \frac{\mu_1}{\mu_2} \sum_{p=0}^{n} \binom{n}{p} \int_{S_0} (\nabla \times \boldsymbol{\Phi}_p^{(2)}(\boldsymbol{r}')) \\ & \cdot (\hat{\boldsymbol{n}} \times \tilde{\gamma}_{n-p}(\boldsymbol{r}, \boldsymbol{r}'))\, \mathrm{d}S(\boldsymbol{r}') + \frac{1}{4\pi} \sum_{p=0}^{n} \binom{n}{p} \int_{V_2} \{ (\frac{\varepsilon_2}{\varepsilon_1} - 1) p(p-1)\, \boldsymbol{\Phi}_{p-2}^{(2)}(\boldsymbol{r}') \\ & \cdot \tilde{\gamma}_{n-p}(\boldsymbol{r}, \boldsymbol{r}') - (1 - \frac{\mu_1}{\mu_2}) \nabla \times \boldsymbol{\Phi}_p^{(2)}(\mathbf{r}') \cdot \tilde{\delta}_{n-p}(\boldsymbol{r}, \boldsymbol{r}') \}\, \mathrm{d}U(\boldsymbol{r}'), \end{aligned} \tag{19}$$

where

$$\tilde{\gamma}_n(\boldsymbol{r}, \boldsymbol{r}') = -\frac{|\boldsymbol{r} - \boldsymbol{r}'|^{n-1}}{n+2} \{ (n+1)\, \tilde{I} - (n-1) \frac{(\boldsymbol{r} - \boldsymbol{r}') \otimes (\boldsymbol{r} - \boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|^2} \} \tag{20}$$

and

$$\tilde{\delta}_n(\boldsymbol{r}, \boldsymbol{r}') = (n-1)\, |\boldsymbol{r} - \boldsymbol{r}'|^{n-3}\, (\boldsymbol{r} - \boldsymbol{r}') \times \tilde{I}. \tag{21}$$

The low-frequency expansion for the scattering amplitude is given by the expression

$$g(\hat{r}, \hat{k}) = -\frac{1}{4\pi}\frac{\mu_1}{\mu_2}\sum_{n=0}^{\infty}\frac{(ik_1)^{n+1}}{n!}\sum_{\rho=0}^{n}\binom{n}{\rho}(-1)^\rho \int_{S_0} \nabla \times \boldsymbol{\Phi}_{n-\rho}^{(2)}(\boldsymbol{r}')$$

$$.(\hat{\boldsymbol{n}} \times (\tilde{I} - \hat{\boldsymbol{r}} \otimes \hat{\boldsymbol{r}}))(\hat{\boldsymbol{r}}.\boldsymbol{r}')^\rho dS(\boldsymbol{r}') - \frac{1}{4\pi}(\frac{\varepsilon_2}{\varepsilon_1} - 1)\sum_{n=0}^{\infty}\frac{(ik_1)^{n+3}}{n!}$$

$$(22) \qquad .\sum_{\rho=0}^{n}\binom{n}{\rho}(-1)^\rho \int_{V_2} \boldsymbol{\Phi}_{n-\rho}^{(2)}(\boldsymbol{r}').(\tilde{I} - \hat{\boldsymbol{r}} \otimes \hat{\boldsymbol{r}})(\hat{\boldsymbol{r}}.\boldsymbol{r}')^\rho dU(\boldsymbol{r}')$$

$$-\frac{1}{4\pi}(1 - \frac{\mu_1}{\mu_2})\sum_{n=0}^{\infty}\frac{(ik_1)^{n+2}}{n!}\sum_{\rho=0}^{n}\binom{n}{\rho}(-1)^\rho \int_{V_2} \nabla \times \boldsymbol{\Phi}_{n-\rho}^{(2)}(\boldsymbol{r}').(\tilde{I} \times \hat{\boldsymbol{r}})(\hat{\boldsymbol{r}}.\boldsymbol{r}')^\rho dU(\boldsymbol{r}').$$

3. The ellipsoidal geometry

In order to reflect the geometrical peculiarities of the scatterer we introduce the ellipsoidal harmonic functions. The ellipsoidal harmonic functions, as it is well-known, form a complete system of eigenfunctions. In what follows, we will give certain definitions about ellipsoidal harmonics. For details about the ellipsoidal harmonics we refer the reader to W. Hobson [4].

The ellipsoidal coordinates (ρ, μ, ν) are related to the Cartesian coordinates (x_1, x_2, x_3) by

$$(23) \qquad \left.\begin{aligned} x_1 &= \frac{\rho\mu\nu}{h_2 h_3} \\ x_2 &= \frac{\sqrt{\rho^2 - h_3^2}\sqrt{\mu^2 - h_3^2}\sqrt{h_3^2 - \nu^2}}{h_1 h_3} \\ x_3 &= \frac{\sqrt{\rho^2 - h_2^2}\sqrt{h_2^2 - \mu^2}\sqrt{h_2^2 - \nu^2}}{h_1 h_2} \end{aligned}\right\},$$

where

$$(24) \qquad h_1^2 = a_2^2 - a_3^2, \quad h_2^2 = a_1^2 - a_3^2, \quad h_3^2 = a_1^2 - a_2^2$$

and

$$0 \leqq \nu^2 \leqq h_3^2 \leqq \mu^2 \leqq h_2^2 \leqq \rho^2 < +\infty.$$

Separation of variables for the Laplace equation in ellipsoidal coordinates produces the interior ellipsoidal harmonics

$$(25) \qquad \mathsf{E}_n^m(\rho, \mu, \nu) = E_n^m(\rho)\, E_n^m(\mu)\, E_n^m(\nu)$$

and the exterior ellipsoidal harmonics

$$(26) \qquad \mathsf{F}_n^m(\rho, \mu, \nu) = F_n^m(\rho)\, E_n^m(\mu)\, E_n^m(\nu),$$

where E_n^m are the Lamé functions of the first kind and

$$F_n^m(\rho) = (2n+1)\, E_n^m(\rho)\, I_n^m(\rho) \tag{27}$$

with

$$I_n^m(\rho) = \int_\rho^{+\infty} \frac{du}{(E_n^m(u))^2 \sqrt{u^2 - h_2^2}\sqrt{u^2 - h_3^2}} \tag{28}$$

are the Lamé functions of the second kind. The index n specifies the degree of the corresponding ellipsoidal harmonic and takes the value $n = 0, 1, 2, 3, \ldots$ while m represents the number of independent harmonic functions of degree n and runs through the values $m = 1, 2, \ldots, 2n+1$. The interior ellipsoidal harmonics of degree 0, 1, and 2 are those we use in the present work and for the sake of completeness we give their exact form, both in ellipsoidal as well as in Cartesian representation :

$$\mathsf{E}_0^1(\rho, \mu, \nu) = 1, \tag{29}$$

$$\left.\begin{aligned} \mathsf{E}_1^1(\rho, \mu, \nu) &= \rho\mu\nu \\ &= x_1 h_2 h_3 \\ \mathsf{E}_1^2(\rho, \mu, \nu) &= \sqrt{\rho^2 - h_3^2}\sqrt{\mu^2 - h_3^2}\sqrt{h_3^2 - \nu^2} \\ &= x_2 h_1 h_3 \\ \mathsf{E}_1^3(\rho, \mu, \nu) &= \sqrt{\rho^2 - h_2^2}\sqrt{h_2^2 - \mu^2}\sqrt{h_2^2 - \nu^2} \\ &= x_3 h_1 h_2 \end{aligned}\right\}, \tag{30}$$

$$\begin{aligned} \mathsf{E}_2^1(\rho, \mu, \nu) &= (\rho^2 - \alpha_1^2 + \Lambda)(\mu^2 - \alpha_1^2 + \Lambda)(\nu^2 - \alpha_1^2 + \Lambda) \\ &= (\Lambda - \alpha_1^2)(\Lambda - \alpha_2^2)(\Lambda - \alpha_3^2)\Big(\sum_{k=1}^{3} \frac{x_k^2}{\Lambda - \alpha_k^2} + 1\Big), \\ \mathsf{E}_2^2(\rho, \mu, \nu) &= (\rho^2 - a_1^2 + \Lambda')(\mu^2 - a_1^2 + \Lambda')(\nu^2 - a_1^2 + \Lambda') \\ &= (\Lambda' - a_1^2)(\Lambda' - a_2^2\Big(\sum_{k=1}^{3} \frac{x_k^2}{\Lambda - a_k^2} + 1\Big), \end{aligned} \tag{31}$$

$$\left.\begin{aligned} \mathsf{E}_2^3(\rho, \mu, \nu) &= \mathsf{E}_1^1(\rho, \mu, \nu)\, \mathsf{E}_1^3(\rho, \mu, \nu). \\ &= x_1 x_2 h_1 h_2 h_3^2 \\ \mathsf{E}_2^4(\rho, \mu, \nu) &= \mathsf{E}_1^1(\rho, \mu, \nu)\, \mathsf{E}_1^2(\rho, \mu, \nu) \\ &= x_1 x_3 h_1 h_2^2 h_3 \\ \mathsf{E}_2^5(\rho, \mu, \nu) &= \mathsf{E}_1^2(\rho, \mu, \nu)\, \mathsf{E}_1^3(\rho, \mu, \nu) \\ &= x_2 x_3 h_1^2 h_2 h_3 \end{aligned}\right\}, \tag{32}$$

where Λ, Λ' are the two roots of the equation

$$\sum_{i=1}^{3} \frac{1}{\Lambda-\alpha_i^2}=0. \tag{33}$$

The exterior ellipsoidal harmonics of degree 0, 1, and 2 are given from (26) when (29) — (32) are used. The Lamé functions of degree 0, 1, and 2 that appear in the expression (28) for the elliptic integrals $I_n^m(\rho)$ are

$$\left.\begin{aligned}
&E_0^1(\rho)=1\\
&E_1^m(\rho)=\sqrt{\rho^2-\alpha_1^2+\alpha_m^2},\quad m=1,\ 2,\ 3\\
&E_2^1(\rho)=\rho^2-\alpha_1^2+\Lambda\\
&E_2^2(\rho)=\rho^2-\alpha_1^2+\Lambda'\\
&E_2^3(\rho)=\rho\sqrt{\rho^2-h_3^2}\\
&E_2^4(\rho)=\rho\sqrt{\rho^2-h_2^2}\\
&E_2^5(\rho)=\sqrt{\rho^2-h_2^2}\sqrt{\rho^2-h_3^2}
\end{aligned}\right\} \tag{34}$$

the set of functions

$$\{E_n^m(\mu)\,E_n^m(\nu):\quad n=0,\ 1,\ 2,\ldots,\quad m=1,\ 2,\ 3,\ldots,2n+1\} \tag{35}$$

forms a complete orthogonal set of surface harmonics on the surface of the ellipsoid

$$\frac{x_1^2}{\rho^2}+\frac{x_2^2}{\rho^2-h_3^2}+\frac{x_3^2}{\rho^2-h_2^2}=1.$$

4. The low-frequency approximations for the electric field

The sequence of problems to which the scattering problem is reduced is described by the inhomogeneous Eq. (15). The coefficient $\boldsymbol{\Phi}_n^{(1)}(r)$ which forms the n-th order low-frequency approximation of our scattering problem is given by the sum

$$\boldsymbol{\Phi}_n^{(1)}(\boldsymbol{r})=\boldsymbol{P}_n^{(1)}(\boldsymbol{r})+\boldsymbol{W}_n^{(1)}(\boldsymbol{r}), \tag{36}$$

where $\boldsymbol{P}_n^{(1)}(\boldsymbol{n})$ is a particular solution of the inhomogeneous equation and $\boldsymbol{W}_n(\boldsymbol{r})$ is the solution of the corresponding homogeneous equation. By substitution in Eq. (15) of the nonvanishing part of the asymptotic form of Eq. (19), as $r\to\infty$ which can be derived from Eq. (19) if we omit the n-th term in the right hand side, we conclude that it provides a particular solution of the inhomogeneous Equation.

So at every step of the low-frequency approximation technique we have to solve the exterior boundary value problem

$$
(37)\qquad
\left.
\begin{aligned}
&\nabla\times\nabla\times \boldsymbol{W}_n^{(1)}(\mathbf{r})=\boldsymbol{0}\\
&\nabla\cdot \boldsymbol{W}_n^{(1)}(\boldsymbol{r})=0,\quad n=0,\ 1,\ldots\\
&\hat{\boldsymbol{n}}\times \boldsymbol{W}_n^{(1)}(\boldsymbol{r}')=\hat{\boldsymbol{n}}\times(\boldsymbol{\Phi}_n^{(2)}(\boldsymbol{r}')-\boldsymbol{P}_n^{(1)}(\boldsymbol{r}')),\quad \boldsymbol{r}'\in S_1\\
&\hat{\boldsymbol{n}}\times(\nabla\times \boldsymbol{W}_n^{(1)}(\boldsymbol{r}'))=\frac{\mu_1}{\mu_2}\hat{\boldsymbol{n}}\times(\nabla\times(\boldsymbol{\Phi}_n^{(2)}(\boldsymbol{r}')-\boldsymbol{P}_n^{(1)}(\boldsymbol{r}'))\\
&\boldsymbol{W}_n^{(1)}(\boldsymbol{r})=0\left(\frac{1}{r}\right),\quad r\to+\infty
\end{aligned}
\right]
$$

Using Stokes' decomposition $\boldsymbol{W}_n^{(1)}(\boldsymbol{r})$ can be written as

$$
(38)\qquad \boldsymbol{W}_n^{(1)}(\boldsymbol{r})=\nabla\, U_n^{(1)}(\boldsymbol{r})+\nabla\times \boldsymbol{V}_n^{(1)}(\boldsymbol{r}),
$$

where every term of the sum in Eq. (38) must be solution of the equations of the problem.

Similar arguments hold for the interior field $\boldsymbol{\Phi}_n^{(2)}(\boldsymbol{r})$. In what follows, we will expose the calculational technique which we will propose in order to overcome all the difficulties arising from the ellipsoidal shape of the scatterer, and moreover, the existence of the core which imposes new boundary conditions on its surface. The use of this technique permit us to evaluate the low-frequency coefficients in a finite number of steps.

We observe that the particular solution $\boldsymbol{P}_n^{(1)}$, as it is given by the asymptotic form of Eq. (19), has expansion in terms of surface ellipsoidal harmonics up to degree n. Since on the surface of the dielectric the tangential components of the electric fields must vary continuously across the boundary and on the surface of the core the tangential component of the electric field must be equal to zero, we conclude that the terms $\boldsymbol{P}_n^{(i)}$, $\boldsymbol{W}_n^{(i)}$, $i=1, 2$, in the representations of $\boldsymbol{\Phi}_n^{(i)}$ have to be up to degree n in terms of surface ellipsoidal harmonics.

So, for all the parts of $\boldsymbol{\Phi}_n^{(i)}(\boldsymbol{r})$, which include the operators ∇, ∇x, we will take expressions, for the functions on which act these operators, in terms of surface ellipsoidal harmonics up to degree $(n+1)$. After the action of the operators we conclude to harmonics up to degree n.

In what follows we will evaluate the zeroth and the first order approximations of the electric fields in low-frequency region. As it is well-known these approximations give us enough information for the total fields [8]. The proposed technique will be clear in the evaluation of the first order approximation, because the zeroth-order approximation as the solution of the electrostatic problem has a well-known form. Nevertheless, the main points of the method characterize the zeroth-order approximation as well.

a) The zeroth-order approximation

The particular solution of the zeroth-order approximation for the exterior field is $\tilde{\boldsymbol{b}}$, that is of degree zero. If we use the well-known representation for the electrostatic problem we have

$$\boldsymbol{\Phi}_0^{(1)}(\boldsymbol{r}) = \hat{\boldsymbol{b}} + \nabla U_0^{(1)}(\boldsymbol{r}), \tag{39}$$

$$\boldsymbol{\Phi}_0^{(2)}(\boldsymbol{r}) = \nabla U_0^{(2)}(\boldsymbol{r}), \tag{40}$$

where the scalar potentials for the exterior and the interior fields are given in terms of second kind and first and second kind ellipsoidal harmonics respectively, as follows:

$$U_0^{(1)}(\boldsymbol{r}) = \alpha_{00}^{(1)1} I_0^1(\rho) + \sum_{m=1}^{3} \alpha_{01}^{(1)m} \mathsf{F}_1^m(\rho, \mu, \nu), \tag{41}$$

$$U_0^{(2)}(\boldsymbol{r}) = \alpha_{00}^{(2)1} I_0^1(\rho) + \sum_{m=1}^{3} \alpha_{01}^{(2)m} \mathsf{F}_1^m(\rho, \mu, \nu) + \sum_{m=1}^{3} b_{01}^{(2)m} \mathsf{E}_1^m(\rho, \mu, \nu). \tag{42}$$

In the evaluation of the $\nabla \mathsf{F}_1^m(\rho, \mu, \nu)$ we will use the general form

$$\nabla \mathsf{F}_n^m(\rho, \mu, \nu) = (2n+1)(\nabla \mathsf{E}_n^m(\rho, \mu, \nu)) I_n^m(\rho) - (2n+1)\frac{\hat{\boldsymbol{\rho}}}{h_\rho} \cdot \frac{E_n^m(\rho, \mu, \nu)}{[E_n^m(\rho)]^2 \sqrt{\rho^2 - h_2^2}\sqrt{\rho^2 - h_3^2}}, \tag{43}$$

where

$$h_\rho = \frac{\sqrt{\rho^2 - \mu^2}\sqrt{\rho^2 - \nu^2}}{\sqrt{\rho^2 - h_2^2}\sqrt{\rho^2 - h_3^2}} \tag{44}$$

is the square root of the ellipsoidal metric coefficient that corresponds to the variable ρ, and

$$\hat{\boldsymbol{\rho}} = \frac{\rho}{h_\rho} \sum_{i=1}^{3} \frac{x_i}{\rho^2 - \alpha_1^2 + \alpha_i^2} \hat{\boldsymbol{x}}_i \tag{45}$$

is the unit curvilinear vector relative to the variable ρ, and $\hat{\boldsymbol{x}}_i$, $i = 1, 2, 3$, are the Cartesian base vectors.

The exterior field $\boldsymbol{\Phi}_0^{(1)}(\boldsymbol{r})$ admits the following representation in terms of surface ellipsoidal harmonics

$$\boldsymbol{\Phi}_0^{(1)}(\boldsymbol{r}) = \{\hat{\boldsymbol{b}} + 3h_1 h_2 h_3 \sum_{m=1}^{3} \frac{\alpha_{01}^{(1)m}}{h_m} I_1^m(\rho)\hat{\boldsymbol{x}}_m\} - \frac{\hat{\boldsymbol{\rho}}}{\sqrt{\rho^2 - \mu^2}\sqrt{\rho^2 - \nu^2}} \{\alpha_{00}^{(1)1} + 3\sum_{m=1}^{3} \frac{\alpha_{01}^{(1)m}}{E_1^m(\rho)} E_1^m(\mu) E_1^m(\nu)\}. \tag{46}$$

We will refer to the first bracket of (46) as the "Cartesian" part of $\boldsymbol{\Phi}_0^{(1)}(\boldsymbol{r})$ and the second bracket as the "ellipsoidal" part of it.

Similar representation, which includes "Cartesian" and "ellipsoidal" part holds for the interior field.

We observe that due to the factor $\hat{\boldsymbol{\rho}}(\rho^2-\mu^2)^{-1/2}(\rho^2-\nu^2)^{-1/2}$ it is not possible to express the second bracket in the right hand side of Eq. (46) in terms of a finite expression of surface ellipsoidal harmonics.

Nevertheless, the boundary conditions involve inner and vector products of the unit normal vector on the surface of the scatterer that is the vector $\hat{\boldsymbol{\rho}}$. So exploiting the above decomposition into "Cartesian" and "ellipsoidal" parts, we apply the boundary conditions and evaluate the solution in a finite number of steps.

After the application of the boundary conditions, by the orthogonality of the surface ellipsoidal harmonics, we conclude to a system for the a's and b's coefficients, the solution of which give us

$$\alpha_{00}^{(1)1}=0, \tag{47}$$

$$\alpha_{01}^{(1)m}=\frac{b_m h_m}{3h_1h_2h_3 I_1^m(\alpha_1)}\left\{\frac{I_1^m(\alpha_1)-I_1^m(\beta_1)}{H_{01}^m}-1\right\}, \quad m=1,\ 2,\ 3, \tag{48}$$

$$H_{01}^m=\alpha_1\alpha_2\alpha_3\left(\frac{\varepsilon_2}{\varepsilon_1}-1\right)I_1^m(\alpha_1)\left\{I_1^m(\alpha_1)-I_1^m(\beta_1)-\frac{1}{\alpha_1\alpha_2\alpha_3}\right\}-I_1^m(\beta_1), \tag{49}$$

$$\alpha_{00}^{(2)1}=0, \tag{50}$$

$$\alpha_{01}^{(2)m}=\frac{b_m h_m}{3h_1h_2h_3 H_{01}^m}, \quad m=1,\ 2,\ 3, \tag{51}$$

$$b_{01}^{(2)m}=-\frac{b_m h_m}{h_1h_2h_3 H_{01}^m}, \quad m=1,\ 2,\ 3. \tag{52}$$

b) The first order approximation

Using the asymptotic form of Eq. (19) and the representation given by Eq. (38) for the first-order approximation we conclude that

$$\boldsymbol{\Phi}_1^{(1)}(\boldsymbol{r})=\hat{\boldsymbol{b}}(\hat{\boldsymbol{k}}\cdot\boldsymbol{r})+\nabla U_1^{(1)}(\boldsymbol{r})+\nabla\times \boldsymbol{V}_1^{(1)}(\boldsymbol{r}), \tag{53}$$

$$\boldsymbol{\Phi}_1^{(2)}(\boldsymbol{r})=\nabla U_1^{(2)}(\boldsymbol{r})+\nabla\times \boldsymbol{V}_1^{(2)}(\boldsymbol{r}). \tag{54}$$

Expressing, now, the scalar potentials $U_1^{(i)}(\boldsymbol{r})$, $i=1,\ 2$, in terms of ellipsoidal harmonics of second kind for the exterior space V_1 and of first and second kind, for the interior space V_2, we have

$$\nabla U_1^{(1)}(\boldsymbol{r})=\alpha_{10}^{(1)m}\nabla I_0^1(\rho)+\sum_{m=1}^{3}\alpha_{11}^{(1)m}\nabla \mathrm{F}_1^m(\rho,\mu,\nu) + \sum_{m=1}^{5}\alpha_{12}^{(2)1}\nabla \mathrm{F}_2^m(\rho,\mu,\nu), \tag{55}$$

$$\nabla U_1^{(2)}(\boldsymbol{r})=\alpha_{10}^{(2)1}\,\nabla I_0^1(\rho)+\sum_{m=1}^{3}\alpha_{11}^{(2)m}\,\nabla \mathrm{F}_1^m(\rho,\mu,\nu)$$

$$+\sum_{m=1}^{5}\alpha_{12}^{(2)m}\,\nabla \mathrm{F}_2^m(\rho,\mu,\nu)+\sum_{m=1}^{3}b_{11}^{(2)m}\,\nabla \mathrm{E}_1^m(\rho,\mu,\nu)$$

$$+\sum_{m=1}^{5}b_{12}^{(2)m}\,\nabla \mathrm{E}_2^m(\rho,\mu,\nu). \tag{56}$$

For the vector potential of the exterior field we have

$$\nabla\times \boldsymbol{V}_1^{(1)}(\boldsymbol{r})=\nabla\times \boldsymbol{A}_1^{(1)}\,\varphi_1^{(1)}(\boldsymbol{r})=\nabla\varphi_1^{(1)}(\boldsymbol{r})\times \boldsymbol{A}_1^{(1)}, \tag{57}$$

where $\varphi_1^{(1)}$ can be expressed in terms of surface ellipsoidal harmonics up to degree two. We can choose $\varphi_1^{(1)}$ in such a way so, after the action of the operator ∇ on it, the result would be a vector of "Cartesian" type. This choice ensures that the first-order field $\boldsymbol{\Phi}_1^{(1)}(\boldsymbol{r})$ can be expressed as the sum of two parts : the "Cartesian" and the "ellipsoidal" one. This decomposition is necessary in order to apply the boundary conditions as we have already seen above. In what follows, taking $\varphi_1^{(1)}(\boldsymbol{r})$ in the form

$$\varphi_1^{(1)}(\boldsymbol{r})=\boldsymbol{r}\cdot\frac{1}{6h_1h_2h_3}\sum_{m=1}^{3}h_m\,\mathrm{F}_1^m(\rho,\mu,\nu)\,\hat{\boldsymbol{x}}_m-\frac{1}{2}I_0^1(\rho) \tag{58}$$

we have

$$\nabla\varphi_1^{(1)}(\boldsymbol{r})=\frac{1}{3h_1h_2h_3}\sum_{m=1}^{3}h_m\,\mathrm{F}_1^m(\rho,\mu,\nu)\,\hat{\boldsymbol{x}}_m \tag{59}$$

and consequently, $\nabla\times \boldsymbol{V}_1^{(1)}(\boldsymbol{r})$ is of Cartesian type only.

Similar arguments hold for the vector potential of the interior field. So, finally we have

$$\boldsymbol{\Phi}_1^{(1)}(\boldsymbol{r})=\hat{\boldsymbol{b}}(\hat{\boldsymbol{k}}\cdot\boldsymbol{r})+\frac{1}{3h_1h_2h_3}\sum_{m=1}^{3}h_m\,\mathrm{F}_1^m(\rho,\mu,\nu)\,\hat{\boldsymbol{x}}_m\times \boldsymbol{A}_1(1)$$

$$+\alpha_{10}^{(1)1}\,\nabla I_0^1(\rho)+\sum_{m=1}^{3}\alpha_{11}^{(1)m}\,\nabla \mathrm{F}_1^m(\rho,\mu,\nu)$$

$$+\sum_{m=1}^{5}\alpha_{12}^{(1)m}\,\nabla \mathrm{F}_2^m(\rho,\mu,\nu), \tag{60}$$

$$\boldsymbol{\Phi}_1^{(2)}(\boldsymbol{r})=\frac{1}{3h_1h_2h_3}\sum_{m=1}^{3}h_m\,\mathrm{F}_1^m(\rho,\mu,\nu)\,\hat{\boldsymbol{x}}_m\times \boldsymbol{A}_1^{(2)}$$

$$+\frac{1}{h_1h_2h_3}\sum_{m=1}^{3}h_m\,\mathrm{E}_1^m(\rho,\mu,\nu)\,\hat{\boldsymbol{x}}_m\times \boldsymbol{B}_1^{(2)}+\alpha_{10}^{(2)1}\,\nabla I_0^1(\rho)$$

$$(61)\qquad + \sum_{m=1}^{3} \alpha_{11}^{(2)m} \nabla F_1^m(\rho,\mu,\nu) + \sum_{m=1}^{5} \alpha_{12}^{(2)m} \nabla F_2^m(\rho,\mu,\nu)$$

$$+ \sum_{m=1}^{3} b_{11}^{(2)m} \nabla E_1^m(\rho,\mu,\nu) + \sum_{m=1}^{5} b_{12}^{(2)m} \nabla E_2^m(\rho,\mu,\nu).$$

After the application of the boundary conditions $\rho=\alpha_1$, $\rho=\beta_1$, by the orthogonality of the surface ellipsoidal harmonics, we can evaluate the coefficients. So we have

$$(62)\qquad \alpha_{10}^{(1)1} = \alpha_{10}^{(2)1} = 0,$$

$$(63)\qquad \alpha_{11}^{(1)m} = \alpha_{11}^{(2)m} = b_{11}^{(2)m} = 0, \quad m=1,\,2,\,3,$$

$$(64)\qquad A_{1m}^{(1)} = \frac{\hat{k}\times\hat{b}\cdot\hat{x}_m}{H_{11}^m}\Big[\Big(1-\frac{\mu_1}{\mu_2}\Big)\Big(\frac{1}{\beta_1\beta_2\beta_3} - I_1^m(\beta_1)\Big) + \Big(1-\frac{\mu_1}{\mu_2}\Big) I_1^m(\alpha_1) - \frac{1}{\alpha_1\alpha_2\alpha_3}\Big], \quad m=1,\,2,\,3,$$

$$(65)\qquad H_{11}^m = \Big(1-\frac{\mu_1}{\mu_2}\Big) I_1^m(\alpha_1)\Big(\frac{1}{\alpha_1\alpha_2\alpha_3} - I_1^m(\alpha_1)\Big) - \Big[\Big(1-\frac{\mu_1}{\mu_2}\Big) I_1^m(\alpha_1) + \frac{\mu_1}{\mu_2}\frac{1}{\alpha_1\alpha_2\alpha_3}\Big]\cdot\Big(\frac{1}{\beta_1\beta_2\beta_3} - I_1^m(\beta_1)\Big),$$

$$(66)\qquad A_{1m}^{(2)} = \frac{\hat{b}\times\hat{k}\cdot\hat{x}_m}{\alpha_1\alpha_2\alpha_3 H_{11}^m}, \quad m=1,\,2,\,3,$$

$$(67)\qquad B_{1m}^{(2)} = \frac{\hat{k}\times\hat{b}\cdot\hat{x}_m}{2\alpha_1\alpha_2\alpha_3 H_{11}^m}\Big[\frac{1}{\beta_1\beta_2\beta_3} - I_1^m(\beta_1)\Big], \quad m=1,\,2,\,3,$$

$$(68)\qquad \alpha_{12}^{(1)1} = \frac{1}{30(\Lambda-\Lambda')I_2^1(\alpha_1)} \sum_{m=1}^{3} \frac{b_m k_m}{\Lambda-\alpha_m^2}\Big(1 - \frac{I_2^1(\alpha_1) - I_2^1(\beta_1)}{H_{12}^1}\Big),$$

$$(69)\qquad H_{12}^1 = (I_2^1(\alpha_1) - I_2^1(\beta_1))\Big[1 + 2\alpha_1\alpha_2\alpha_3\Lambda\Big(\frac{\varepsilon_2}{\varepsilon_1} - 1\Big) I_2^1(\alpha_1)\Big] - \frac{\varepsilon_2}{\varepsilon_1} I_2^1(\alpha_1).$$

$\alpha_{12}^{(1)2}$ can be obtained by $\alpha_{12}^{(1)1}$, if we interchange Λ with Λ' and I_2^1 with I_2^2. We will denote this, by

$$(70)\qquad \alpha_{12}^{(1)2} = \alpha_{12}^{(1)1} \quad \Lambda \leftrightarrow \Lambda', \quad I_2^1 \leftrightarrow I_2^2,$$

$$(71)\qquad \alpha_{12}^{(2)1} = -\frac{1}{30(\Lambda-\Lambda')H_{12}^1} \sum_{m=1}^{3} \frac{b_m k_m}{\Lambda-\alpha_m^2},$$

(72) $$\alpha_{12}^{(2)2}=\alpha_{12}^{(2)2} \quad \Lambda \leftrightarrow \Lambda', \quad I_2^1 \leftrightarrow I_2^2,$$

(73) $$b_{12}^{(2)1}=\frac{I_2^1(\beta_1)}{6(\Lambda-\Lambda')H_{12}^1}\sum_{m=1}^{3}\frac{b_m k_m}{\Lambda-\alpha_m^2},$$

(74) $$b_{12}^{(2)2}=b_{12}^{(2)1} \quad \Lambda \leftrightarrow \Lambda', \quad I_2^1 \leftrightarrow I_2^2,$$

$$\alpha_{12}^{(1)(m+n)}=\frac{1}{10h_1h_2h_3h_k H_{12}^{m+n}}\{2(I_2^{m+n}(\alpha_1)-I_2^{m+n}(\beta_1))$$

$$(b_mk_n\alpha_n^2+b_nk_m\alpha_m^2)-\frac{\varepsilon_2}{\varepsilon_1}((\alpha_m^2+\alpha_n^2)(I_2^{m+n}(\alpha_1)$$

$$-I_2^{m+n}(\beta_1))-\frac{1}{\alpha_1\alpha_2\alpha_3})(k_mb_n+k_nb_m)+[2(I_2^{m+n}(\alpha_1)$$

$$-I_2^{m+n}(\beta_1))(\alpha_n^2 I_1^n(\alpha_1)-\alpha_m^2 I_1^m(\alpha_1))-\frac{\varepsilon_2}{\varepsilon_1}(I_1^n(\alpha_1)-I_1^m(\alpha_1))$$

$$((\alpha_m^2+\alpha_n^2)(I_2^{m+n}(\alpha_1)-I_2^{m+n}(\beta_1))-\frac{1}{\alpha_1\alpha_2\alpha_3})]A_{1k}^{(1)}$$

$$+2\frac{\varepsilon_2}{\varepsilon_1}h_k^2(I_2^{m+n}(\alpha_1)-I_2^{m+n}(\beta_1))B_{1k}^{(2)}-\frac{\varepsilon_2}{\varepsilon_1}[(I_2^{m+n}(\alpha_1)$$

$$-I_2^{m+n}(\beta_1))(2(\alpha_n^2 I_1^n(\alpha_1)-\alpha_m^2 I_1^m(\alpha_1))-(\alpha_n^2+\alpha_m^2)$$

(75) $$(I_1^n(\alpha_1)-I_1^m(\alpha_1)))+\frac{1}{\alpha_1\alpha_2\alpha_3}((I_1^n(\alpha_1)-I_1^m(\alpha_1))-(I_1^n(\beta_1)-I_1^m(\beta_1)))]A_{1k}^{(2)}\},$$

$$H_{12}^{m+n}=[I_2^{m+n}(\alpha_1)-I_2^{m+n}(\beta_1)][(\frac{\varepsilon_2}{\varepsilon_1}-1)I_2^{m+n}(\alpha_1)(\alpha_m^2+\alpha_n^2)$$

(76) $$+\frac{1}{\alpha_1\alpha_2\alpha_3}]-\frac{\varepsilon_2}{\varepsilon_1}\frac{I_2^{m+n}(\alpha_1)}{\alpha_1\alpha_2\alpha_3}, \quad m,\ n=1,\ 2,\ 3, \quad k\neq m\neq n\neq k$$

$$\alpha_{12}^{(2)(m+n)}=\frac{1}{10h_1h_2h_3h_k H_{12}^{m+n}}\{2I_2^{m+n}(\alpha_1)(b_mk_n\alpha_n^2+b_nk_m\alpha_m^2)$$

$$-(k_mb_n+k_nb_m)(I_2^{m+n}(\alpha_1)(\alpha_m^2+\alpha_n^2)-\frac{1}{\alpha_1\alpha_2\alpha_3})$$

$$+[2I_2^{m+n}(\alpha_1)(\alpha_n^2 I_1^n(\alpha_1)-\alpha_m^2 I_1^m(\alpha_1))-(I_1^n(\alpha_1)-I_1^m(\alpha_1))$$

$$(I_2^{m+n}(\alpha_1)(\alpha_m^2+\alpha_n^2)-\frac{1}{\alpha_1\alpha_2\alpha_3}]A_{1k}^{(1)}+2\frac{\varepsilon_2}{\varepsilon_1}h_k^2 I_2^{m+n}(\alpha_1)B_{1k}^{(2)}$$

$$+[(\frac{\varepsilon_2}{\varepsilon_1}-1)I_2^{m+n}(\alpha_1)(\alpha_m^2+\alpha_n^2)(I_1^n(\beta_1)-I_1^m(\beta_1))$$

$$+(I_1^n(\alpha_1)-I_1^m(\alpha_1))(I_2^{m+n}(\alpha_1)(\alpha_m^2+\alpha_n^2)-\frac{1}{\alpha_1\alpha_2\alpha_3})$$

$$+\frac{1}{\alpha_1\alpha_2\alpha_3}(I_1^n(\beta_1)-I_1^m(\beta_1))-2\frac{\varepsilon_2}{\varepsilon_1}I_2^{m+n}(\alpha_1)(\alpha_n^2 I_1^n(\alpha_1)$$

$$-\alpha_m^2 I_1^m(\alpha_1))]\,A_{1k}^{(2)}\},\quad m,\ n=1,\ 2,\ 3,\quad k\neq m\neq n\neq k, \tag{77}$$

$$b_{12}^{(2)(m+n)}=\frac{I_2^{m+n}(\beta_1)}{2h_1h_2h_3h_k\,H_{12}^{m+n}}\{(k_m b_n+k_n b_m)(I_2^{m+n}(\alpha_1)$$

$$(\alpha_m^2+\alpha_n^2)-\frac{1}{\alpha_1\alpha_2\alpha_3})-2I_2^{m+n}(\alpha_1)(b_m k_n\alpha_n^2+b_n k_m\alpha_m^2)$$

$$-[2I_2^{m+n}(\alpha_1)(\alpha_n^2 I_1^n(\alpha_1)-\alpha_m^2 I_1^m(\alpha_1))-(I_1^n(\alpha_1)-I_1^m(\alpha_1))$$

$$(I_2^{m+n}(\alpha_1)(\alpha_m^2+\alpha_n^2)-\frac{1}{\alpha_1\alpha_2\alpha_3})]\,A_{1k}^{(1)}-2\frac{\varepsilon_2}{\varepsilon_1}h_k^2\,I_2^{m+n}(\alpha_1)\,B_{1k}^{(2)}$$

$$-[(\frac{\varepsilon_2}{\varepsilon_1}-1)(I_1^n(\beta_1)-I_1^m(\beta_1))\frac{I_2^{m+n}(\alpha_1)}{I_2^{m+n}(\beta_1)}(I_2^{m+n}(\alpha_1)(\alpha_n^2+\alpha_m^2)$$

$$-\frac{1}{\alpha_1\alpha_2\alpha_3})+(I_1^n(\alpha_1)-I_1^m(\alpha_1))(I_2^{m+n}(\alpha_1)(\alpha_n^2+\alpha_m^2)-\frac{1}{\alpha_1\alpha_2\alpha_3})$$

$$-2\frac{\varepsilon_2}{\varepsilon_1}I_2^{m+n}(\alpha_1)(\alpha_n^2 I_1^n(\alpha_1)-\alpha_m^2 I_1^m(\alpha_1))]\,A_{1k}^{(2)}\},\quad m,\ n=1,\ 2,\ 3,\quad k\neq m\neq n\neq k. \tag{78}$$

5. The normalized scattering amplitude and the scattering cross-section

Knowing the exact forms of the first two low-frequency fields it is possible to evaluate the normalized spherical scattering amplitude up to the term k^3 by using the expansion [5]

$$g(\hat{r},\hat{k})=(ik_1)^3\{-\frac{1}{8\pi}\frac{\mu_1}{\mu_2}\int_{S_0}\nabla\times\boldsymbol{\Phi}_2^{(2)}(\boldsymbol{r}')\times\hat{\boldsymbol{n}}\,dS(\boldsymbol{r}')$$

$$+\frac{1}{4\pi}\frac{\mu_1}{\mu_2}\hat{r}\cdot\int_{S_0}\boldsymbol{r}'\otimes\nabla\times\boldsymbol{\Phi}_1^{(2)}(\boldsymbol{r}')\times\hat{\boldsymbol{n}}\,dS(\boldsymbol{r}')$$

$$-\frac{1}{4\pi}(\frac{\varepsilon_2}{\varepsilon_1}-1)\int_{V_2}\boldsymbol{\Phi}_0^{(2)}(\boldsymbol{r}')\,dU(\boldsymbol{r}')\cdot(\tilde{I}-\hat{r}\otimes\hat{r})$$

(79) $$-\frac{1}{4\pi}(1-\frac{\mu_1}{\mu_2})\int_{V_2}\nabla\times\boldsymbol{\Phi}_1^{(2)}(\boldsymbol{r}')\,dU(\boldsymbol{r}').\tilde{I}\times\hat{\boldsymbol{r}}\}+O(k_1^4),\quad k_1\to 0.$$

The integrals in (79) can be evaluated as follows:

(80) $$\int_{S_0}\boldsymbol{r}'\otimes\nabla\times\boldsymbol{\Phi}_1^{(2)}(\boldsymbol{r}')\times\hat{\boldsymbol{n}}\,dS(\boldsymbol{r}')=\frac{4\pi}{3\alpha_1\alpha_2\alpha_3}\sum_{m=1}^{3}\frac{1}{H_{11}^m}(\hat{\boldsymbol{b}}\times\hat{\boldsymbol{k}}.\hat{\boldsymbol{x}}_m)\hat{\boldsymbol{x}}_m\times\tilde{I},$$

(81) $$\int_{V_2}\boldsymbol{\Phi}_0^{(2)}(\boldsymbol{r}')\,dU(\boldsymbol{r}')=\frac{4\pi}{3}\alpha_1\alpha_2\alpha_3\sum_{m=1}^{3}\frac{[I_1^m(\alpha_1)-I_1^m(\beta_1)]}{H_{01}^m}b_m\hat{\boldsymbol{x}}_m,$$

$$\int_{V_2}(\nabla\times\boldsymbol{\Phi}_1^{(2)}(\boldsymbol{r}')\,dU(\boldsymbol{r}')=\frac{4\pi}{3}\sum_{m=1}^{3}[(I_1^m(\beta_1)-I_1^m(\alpha_1))$$

(82) $$+(\frac{1}{\alpha_1\alpha_2\alpha_3}-\frac{1}{\beta_1\beta_2\beta_3})]\frac{\hat{\boldsymbol{k}}\times\hat{\boldsymbol{b}}.\hat{\boldsymbol{x}}_m}{H_{11}^m}\hat{\boldsymbol{x}}_m.$$

For the evaluation of the integral which contains the second-order approximation of the electric field, following the above prescribed technique for the evaluation of the low-frequency coefficients we must work with ellipsoidal harmonics of third degree. We prefer to solve a similar as before problem for the evaluation of the first order approximation of the magnetic field and exploit the relation between the electric and magnetic low-frequency fields [5]

(83) $$\nabla\times\boldsymbol{\Phi}_2^{(2)}(\boldsymbol{r})=\frac{2\mu_2}{\sqrt{\mu_1\varepsilon_1}}\Psi_1^{(2)}(\boldsymbol{r}),\quad \boldsymbol{r}\in V_2.$$

After the evaluation of the coefficients for $\Psi_1^{(2)}(\boldsymbol{r})$, we conclude that

(84) $$\int_{S_0}\nabla\times\boldsymbol{\Phi}_2^{(2)}(\boldsymbol{r}')\times\hat{\boldsymbol{n}}\,\mathrm{d}S(\boldsymbol{r}')=\frac{8\pi}{3}\frac{\mu_2}{\mu_1}\sum_{m=1}^{3}\frac{b_m}{G_{11}^m}\hat{\boldsymbol{x}}_m,$$

where

(85) $$G_{11}^m=(I_1^m(\beta_1)-I_1^m(\alpha_1))[(1-\frac{\varepsilon_1}{\varepsilon_2})\alpha_1\alpha_2\alpha_3I_1^m(\alpha_1)+\frac{\varepsilon_1}{\varepsilon_2}]+I_1^m(\alpha_1).$$

The scattering cross-section is given by the relation [5]

$$\sigma=k_1^4\{\frac{1}{24\pi}\frac{\mu_1^2}{\mu_2^2}|\int_{S_0}\nabla\times\boldsymbol{\Phi}_2^{(2)}(\boldsymbol{r}')\times\hat{\boldsymbol{n}}\,\mathrm{d}S(\boldsymbol{r}')|^2$$

$$+\frac{1}{60\pi}\frac{\mu_1^2}{\mu_2^2}\|\int_{S_0}\boldsymbol{r}'\otimes(\nabla\times\boldsymbol{\Phi}_1^{(2)}(\boldsymbol{r}')\times\hat{\boldsymbol{n}})\,\mathrm{d}S(\boldsymbol{r}')\|^2$$

$$+\frac{7}{60\pi}\frac{\mu_1^2}{\mu_2^2}\left|\int_{S_0}(\nabla\times\boldsymbol{\Phi}_1^{(2)}(\boldsymbol{r}')\times\hat{\boldsymbol{n}}).\boldsymbol{r}'\,dS(\boldsymbol{r}')\right|^2+\frac{1}{6\pi}\left(\frac{\varepsilon_2}{\varepsilon_1}-1\right)^2\left|\int_{V_2}\boldsymbol{\Phi}_0^{(2)}(\boldsymbol{r}')\,dU(\boldsymbol{r}')\right|^2$$

(86)

$$+\frac{1}{6\pi}\left(1-\frac{\mu_1}{\mu_2}\right)^2\left|\int_{V_2}\nabla\times\boldsymbol{\Phi}_1^{(2)}(\boldsymbol{r}')\,dU(\boldsymbol{r}')\right|^2$$

$$+\frac{1}{6\pi}\frac{\mu_1}{\mu_2}\left(\frac{\varepsilon_2}{\varepsilon_1}-1\right)\int_{S_0}\nabla\times\boldsymbol{\Phi}_2^{(2)}(\boldsymbol{r}')\times\hat{\boldsymbol{n}}\,dS(\boldsymbol{r}').\int_{V_2}\boldsymbol{\Phi}_0^{(2)}(\boldsymbol{r}')\,dU(\boldsymbol{r}')$$

$$+\frac{1}{6\pi}\frac{\mu_1}{\mu_2}\left(1-\frac{\mu_1}{\mu_2}\right)\int_{S_0}(\nabla\times\boldsymbol{\Phi}_1^{(2)}(\boldsymbol{r}')\times\hat{\boldsymbol{n}})\times\boldsymbol{r}'\,dS(\boldsymbol{r}').\int_{V_2}\nabla\times\boldsymbol{\Phi}_1^{(2)}(\boldsymbol{r}')\,dU(\boldsymbol{r}')\}+O(k_1^6).$$

The integrals that appear in (86) assume the values

$$\int_{S_0}\boldsymbol{r}'.\nabla\times\boldsymbol{\Phi}_1^{(2)}(\boldsymbol{r}')\times\hat{\boldsymbol{n}}\,dS(\boldsymbol{r}')=0, \tag{87}$$

$$\int_{S_0}(\nabla\times\boldsymbol{\Phi}_1^{(2)}(\boldsymbol{r}')\times\hat{\boldsymbol{n}})\times\boldsymbol{r}'\,dS(\boldsymbol{r}')=\frac{8\pi}{3\alpha_1\alpha_2\alpha_3}\sum_{m=1}^{3}\frac{\hat{\boldsymbol{k}}\times\hat{\boldsymbol{b}}.\hat{\boldsymbol{x}}_m}{H_{11}^m}\hat{\boldsymbol{x}}_m. \tag{88}$$

The integrals given by Eqs (87, 88) are the scalar and vector invariants respectively of the dyadic given by Eq. (80).

By direct substitution of Eqs (81, 82, 84, 87, 88) in (86) we derive the leading term approximation for the scattering cross-section in low-frequency region.

6. Physical and geometrical degenerate cases

The physical interpretation of the mathematical problem analyzed in this work involves a plane harmonic electromagnetic wave, that propagates in the three-dimensional Euclidean space where there exists a dielectric with the shape of a general triaxial ellipsoid with a confocal triaxial ellipsoidal core, which is a perfect conductor. The existence of the ellipsoidal scatterer, which is arbitrarily oriented with respect to the propagation vector of the incident plane wave, disturbs the incident wave, and as a result a much more complicated wave field is established.

The zeroth- and the first-order approximation coefficients of the electric fields are determined. The real difficulty of the problem is focused on the evaluation of the first-order low-frequency approximation, which demands many mathematical techniques. The leading term approximations for the normalized scattering amplitude and the scattering cross-section are also given.

The physics of the problem is determined by the values of ε_1, ε_2, μ_1, μ_2. The special cases of scattering by a dielectric or a perfect conductor correspond to physically degenerate cases of our general problem. This has been achieved since all the types of boundary conditions have been incorporated on the surfaces of the two ellipsoids. In what follows we will present results for these special cases.

a) Dielectric

Considering the limit case

$$\beta_1 \to h_2, \quad \beta_2 \to h_1, \quad \beta_3 \to 0,$$

we have $V_2 \to 0$. Taking the limits as $\beta_1 \to h_2$ of the elliptic integrals and evaluating the limits of all the coefficients we conclude to the solution for the dielectric scatterer, we present the result for the zeroth-order approximation.

$$(89)\quad \boldsymbol{\Phi}_0^{(1)}(\boldsymbol{r})=\hat{\boldsymbol{b}}-\sum_{m=1}^{3}\frac{b_m h_m}{3h_1h_2h_3}\frac{\alpha_1\alpha_2\alpha_3(\frac{\varepsilon_2}{\varepsilon_1}-1)}{\alpha_1\alpha_2\alpha_3(\frac{\varepsilon_2}{\varepsilon_1}-1)I_1^m(\alpha_1)+1}\cdot\nabla \mathrm{F}_1^m(\rho,\mu,\nu),$$

$$(90)\quad \boldsymbol{\Phi}_0^{(2)}(\boldsymbol{r})=\sum_{m=1}^{3}\frac{b_m h_m}{h_1h_2h_3}\frac{1}{\alpha_1\alpha_2\alpha_3(\frac{\varepsilon_2}{\varepsilon_1}-1)I_1^m(\alpha_1)+1}\nabla \mathrm{E}_1^m(\rho,\mu,\nu).$$

The above solution is in agreement with already existing results.

b) Perfect conductor

Taking $\varepsilon_1=\varepsilon_2$ and $\mu_1=\mu_2$, we have the case of the perfect ellipsoidal conductor with $\rho=\beta_1$ and the result for example, for the zeroth-order low-frequency approximation is

$$(91)\quad \boldsymbol{\Phi}_0^{(2)}(\boldsymbol{r})=\hat{\boldsymbol{b}}-\sum_{m=1}^{3}\frac{b_m h_m}{3h_1h_2h_3 I_1^m(\beta_1)}\nabla \mathrm{F}_1^m(\rho,\mu,\nu).$$

As geometrical degenerate cases can be considered the spheroids, the needle, the disc and the sphere.

A prolate spheroid is obtained whenever $\alpha_1>\alpha_2=\alpha_3$, while the case of an oblate spheroid corresponds to $\alpha_1<\alpha_2=\alpha_3$.

The elliptic integrals can be evaluated in closed form for spheroids

$$(92)\quad I_0^1(\rho)=\frac{1}{h_3}\begin{cases}\dfrac{1}{2}\ln\left(\dfrac{\rho+h_3}{\rho-h_3}\right), & \alpha_1>\alpha_2\\ \hline \dfrac{1}{i}\tan^{-1}\left(\dfrac{ih_3}{\rho}\right), & \alpha_1<\alpha_2,\end{cases}$$

$$(93)\quad I_1^1(\rho)=\frac{1}{h_3^2}\left(I_0^1(\rho)-\frac{1}{\rho}\right),$$

$$(94)\quad I_1^2(\rho)=I_1^3(\rho)=-\frac{1}{2h_3^2}\left(I_0^1(\rho)-\frac{\rho}{\rho^2-h_3^2}\right),$$

$$I_2^1(\rho)=\frac{9}{4h_3^4}(I_0^1(\rho)-\frac{3\rho}{3\rho^2-h_3^2}), \tag{95}$$

$$I_2^2(\rho)=I_2^5(\rho)=\frac{3}{8h_3^4}(I_0^1(\rho)-\frac{\rho(3\rho^2-5h_3^2)}{3(\rho^2-h_3^2)^2}), \tag{96}$$

$$I_2^3(\rho)=I_2^4(\rho)=-\frac{3}{2h_3^4}(I_0^1(\rho)-\frac{3\rho^2-2h_3^2}{3\rho(\rho^2-h_3^2)}), \tag{97}$$

where

$$\rho=h_3\begin{matrix}\cos h\omega\\ \\ i\sin h\omega\end{matrix}=\begin{cases}\sqrt{\alpha_1^2-\alpha_2^2}\cos h\omega, & \alpha_1>\alpha_2\\ \\ \sqrt{\alpha_2^2-\alpha_1^2}\sin h\omega, & \alpha_1<\alpha_2\end{cases} \tag{98}$$

and (ω, , φ) are the spheroidal coordinates which are related to the Cartesian coordinates (x_1, x_2, x_3) by

$$\begin{aligned} x_1&=\rho\cos\vartheta & ,\quad & \omega\in[0,\ +\ \infty)\\ x_2&=\sqrt{\rho^2-h_3^2}\sin\vartheta\cos\varphi, & & \vartheta\in[0,\ \pi]\\ x_3&=\sqrt{\rho^2-h_3^2}\sin\vartheta\sin\varphi, & & \varphi\in[0,\ 2\pi). \end{aligned} \tag{99}$$

For $\rho=\alpha_1$ we obtain

$$I_0^1=\frac{1}{\alpha_2}\begin{cases}[(\frac{\alpha_1}{\alpha_2})^2-1]^{-1/2} & \cos h^{-1}(\frac{\alpha_1}{\alpha_2}), \quad \alpha_1>\alpha_2\\ \\ [1-(\frac{\alpha_1}{\alpha_2})^2]^{-1/2} & \cos^{-1}(\frac{\alpha_1}{\alpha_2}), \quad \alpha_1>\alpha_2\end{cases} \tag{100}$$

and through (93)—(97) all the other elliptic integrals can be expressed as functions of the ratio $\frac{\alpha_1}{\alpha_2}$, whenever $\rho=\alpha_1$.

Having the values of the elliptic integrals we can substitute them in the corresponding expressions and obtain the results for an oblate or a prolate spheroid, as the case maybe.

The needle-shaped scatterer can be approximated by a prolate spheroid, where $\alpha_1\gg\alpha_2=\alpha_3$. In this case

$$I_0^1\sim\frac{1}{\alpha_2}\frac{\ln 2(\frac{\alpha_1}{\alpha_2})}{(\frac{\alpha_1}{\alpha_2})}, \quad \frac{\alpha_1}{\alpha_2}\to+\infty. \tag{101}$$

In the case, where $\alpha_1 \ll \alpha_2 = \alpha_3$, the oblate spheroid takes the shape of a circular disc and

$$I_0^1 \sim \frac{\pi}{2\alpha_2}, \quad \frac{\alpha_1}{\alpha_2} \to 0+. \tag{102}$$

The sphere is the shape that corresponds to radial symmetry and comes out of the case, where $\alpha_1 = \alpha_2 = \alpha_3 = \alpha$.

The elliptic integrals assume the following values

$$I_0^1(\rho) = \frac{1}{\rho}, \tag{103}$$

$$I_1^n(\rho) = \frac{1}{3\rho^3}, \quad n = 1, 2, 3, \tag{104}$$

$$I_2^n(\rho) = \frac{1}{5\rho^5}, \quad n = 1, 2, 3, 4, 5. \tag{105}$$

We also obtain that $\rho = r$, $\mu = \nu = 0$ and $\Lambda = \Lambda' = \alpha^2$. In order to evaluate the undetermined forms in the various expressions it is enough to approximate the sphere, say by a prolate spheroid setting $\alpha_1 = \alpha(1+\varepsilon)$, $\varepsilon > 0$, $\alpha_2 = \alpha_3 = \alpha$ and obtain the case of a sphere in the limit as $\varepsilon \to 0+$. Obviously, the combination of the physically and the geometrically degenerate cases give us a certain number of special scattering problems which can be solved by our general approach.

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