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Best Onesided Approximations and Mean Approximations by Interpolation Polynomials of Periodic Functions

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0. Notations

We shall consider the functions defined on R^d, d-integer, which are 2π-periodic on every variable. The norm of the element $x=(x_1, x_2, \dots, x_d)\in R^d$ is $\|x\|=\max\{|x_s| : s=1, 2, \dots, d\}$ and the δ-neighbourhood $(\delta>0)$ of the point x is given by

$$(1)\qquad U(\delta, x)=\{y : y\in R^d,\ \|x-y\|\leqq\delta/2\}.$$

With T_n^d we denote the set of all trigonometric polynomials of d variables, which are polynomials of degree n on every variable. Set $\Pi^d=[-\pi, \pi]^d$. The space $L_p(\Pi^d)=L_p$ is equipped with the following norm $(f\in L_p(\Pi^d))$

$$\|f\|_p=\|f\|_{p(\Pi^d)}=[(2\pi)^{-d}\int_{\Pi^d}|f(x)|^p\,dx]^{1/p},\quad 1\leqq p<\infty;$$

$$\|f\|_\infty=\|f\|_{\infty(\Pi^d)}=\sup\{|f(x)| : x\in\Pi^d\}.$$

By $L_\infty(\Pi^d)$ we denote the set of all bounded and measurable functions f with the norm $\|\cdot\|_\infty$ and by $C(\Pi^d)$ — the space of all continuous functions with the same norm $\|\cdot\|_\infty$. For the functions from $L_\infty(\Pi^d)$ we introduce the following (local-global) quasi-norms (see [2]) $(\delta>0, 1\leqq p\leqq\infty)$

$$(2)\qquad \|f\|_{\delta,p}=\|f\|_{\delta,p(\Pi^d)}=\|\ \|f\|_{\infty(U(\delta,.))}\|_{p(\Pi^d)}=\|f_\delta\|_{p(\Pi^d)},$$

where

$$(3)\qquad f_\delta(x)=\sup\{|f(t)| : t\in U(\delta, x)\}.$$

It is easy to see that for fixed $\delta>0$ and $1\leqq p\leqq\infty$ the norm in (2) satisfies all of the norm's axioms. The set of functions $L_\infty(\Pi^d)$ equipped with the norm (2) we denote by $L_{\delta,p}(\Pi^d)=L_{\delta,p}$.

Let $a=(a_1, a_2, \dots, a_d)$ be a multiindex. We denote by $D^a=D^{a_1}\dots D^{a_d}$ the differential operator in R^d (see [11], p. 140), where $D^{a_s}=\partial^{a_s}/\partial x_s^{a_s}$, $s=1, \dots, d$. For given integers m and d, $m \geqq 0$, we set $N^d_m = \{0, 1, \ldots, m\}^d$ — the set of all different multiindices of dimension d with components which can have the values from 0 to m.

For a given point $x_0 \in \mathbb{R}^d$ we consider the following equispaced set of points $\{x_j\}_{j \in N^d_{2n}}$, where

$$x_j = (x_{j_1}, x_{j_2}, \ldots, x_{j_d}) = x_0 + 2\pi j/(2n+1) \in \mathbb{R}^d, \quad j = (j_1, j_2, \ldots, j_d) \in N^d_{2n}.$$

For the functions f in $L_\infty(\Pi^d)$ we shall use also the following net norm $(1 \leqq p \leqq \infty)$

$$\| f \| e^p_{2n+1} = [(2n+1)^{-d} \sum_{j \in N^d_{2n}} |f(x_j)|^p]^{1/p}.$$

The best approximation of a function $f \in L_p(\Pi^d)$ with trigonometric polynomials from T^d_n in the metric of the space L_p is given by

$$E_n(f)_p = \inf \{ \| f - T \|_{p(\Pi^d)} : T \in T^d_n \},$$

and the best approximation of a function $f \in L_{\delta,p}(\Pi^d)$ with polynomials from T^d_n in metric (2) is given by

$$E_n(f)_{\delta,p} = \inf \{ \| f - T \|_{\delta,p(\Pi^d)} : T \in T^d_n \}.$$

The best onesided approximation ([13], p. 242) of a function $f \in L_\infty(\Pi^d)$ with polynomials from T^d_n in the metrics of the spaces L_p or $L_{\delta,p}$ are respectively given by

$$\tilde{E}_n(f)_p = \inf \{ \| T^+ - T^- \|_{p(\Pi^d)} : T^\pm \in T^d_n,\ T^-(x) \leqq f(x) \leqq T^+(x),\ x \in \mathbb{R}^d \},$$

$$\tilde{E}_n(f)_{\delta,p} = \inf \{ \| T^+ - T^- \|_{\delta,p(\Pi^d)} : T^\pm \in T^d_n,\ T^-(x) \leqq f(x) \leqq T^+(x),\ x \in \mathbb{R}^d \}.$$

For characterization of the structural properties for a given function f from L_p or L_∞ we shall use the following moduli (see [11], p. 145, [13], p. 18, [12] and [13])

$$\omega_k(f, \delta)_p = \sup \{ \| \Delta^k_h f(\cdot) \|_p : |h| \leqq \delta \} \tag{4}$$

$$\tau_k(f, \delta)_p = \| \omega(f, \cdot, \delta) \|_p, \tag{5}$$

where

$$\Delta^k_h f(x) = \sum_{i=0}^{n} (-1)^{k+i} \binom{k}{i} f(x+ih), \quad x, h \in \mathbb{R}^d,$$

$$\omega_k(f, x, \delta) = \sup \{ |\Delta^k_h f(t)| : t, t+kh \in U(k\delta, x) \}, \quad x \in \mathbb{R}^d.$$

$B^0_{p,q}(\Pi^d)$ denotes the Besov space equipped with the norm generated by moduli (4) or the equivalent norm generated by $E_n(f)_p$ (see [1], p. 254 or [11], pp. 159, 212).

In the paper d, n, k (integers) and p, $1 \leq p \leq \infty$, are fixed numbers which will be used for the dimension of the space R^d, for the degree of the approximating polynomials, for the order of moduli (4) or (5) and for the metric of the spaces L_p. By c we denote positive numbers which may differ at each occurrence. If c depends from some parameters, we indicate the dependence using indices.

The unique trigonometric polynomial from T_n^d interpolating a given function $f \in L_\infty(\Pi^d)$ at the points $\{x_j\}_{j \in N_{2n}^d}$ is denoted by $I_n(f)$.

If $t=(t_1, t_2, \ldots, t_d) \in \mathrm{R}^d$ and $u \in \mathrm{R}^1$ then we denote by $D_n(u)=D_{n,1}(u) = \dfrac{\sin(n+1/2)u}{2\sin(u/2)}$ — the Dirichle kernel, by

$$\Phi_n(u)=\Phi_{n,1}(u)=\frac{\sin^2 nu/2}{\sin^2 u/2} \cdot \sin^2 \pi/2n \tag{6}$$

the properly normalized Fejer kernel, by $D_{n,d}(t)=\Pi_{s=1}^d D_n(t_s)$ and by $\Phi_{n,d}(t)=\Pi_{s=1}^d \Phi_n(t_s)$ — their corresponding d-dimensional analogs. Note that

$$\Phi_{n,d} \in T_{n-1}^d. \tag{7}$$

The interpolating polynomial $I_n(f)$ has the representation (see [8], p. 10)

$$I_n(f,x)=(2/(2n+1))^{-d} \sum_{j \in N_{2n}^d} f(x_j) D_{n,d}(x-x_j).$$

1. Main results

Inequalities between the quantities $E_n(f)_p$, $E_n(f)_{2\pi/n,p}$, $\tilde{E}_n(f)_p$, $\| I_n(f)-f \|_p$ and $\| I_n(f)-f \|_{2\pi/n,p}$ (note the usage of the defined above norm (2) with $\delta=2\pi/n$) are obtained in this article. These inequalities and Theorems A and B given below will imply characterizations of the orders of convergence of these quantities via moduli of type (4) or (5) of function f.

Theorem A (see e. g. [11], pp. 189, 195). *For every $f \in L_p(\Pi^d)$ we have*

$$E_n(f)_p \leq c_{k,d}\, \omega_k(f, 1/n)_p;$$

$$\omega_k(f, 1/n)_p \leq c_{k,d}\, n^{-k} \sum_{\nu=0}^{n} (\nu+1)^{k-1} E_\nu(f)_p.$$

Theorem B ([13], pp. 242-257, [5], [3]). *For every $f \in L_\infty \in (\Pi^d)$ we have*

$$\tilde{E}_n(f)_p \leq c_{k,d}\, \tau_k(f, 1/n)_p;$$

$$\tau_k(f, 1/n)_p \leq c_{k,d}\, n^{-k} \sum_{\nu=0}^{n} (\nu+1)^{k-1} \tilde{E}_\nu(f)_p.$$

In the paper we prove

Theorem 1. *For every* $f \in L_\infty(\Pi^d)$ *we have*

$$\tilde{E}_n(f)_p \leqq c_d E_n(f)_{2\pi/n,p} \leqq c_d \tilde{E}_n(f)_p.$$

Theorem 2. *If* $f \in B^{d/p}_{p,1}(\Pi^d)$ *then* $f = F$ a.e. *on* Π^d *for some* $F \in C(\Pi^d)$ *and*

$$\tilde{E}_n(F)_p \leqq c_d n^{-d/p} \sum_{\nu=n}^{\infty} \nu^{d/p-1} E_\nu(f)_p.$$

Theorem 3. *If* $f \in L_\infty(\Pi^d)$ *and* $1 < p < \infty$ *then*

$$\tilde{E}_n(f)_p \leqq c_d \| I_n(f) - f \|_{2\pi/n,p} \leqq c_{d,p} \tilde{E}_n(f)_p.$$

Theorem 4. *If* $f \in L_\infty(\Pi^d)$ *then*

$$\| I_n(f) - f \|_1 \leqq \| I_n(f) - f \|_{2\pi/n,1} \leqq c_d \tilde{E}_n(f)_1 \log^d(1+n).$$

Theorem 5. *If* $f \in L_\infty(\Pi^d)$ *and* $1 \leqq p \leqq q \leqq \infty$ *then*

$$\tilde{E}_n(f)_q \leqq c_d n^{d/p-d/q} \tilde{E}_n(f)_p.$$

Theorems 1-3 and Theorems A and B immediately imply

Corollary 1 (a similar result is proved in [4]). *Let* $f \in B^{d/p}_{p,1}(\Pi^d)$ *and let* $F \in C(\Pi^d)$ *be such that* $f = F$ a.e. *on* Π^d. *Then*

$$\tilde{E}_n(F)_p \leqq c_{d,k} n^{-d/p} \int_0^{1/n} t^{d/p-1} \omega_k(f,t)_p \, dt$$

and

$$\tilde{E}_n(F)_p = 0(n^{-\rho}) \Leftrightarrow E_n(f)_p = 0(n^{-\rho}) \Leftrightarrow \omega_k(f,\delta)_p = 0(\delta^\rho) \Leftrightarrow \tau_k(F,\delta)_p = 0(\delta^\rho),$$
$$d/p < \rho < k.$$

Corollary 2. *If* $f \in L_\infty(\Pi^d)$, *then for every* $0 < \rho < k$ *we have*

$$\tilde{E}_n(f)_p = 0(n^{-\rho}) \Leftrightarrow E_n(f)_{2\pi/n,p} = 0(n^{-\rho}) \Leftrightarrow \tau_k(f,\delta)_p = 0(\delta^\rho).$$

Corollary 3. *If* $f \in L_\infty(\Pi^d)$ *and* $1 < p < \infty$, *then*

$$\| I_n(f) - f \|_p \leqq c_{d,p} \tilde{E}_n(f)_p.$$

2. Auxiliary results

Let $A_0 \supset A_1$ be two quasinormed spaces. Peetre K-functional (see e.g. [6], p. 54) for A_0 and A_1 is defined by ($f \in A_0$, real $t > 0$)

$$K(f,t;A_0,A_1) = \inf\{ \|f-g\|_{A_0} + t \,.\, \|g\|_{A_1} : g \in A_1 \}.$$

The following lemma asserts that spaces $L_{\delta,p}$ for a fixed δ possess the interpolating property, i.e. for the spaces $L_{\delta,p}(\delta - \text{fixed})$ an analog of Riesz — Thorin theorem (see e.g. [6], p. 10) holds.

Lemma 1. *Let $f \in L_\infty(\Pi^d)$ and $\delta > 0$. Then*

$$(8) \qquad K(f,t\,;L_{\delta,p},L_{\delta,\infty}) = K(f_\delta,t\,;L_p,L_\infty) \sim \Big[\int_0^{t^p} (f_\delta)^*(s)^p\,ds\Big]^{1/p},$$

with equivalence constants depending only on d and p, where $f_\delta(x) = \sup\{|f(y)| : y \in U(\delta,x)\}$ (cf. (3)), and g^* denotes the non-increasing rearrangement of the function g.

Proof. The equivalence in (8) is well-known (see e.g. [8], p. 142).
In order to prove the equality in (8) we show first that

$$(9) \qquad K(f,t\,;L_{\delta,p},L_{\delta,\infty}) \leqq K(f_\delta,t\,;L_p,L_\infty).$$

Let (see (3))
$f_\delta(x) = g_1(x) + g_2(x)$, $x \in \Pi^d$, where $g_1 \in L_p(\Pi^d)$, $g_2 \in L_\infty(\Pi^d)$. Denote $E_f = \{y : y \in \Pi^d, |f(y)| \geqq \|g_2\|_{\infty(\Pi^d)}\}$ and

$$f_1(x) = \begin{cases} f(x) - \|g_2\|_{\infty(\Pi^d)} f(x)/|f(x)| & \text{for } x \in E_f, \\ 0 & \text{for } x \in \Pi^d \setminus E_f, \end{cases}$$

$f_2(x) = f(x) - f_1(x)$, $x \in \Pi^d$. Functions f_1 and f_2 are continued 2π-periodically to R^d. Let us denote the δ-neighbourhood of set E_f by $E_{f,\delta} = \{z : z \in \Pi^d, z \in U(\delta,y), y \in E_f\}$. From definition (3) of function f_δ and from $|f(y)| \geqq \|g_2\|_{\infty(\Pi^d)}$ for $y \in E_f$ we have

$$(f_1)_\delta(x) = f_\delta(x) - \|g_2\|_{\infty(\Pi^d)} \text{ for } x \in E_{f,\delta},$$

$$(f_1)_\delta(x) = 0 \text{ for } x \in \Pi^d \setminus E_{f,\delta},$$

$$\|f_2\|_{\infty(\Pi^d)} = \|g_2\|_{\infty(\Pi^d)}.$$

Therefore ($f = f_1 + f_2$ and $f_\delta = g_1 + g_2$)

$$\begin{aligned} K(f,t\,;L_{\delta,p},L_{\delta,\infty}) &\leqq \|f_1\|_{\delta,p(\Pi^d)} + t\,\|f_2\|_{\infty(\Pi^d)} = \|f_1\|_{\delta,p(\Pi^d)} + t\,\|g_2\|_{\infty(\Pi^d)} \\ &= \Big((2\pi)^{-d} \int_{E_{f,\delta}} (f_1)_\delta(x)^p\,dx\Big)^{1/p} + t\,\|g_2\|_{\infty(\Pi^d)} \\ &= \Big((2\pi)^{-d} \int_{E_{f,\delta}} (f_\delta(x) - \|g_2\|_{\infty(\Pi^d)})^p\,dx\Big)^{1/p} + t\,\|g_2\|_{\infty(\Pi^d)} \\ &\leqq \Big((2\pi)^{-d} \int_{E_{f,\delta}} (f_\delta(x) - g_2(x))^p\,dx\Big)^{1/p} + t\,\|g_2\|_{\infty(\Pi^d)} \\ &= \Big((2\pi)^{-d} \int_{E_{f,\delta}} g_1(x)^p\,dx\Big)^{1/p} + t\,\|g_2\|_{\infty(\Pi^d)} \\ &\leqq \|g_1\|_{p(\Pi^d)} + t\,\|g_2\|_{\infty(\Pi^d)}. \end{aligned}$$

This proves (9) because of the arbitrariness of the decomposition $f_\delta=g_1+g_2$. Now we prove the inequality opposite to (9), i.e.

$$K(f_\delta,t;L_p,L_\infty)\leqq K(f,t;L_{\delta,p},L_{\delta,\infty}). \tag{10}$$

Let $f(x)=f_1(x)+f_2(x)$, $x\in\Pi^d$, where $f_1\in L_{\delta,p}(\Pi^d)$, $f_2\in L_\infty(\Pi^d)$. To f we put into correspondence the function $f_\delta(x)$ using (3) and define $E=\{y:y\in\Pi^d, f_\delta(y)\geqq\|f_2\|_{\infty(\Pi^d)}\}$,

$$g_1(x)=\begin{cases} f_\delta(x)-\|f_2\|_{\infty(\Pi^d)} & \text{for } x\in E,\\ 0 & \text{for } x\in\Pi^d\setminus E,\end{cases}$$

$g_2(x)=f_\delta(x)-g_1(x)$, $x\in\Pi^d$. Obviously $\|g_2\|_{\infty(\Pi^d)}=\|f_2\|_{\infty(\Pi^d)}$. Using $|\|f\|-\|g\||\leqq\|f-g\|$ and $g_\delta(x)\leqq\|g\|_{\infty(\Pi^d)}$, $x\in\Pi^d$, we get $(f_\delta=g_1+g_2,\ f=f_1+f_2)$

$$\begin{aligned}K(f_\delta,t;L_p,L_\infty)&\leqq\|g_1\|_{p(\Pi^d)}+t\|g_2\|_{\infty(\Pi^d)}\\ &=((2\pi)^{-d}\int_E g_1(x)^p\,dx)^{1/p}+t\|f_2\|_{\infty(\Pi^d)}\\ &=((2\pi)^{-d}\int_E (f_\delta(x)-\|f_2\|_{\infty(\Pi^d)})^p\,dx)^{1/p}+t\|f_2\|_{\infty(\Pi^d)}\\ &\leqq((2\pi)^{-d}\int_E (f_\delta(x)-(f_2)_\delta(x))^p\,dx)^{1/p}+t\|f_2\|_{\infty(\Pi^d)}\\ &\leqq((2\pi)^{-d}\int_E ((f-f_2)_\delta(x))^p\,dx)^{1/p}+t\|f_2\|_{\infty(\Pi^d)}\\ &=((2\pi)^{-d}\int_E ((f_1)_\delta(x))^p\,dx)^{1/p}+t\|f_2\|_{\infty(\Pi^d)}\\ &\leqq\|f_1\|_{\delta,p(\Pi^d)}+t\|f_2\|_{\infty(\Pi^d)}.\end{aligned}$$

This proves (10) because of the arbitrariness of the decomposition $f=f_1+f_2$. Now (9) and (10) imply (8).

Applying the real method for interpolation to the couple $(L_{\delta,p_0},L_{\delta,p_1})$, $1\leqq p_0<p_1\leqq\infty$, and using the reiteration theorem for the real method for interpolation (see [6], p. 66, 144) and the previous lemma, we get that an analog of the Riesz – Thorin theorems (see e.g. [6], p. 10) holds for the spaces $L_{\delta,p}$ with different p's.

Lemma 2. *For every $f\in L_\infty(\Pi^d)$ we have*

1° $\|f\|_{p(\Pi^d)}\leqq\|f\|_{\delta,p(\Pi^d)}\leqq\|f\|_{\infty(\Pi^d)}=\|f\|_{\delta,\infty(\Pi^d)}$;

2° $\|f\|_{\delta,p(\Pi^d)}\leqq\|f\|_{\delta',p'(\Pi^d)}$, $\delta\leqq\delta'$, $p\leqq p'$;

3° $\|f(\cdot+h)\|_{\delta,p(\Pi^d)}=\|f(\cdot)\|_{\delta,p(\Pi^d)}$;

4° $\|f\|_{m\delta,p(\Pi^d)}\leqq m^{d/p}\|f\|_{\delta,p(\Pi^d)}$ *(natural m)*;

5° $\|f\|_{e^p_{2n+1}}\leqq\|f\|_{\frac{2\pi}{2n+1},p}$ $(1\leqq p<\infty)$.

Proof. Properties 1°-3° immediately follow from the definitions of the norms included. Property 4° is obvious (as an equality) for $p=\infty$. Therefore, in view of the interpolation property of spaces $L_{\delta,p}$, the validity of 4° for every p will follow from its validity for $p=1$. So we shall prove

$$\|f\|_{m\delta,1(\Pi^d)}\leqq m^d\,\|f\|_{\delta,1(\Pi^d)}.$$

Denote by $e=(1,\ 1,\dots,1)$ the unitary vector in R^d. Taking into account property 3° of the same lemma, we have

$$\|f\|_{m\delta,1(\Pi^d)}=(2\pi)^{-d}\int_{\Pi^d}\sup\{|f(x+t)|:t\in U(m\delta,0)\}\,dx$$

$$\leqq(2\pi)^{-d}\int_{\Pi^d}\sum_{j\in N^d_{m-1}}\sup\{|f(x-(m-1)\delta e/2+\delta j+t)|:t\in \dot U(\delta,0)\}\,dx$$

$$=\sum_{j\in N^d_{m-1}}(2\pi)^{-d}\int_{\Pi^d}\sup\{|f(x-(m-1)\delta e/2+\delta j+t)|:t\in U(\delta,0)\}\,dx=m^d\,\|f\|_{\delta,1(\Pi^d)}.$$

In order to prove property 5° we note that $x_j\in U(2\pi/(2n+1),x)$ whenever $x\in U(2\pi/(2n+1),x_j)$. Therefore, using that mes $U(2\pi/(2n+1),x)=(2\pi/(2n+1))^d$, $x\in\mathsf{R}^d$, we get

$$\|f\|_{\frac{2\pi}{2n+1},p}=[(2\pi)^{-d}\int_{\Pi^d}\sup\{|f(x+t)|:t\in U(\frac{2\pi}{2n+1},0)\}^p\,dx]^{1/p}$$

$$=[(2\pi)^{-d}\sum_{j\in N^d_{2n}}\int_{U(2\pi/(2n+1),x_j)}\sup\{|f(x+t)|:t\in U(\frac{2\pi}{2n+1},0)\}^p\,dx]^{1/p}$$

$$=[(2\pi)^{-d}\sum_{j\in N^d_{2n}}\int_{U(2\pi/(2n+1),0)}\sup\{|f(x_j+x+t)|:t\in U(\frac{2\pi}{2n+1},0)\}^p\,dx]^{1/p}$$

$$\geqq[(2\pi)^{-d}\sum_{j\in N^d_{2n}}|f(x_j)|^p[\frac{2\pi}{2n+1}]^d]^{1/p}=[(2n+1)^{-d}\sum_{j\in N^d_{2n}}|f(x_j)|^p]^{1/p}=\|f\|_{e^p_{2n+1}}.$$

The lemma is proved.

In the sequel an important role will play

Lemma 3. *If $T\in T^d_n$ then*

$$\|T\|_{\delta,p(\Pi^d)}\leqq(1+\delta n)^{d/p}\,\|T\|_{p(\Pi^d)}.$$

Proof. For $p=\infty$ this statement is obvious. We shall prove it for $p=1$. For every $x\in\mathsf{R}^d$ we denote by ξ_x this point from $U(\delta,x)$ for which $\sup\{|T(y)|:y\in U(\delta,x)\}=|T(\xi_x)|$. Then we have

$$| \| T \|_{\delta,1} - \| T \|_1 | = (2\pi)^{-d} \int_{\Pi^d} (\sup\{|T(y)| : y \in U(\delta, x)\} - |T(x)|)\, dx$$

$$= (2\pi)^{-d} \int_{\Pi^d} (|T(\xi_x)| - |T(x)|)\, dx \leqq (2\pi)^{-d} \int_{\Pi^d} |T(\xi_x) - T(x)|\, dx$$

$$\leqq (2\pi)^{-d} \int_{\Pi^d} \sum_{\substack{|\alpha| \geqq 1 \\ \alpha_s = 0,1}} \int_{U(\delta,0)^\alpha} |D^\alpha T(x + v^\alpha)|\, dv^\alpha\, dx$$

$$= (2\pi)^{-d} \sum_{\substack{|\alpha| \geqq 1 \\ \alpha_s = 0,1}} \int_{\Pi^d} |D^\alpha T(x)|\, dx\, \delta^{|\alpha|}$$

$$= \sum_{\substack{|\alpha| \geqq 1 \\ \alpha_s = 0,1}} \delta^{|\alpha|} \| D^\alpha T \|_{1(\Pi^d)} \leqq \sum_{\substack{|\alpha| \geqq 1 \\ \alpha_s = 0,1}} (\delta n)^{|\alpha|} \| T \|_{1(\Pi^d)} = ((1 + \delta n)^d - 1) \| T \|_{1(\Pi^d)},$$

where for the difference $T(\xi_x) - T(x)$ we use the representation from [7], p. 114 (107) or a similar representation from [12] (the integration goes on these variables from $v = (v_1, v_2, \ldots, v_d)$ for which the corresponding components of the multiindex a are equal to 1). This proves the lemma for $p = 1$, and the interpolating property of the spaces $L_{\delta,p}$ implies its validity for any p, $1 \leqq p \leqq \infty$.

Property 1° in Lemma 2 and the last lemma imply

Corollary 4. *Let* $T \in T_n^d$, $\delta n \leqq \mu = \text{const}$. *Then*

$$\| T \|_{p(\Pi^d)} \leqq \| T \|_{\delta,p(\Pi^d)} \leqq c_{d,\mu} \| T \|_{p(\Pi^d)}.$$

Using Bernstein inequality for trigonometric polynomials (see e. g. [11], p. 98) from corollary 4, we get

Corollary 5. *Let* $T \in T_n^d$, $\delta n \leqq \mu = \text{const}$ *and let* α *be a fixed multiindex. Then*

$$\| D^\alpha T \|_{\delta,p(\Pi^d)} \leqq c_{d,\mu}\, n^{|\alpha|} \| T \|_{\delta,p(\Pi^d)}.$$

In the sequel we make use of the following properties of Fejer kernels (6).

Lemma 4 (see [3]). *We have*

(11) $$\Phi_{n,d}(t) \geqq 0 \text{ for } t \in \mathrm{R}^d;$$

(12) $$\Phi_{n,d}(t) \geqq 1 \text{ for } t \in U(2\pi/n, 0);$$

(13) $$\| \Phi_{n,d} \|_{1(\Pi^d)} = (n \sin^2 \pi/2n)^d \leqq (\pi/2)^{2d}/n^d = c_d/n^d.$$

3. Proofs of the theorems

Proof of Theorem 1. Let $T^+, T^- \in T_n^d$ be such that $T^+(x) \leqq f(x) \leqq T^-(x)$ for every $x \in \mathrm{R}^d$ and $\| T^+ - T^- \|_p = \tilde{E}_n(f)_p$. Then using Corollary 4 with $\delta = 2\pi/n$ and $\mu = 2\pi$, we get

(14) $$E_n(f)_{2\pi/n,p} \leqq \tilde{E}_n(f)_{2\pi/n,p} \leqq \| T^+ - T^- \|_{2\pi/n,p} \leqq c_d \| T^+ - T^- \|_p = c_d \tilde{E}_n(f)_p.$$

Now we shall prove the inequality opposite to (14), i.e.

$$\tilde{E}_n(f)_p \leqq c_d E_n(f)_{2\pi/n,p}. \tag{15}$$

Let $Q_n \in T_n^d$ be such that $\|f-Q_n\|_{2\pi/n,p} = E_n(f)_{2\pi/n,p}$. Set (cf. [3])

$$Q_n^{\pm}(f;x) = Q_n(x) \pm (n/2\pi)^d \int_{\Pi^d} \Phi_{n,d}(x-t) \sup\{|f(y)-Q_n(y)| : y \in U(2\pi/n, t)\}\, dt.$$

Obviously (see (7)) $Q_n^{\pm} \in T_n^d$. We shall show that for every $x \in \mathrm{R}^d$ the inequalities

$$Q_n^-(f;x) \leqq f(x) \leqq Q_n^+(f;x) \tag{16}$$

hold. Indeed, taking into account (11), (12) and mes $\{U(2\pi/n, 0)\} = (2\pi/n)^d$, we get

$$Q_n^+(f;x) = Q_n(x) + (n/2\pi)^d \int_{\Pi^d} \Phi_{n,d}(v) \sup\{|f(y)-Q_n(y)| : y \in U(2\pi/n, x-v)\}\, dv$$

$$\geqq Q_n(x) + (n/2\pi)^d \int_{U(2\pi/n,0)} \Phi_{n,d}(v) \sup\{|f(y)-Q_n(y)| : y \in U(2\pi/n, x-v)\}\, dv$$

$$\geqq Q_n(x) + (n/2\pi)^d |f(x)-Q_n(x)| (2\pi/n)^d \geqq Q_n(x) + f(x) - Q_n(x) = f(x).$$

The other inequality in (16), $Q_n^-(f;x) \leqq f(x)$, can be proved in a similar way.
Using the generalized Minkowski inequality and inequality (13), we get

$$\|Q_n^+ - Q_n^-\|_{p(\Pi^d)} = 2\,\|(n/2\pi)^d \int_{\Pi^d} \Phi_{n,d}(x-t) \sup\{|f(y)-Q_n(y)| : y \in U(2\pi/n, t)\}\, dt\,\|_p$$

$$\leqq 2(n/2\pi)^d \int_{\Pi^d} \Phi_{n,d}(v) \Big[(2\pi)^{-d} \int_{\Pi^d} \sup\{|f(y)-Q_n(y)| : y \in U(2\pi/n, x-v)\}^p\, dx\Big]^{1/p} dv$$

$$= 2n^d \|\Phi_{n,d}\|_1 \|f-Q_n\|_{2\pi/n,p} = c_d E_n(f)_{2\pi/n,p}.$$

Thus, in view of (16), we have

$$\tilde{E}_n(f)_p \leqq \|Q_n^+ - Q_n^-\|_p \leqq c_d E_n(f)_{2\pi/n,p}$$

and this proves inequality (15) and completes the proof of Theorem 1.

P r o o f o f T h e o r e m 2. Let $Q_\nu \in T^d$ be such that $E_\nu(f)_p = \|f - Q_\nu\|_{p(\Pi^d)}$, $\nu = 0, 1, \dots$. Because of $f \in B_{p,1}^{d/p}$ there is $F \in C(\Pi^d)$ such that $f = F$ a.e. on R^d (see [10], [9], [4]). As

$$\sum_{\nu=1}^{N} (Q_{n2^\nu} - Q_{n2^{\nu-1}}) = Q_{n2^N} - Q_n \quad \text{and} \quad \|F - Q_{n2^N}\|_\infty \to 0\ (N \to \infty),$$

we have

$$F(x) - Q_n(x) = \sum_{\nu=1}^{\infty} (Q_{n2^\nu}(x) - Q_{n2^{\nu-1}}(x))$$

for every $x \in \mathbb{R}^d$. Then, using Theorem 1 and Lemma 3, we get

$$\tilde{E}_n(F)_p \leqq c_d E_n(F)_{2\pi/n,p} = c_d E_n(F - Q_n)_{2\pi/n,p} \leqq c_d \| F - Q_n \|_{2\pi/n,p}$$

$$\leqq c_d \sum_{\nu=1}^{\infty} \| Q_{n2^\nu} - Q_{n2^{\nu-1}} \|_{2\pi/n,p}$$

$$\leqq c_d \sum_{\nu=1}^{\infty} (1 + 2\pi . 2^\nu)^{d/p} \| Q_{n2^\nu} - Q_{n2^{\nu-1}} \|_p$$

$$\leqq c_d n^{-d/p} \sum_{\nu=1}^{\infty} (n2^\nu)^{d/p} (\| Q_{n2^\nu} - f \|_p + \| f - Q_{n2^{\nu-1}} \|_p)$$

$$= c_d n^{-d/p} \sum_{\nu=1}^{\infty} (n2^\nu)^{d/p} (E_{n2^\nu}(f)_p + E_{n2^{\nu-1}}(f)_p) \leqq c_d n^{-d/p} \sum_{j=n}^{\infty} j^{d/p-1} E_j(f)_p.$$

Theorem 2 is proved.

Proof of Theorem 3. In view of Theorem 1 one only has to prove the inequality $(1 < p < \infty)$

$$\| f - I_n(f) \|_{2\pi/n,p} \leqq c_{p,d} \tilde{E}_n(f)_p. \tag{17}$$

It will be a simple consequence of the inequality

$$\| I_n(f) \|_{2\pi/n,p} \leqq c_{p,d} \| f \|_{2\pi/n,p} (1 < p < \infty). \tag{18}$$

In order to establish (18) we make use of the following inequality due to Marcinkiewicz (see e. g. [8], p. 46, where a proof for $d=1$ is given; the case $d>1$ can be similarly obtained)

$$\| T \|_p \leqq c_{p,d} \| T \|_{e^p_{2n+1}} \quad \text{for } T \in T^d_n \text{ and } 1 < p < \infty, \tag{19}$$

of the equivalence of the norms $\| T \|_p$ and $\| T \|_{2\pi/n,p}$ for $T \in T^d_n$ (see Corollary 4) and of the interpolation conditions $I_n(f, x_j) = f(x_j)$ for $j \in N^d_{2n}$. Indeed, using (19) for $T = I_n(f)$ and property 5° in Lemma 2, we get

$$\| I_n(f) \|_{2\pi/n,p} \leqq c_d \| I_n(f) \|_{p(\Pi^d)} \leqq c_{d,p} \| I_n(f) \|_{e^p_{2n+1}} = c_{d,p} \| f \|_{e^p_{2n+1}}$$

$$\leqq c_{d,p} \| f \|_{2\pi/(2n+1),p} \leqq c_{d,p} \| f \|_{2\pi/n,p}.$$

This proves (18). In order to obtain inequality (17) we consider the polynomial $Q_n \in T^d_n$ for which $\| f - Q_n \|_{2\pi/n,p} = E_n(f)_{2\pi/n,p}$. Then

$$\| f - I_n(f) \|_{2\pi/n,p} \leqq \| f - Q_n \|_{2\pi/n,p} + \| I_n(f - Q_n) \|_{2\pi/n,p} \leqq E_n(f)_{2\pi/n,p} + c_{d,p} \| f - Q_n \|_{2\pi/n,p}$$

$$\leqq c_{d,p} E_n(f)_{2\pi/n,p} \leqq c_{p,d} \tilde{E}_n(f)_p.$$

Theorem 3 is proved.

Proof of Theorem 4. Because of $\| D_{n,1} \|_1 = 0(\log(1+n))$, taking into account properties 1° and 5° from Lemma 2 and Corollary 4, we have

$$\| I_n(f) \|_{2\pi/n,1} \leqq c_d \| I_n(f) \|_{1(\Pi^d)} \leqq c_d \| D_{n,d} \|_1 \| f \|_{e^1_{2n+1}} = c_d \| D_{n,1} \|_1^d \| f \|_{e^1_{2n+1}}$$
$$\leqq c_d \log^d(1+n) \| f \|_{2\pi/(2n+1),1} \leqq c_d \log^d(1+n) \| f \|_{2\pi/n,1}.$$

Let polynomial $Q_n \in T_n^d$ be such that $\| f - Q_n \|_{2\pi/n,1} = E_n(f)_{2\pi/n,1}$. Then

$$\| f - I_n(f) \|_{2\pi/n,1} \leqq \| f - Q_n \|_{2\pi/n,1} + \| I_n(f - Q_n) \|_{2\pi/n,1}$$
$$\leqq E_n(f)_{2\pi/n,1} + c_d \log^d(1+n) \| f - Q_n \|_{2\pi/n,1}$$
$$\leqq c_d \log^d(1+n) E_n(f)_{2\pi/n,1} \leqq c_p \log^d(1+n) \tilde{E}_n(f)_1.$$

This proves Theorem 4.

Proof of Theorem 5. Let $Q_n^{\pm} \in T_n^d$, $Q_n^-(x) \leqq f(x) \leqq Q_n^+(x)$ for $x \in \mathbb{R}^d$ be such that $\| Q_n^+ - Q_n^- \|_p = \tilde{E}_n(f)_p$. Then, using Nikol'skii inequality (see [11], p. 132), we get

$$\tilde{E}_n(f)_q \leqq \| Q_n^+ - Q_n^- \|_q \leqq c_d n^{d/p - d/q} \| Q_n^+ - Q_n^- \|_p = c_d n^{d/p - d/q} \tilde{E}_n(f)_p.$$

This proves the theorem.

4. Notes and remarks

4.1. Theorem 4 and Corollary 3 in the case of interpolating polynomials $I_n(f)$ are analogs of the well-known inequalities for the L_p-deviations of the partial sums of Fourier series of a given function f from the function f itself with the natural replacement of $E_n(f)_p$ with $\tilde{E}_n(f)_p$.

4.2. In view of properties 2° and 4° Lemma 2 the norm $\| \cdot \|_{2\pi/n,p}$ is equivalent to the norms $\| \cdot \|_{\mu/n,p}$ with constant of equivalence depending only on μ and d. In particular one can set $\mu = 1$. Thus, the statements of Theorems 1, 3 and 4 remain valid if one replaces in them $\| \cdot \|_{2\pi/n,p}$ with $\| \cdot \|_{1/n,p} (1 \leqq p \leqq \infty)$.

4.3. Norms of the type of $\| \cdot \|_{\delta,p}$ have been considered in [2] where in another way (using the retract theorem) the interpolating property of the spaces $L_{\delta,p}$ for a fixed δ has been proved.

4.4. An analysis of the proof of Lemma 1 shows that its statement remains valid if one consider non-periodic functions and the δ-neighbourhood $U(\delta, x)$ of a point x is defined in a way different from this one in (1). Thus, the interpolating property of spaces of the type $L_{\delta,p}$ (δ fixed) takes place in more general situations, for example if one consider $U^*(\delta, x) = \{ y : y \in [0, 1],\ |x - y| \leqq \delta \sqrt{x(1-x)} + \delta^2 \}$, $x \in [0, 1]$, instead of $U(\delta, x)$.

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