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Continuity Modulo Sets of Measure Zero

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It is the purpose of this note to give a solution for the following problem. Given a measurable function f and a measure μ, how do you tell if there is a continuous function g such that $f=g$ a.e. (μ), and how do you find g if it exists? For example, on the real line with Lebesgue measure, the characteristic function of the set of rational numbers is nowhere continuous, but is equal a.e. to the (continuous) constant 1 function. On the other hand, the characteristic function of the set of positive real numbers is continuous everywhere except at one point, but cannot be made everywhere continuous by changing it on a set of measure zero. We will show that the latter example is prototypical, in the sense that if a function f cannot be made everywhere continuous by changing it on a set of measure zero, then there is a point at which the function cannot be made continuous by changing it on a set of measure zero.

The solution should not depend on the measure, but only on the sets of measure zero. We therefore suppose that X is a Hausdorff topological space and that $\mathfrak{M}$ is a σ-algebra of subsets of X that contains the Borel sets. As for the sets of measure zero, we suppose that ζ is a hereditary σ-subring of $\mathfrak{M}$ (i.e., ζ is closed under countable unions, and $A\subset B$ and $B\in\zeta$ implies that $A\in\zeta$). To avoid topological annoyances we also assume that ζ contains no nonempty open subset of X. We use the familiar notation "a.e. (ζ)" to mean "except on a set in ζ".

We also need to assume some regularity condition on the suppressed measure μ. Recall that a measure μ is *inner regular* if $\mu(E)$ is the supremum of $\{\mu(K): K\subset E, K \text{ is compact}\}$. This is equivalent to saying that every set with positive measure contains a compact set of positive measure. We require a much weaker condition, one that replaces compact sets with Lindelöf sets. (Recall that a set is Lindelöf if every open cover has a countable subcover.) We say that ζ is *Lindelöf-regular* if, for each $A\in\mathfrak{M}\backslash\zeta$, there is a Lindelöf subset K of A in $\mathfrak{M}\backslash\zeta$.

Throughout, Y is a separable metric space and $f: X\to Y$ is Borel measurable, i.e. $f^{-1}(B)\in\mathfrak{M}$ for every Borel subset B of Y. The key idea is that if there is a continuous $g: X\to Y$ such that $\{x: f(x)\neq g(x)\}$ is in ζ, then $f(a)=g(a)$ whenever $\{a\}$ is not in ζ, and, "modulo sets of measure zero", $\lim_{x\to a}f(x)=g(a)$ for every a in X. We define a relation $\zeta\text{-}\lim_{x\to a}f(x)=L$ for a in X and L in Y to mean that for every neighborhood U of L there is an E in ζ such that $f^{-1}(U)\cup E\cup\{a\}$ is a neighborhood of a; equivalently, for every open set U containing L, there is an open set V containing a such that $\{x\in V\backslash\{a\}: f(x)\notin U\}\in\zeta$. If a is an isolated point of X, then $\zeta\text{-}\lim_{x\to a}f(x)=L$ for every L in Y; otherwise, there is at most one possible ζ-limit.

A folk theorem says that if $f:(-\infty, \infty)\rightarrow(-\infty, \infty)$ has the property that $\lim_{x\rightarrow a}f(x)=g(a)$ exists for every a in $(-\infty, -\infty)$, then the function g is continuous and $\{a\in(-\infty, -\infty):f(a)\neq g(a)\}$ is countable. This phenomenon was studied for topological spaces in [2], which contains a characterization of regular spaces. The proof in [2] easily adapts to our setting.

Theorem 1. *Suppose $f:X\rightarrow Y$ is Borel measurable. The following are equivalent:*

(1) *There is a continuous function $g:X\rightarrow Y$ such that $f=g$ a.e. (ζ);*

(2) *For each a in X, ζ-$\lim_{x\rightarrow a}f(x)$ exists in Y and equals $f(a)$ when $\{a\}\notin\zeta$.*

Proof. The proof that (1) implies (2) is immediate from the definitions. For the converse, assume that (2) is true and define $g(a)=f(a)$ if $\{a\}\notin\zeta$, and $g(a)=\zeta-\lim_{x\rightarrow a}f(x)$ otherwise. We first show that g is continuous. Suppose $a\in X$ and D is a closed disk centered at $g(a)$. Since $\zeta-\lim_{x\rightarrow a}f(x)=g(a)$, there is an open set V containing a and an E in ζ such that $V\subset f^{-1}(D)\cup E\cup\{a\}$. We will prove g is continuous by showing that $g(V)\subset D$. Suppose $b\in V\backslash\{a\}$. If $\{b\}\notin\zeta$, then $b\in f^{-1}(D)$, and $g(b)=f(b)\in D$. Suppose $\{b\}\in\zeta$, and let U be any open set containing $g(b)$. Then there is an open set W containing b and an F in ζ such that $W\subset f^{-1}(U)\cup F$. Since $V\cap W$ is a nonempty open set, we know that $V\cap W$ is not contained in $E\cup F$, which implies that $f^{-1}(D)\cap f^{-1}(U)$ is nonempty. Since $D\cap U$ is nonempty for every open set U containing $g(b)$ and D is closed, $g(b)$ must be in D.

We now show that $\{x:f(x)\neq g(x)\}\in\zeta$; this set clearly contains no x with $\{x\}\notin\zeta$. For each positive integer let $S_n=\{x:\mathrm{dist}(f(x),g(x))>1/n\}$, and suppose $b\in S_n$. Let D be the disk centered at $g(b)$ with radius $1/2n$. Then there is an open set V containing b such that $\{x\in V:f(x)\notin D\}\in\zeta$. If $V_b=g^{-1}(D)\cap V$, then $V_b\cap S_n\subset\{x\in V:f(x)\notin D\}$ (see the definition of D). Thus $V_b\cap S_n\in\zeta$. It follows that every Lindelöf subset of S_n is in ζ, which, by Lindelöf regularity, implies $S_n\in\zeta$. Since ζ is closed under countable unions, $\{x:f(x)\neq g(x)\}\in\zeta$. □

It is perhaps surprising that condition (2) in the above theorem is a local one.

Corollary. *If ζ and f are as above, and if, for each a in X, there is a measurable function $g_a:X\rightarrow Y$ that is continuous at a, and such that $\{x:f(x)\neq g_a(x)\}\in\zeta$, then there is a continuous $g:X\rightarrow Y$ such that $\{x:f(x)\neq g(x)\}\in\zeta$.*

Remarks (1). The proof that g is continuous in Theorem 1 does not depend on ζ being closed under countable unions or on ζ being Lindelöf regular. All that is needed is that ζ be closed under finite unions.

(2) It is easy to see from the above proof that Theorem 1 and its corollary remain valid when ζ is not assumed to be Lindelöf-regular, but, for a given infinite cardinal m, ζ is closed under all unions of collections with cardinality less than m and each A in $\mathfrak{M}\backslash\zeta$ contains a K in $\mathfrak{M}\backslash\zeta$ with the property that every open cover of K contains a subcover of cardinality less than m. □

Now suppose that $f:X\rightarrow Y$ is Borel measurable. Since Y is separable, the union W of all open subsets U of Y for which $f^{-1}(U)\in\zeta$ is Lindelöf. Hence $f^{-1}(W)\in\zeta$. We call the complement $Y\backslash W$ the ζ-essential range of f, denoted by $\mathfrak{R}_\zeta(f)$. Equivalently, a point y is in $\mathfrak{R}_\zeta(f)$ if and only if, for every open set U containing y, $f^{-1}(U)\notin\zeta$. More generally, if $V\in\mathfrak{M}$, we define $\mathfrak{R}_\zeta(f|V)$ to mean the ζ_V-essential range, where $\zeta_V=\{E\cap V:E\in\zeta\}$. If $a\in X$, let $\mathfrak{N}_a$ denote the family of neighborhoods of a. Since $\mathfrak{N}_a$ is directed by $\supset$, we can talk of limits indexed by $\mathfrak{N}_a$. The following result relates ζ-limits with ζ-essential ranges.

Theorem 2. *Suppose $f: X \to Y$ is Borel measurable, $a \in X$, and $L \in Y$. The following are equivalent:*
(1) $\zeta - \lim_{x \to a} f(x) = L$;
(2) *The following conditions hold:*

(i) $\cap\{\mathfrak{R}_\zeta(f|_V) : V \in \mathfrak{N}_a\} = L$,
(ii) $\lim_{V \in \mathfrak{N}_{(a)}} \operatorname{diam}(\mathfrak{R}_\zeta(f|_V)) = 0$.

Corollary. *Suppose $f: X \to Y$ is Borel measurable. If Y is complete, then there is a continuous function $g: X \to Y$ with $\{x : f(x) \neq g(x)\} \in \zeta$ if and only if, for each a in X, $\lim_{V \in \mathfrak{N}_{(a)}} \operatorname{diam}(\mathfrak{R}_\zeta(f|_V)) = 0$. If Y is compact then there exists such a g if and only if, for each a in X,* $\cap\{\mathfrak{R}_\zeta(f|_V) : V \in \mathfrak{N}_a\}$ contains at most one point.

We now show that our results can be used to give a new proof of a theorem about normal operators on a separable Hilbert space. Suppose that T is a normal operator on a separable Hilbert space H. A theorem of Fuglede [1] says that any operator on H that commutes with T must commute with the adjoint (conjugate transpose) T^* of T. A theorem of von Neumann [1] says that if A is an operator that commutes with every operator that commutes with both T and T^*, then $A = f(T)$ for some bounded complex Borel function f defined on the spectrum $\sigma(T)$ of T. An asymptotic version [3] of Fuglede's theorem says that if $\{S_n\}$ is a bounded sequence of operators such that $\| S_n T - T S_n \| \to 0$, then $\| S_n T^* - T^* S_n \| \to 0$. We will prove an asymptotic version of von Neumann's theorem. (This is a special case of a result in [4].)

Proposition. *Suppose that T is a normal operator on a separable Hilbert space H, and A is an operator on H such that $\| S_n A - A S_n \| \to 0$ for every bounded sequence $\{S_n\}$ of operators such that $\| S_n T - T S_n \| \to 0$. Then $A = f(T)$ for some continuous complex function f on $\sigma(T)$.*

Proof. Let E be the spectral measure of T, and let $\mathfrak{M}$ be the collection of sets of the form $B \Delta C$ with B a Borel subset of $\sigma(T)$ and $C \in \zeta$, where ζ is the collection of all subsets of sets with E-measure zero. By considering constant sequences $\{S_n\}$, we see from von Neumann's theorem that $A = f(T)$ for some bounded Borel function f. We wish to show that there is a continuous function g on $\sigma(T)$ such that $f = g$ a.e. (E). Assume not. Since the closure of $f(\sigma(T))$ is compact, the corollary to Theorem 2 implies that there is a point a in $\sigma(T)$ such that there are distinct complex numbers α, β in $\cap\{\mathfrak{R}_\zeta(f|_V) : V \in \mathfrak{N}_a\}$. By replacing f by $(f-\alpha)/(\beta-\alpha)$ we can assume that $\alpha = 0$ and $\beta = 1$. For each $n \geqq 2$, let $F_n = f^{-1}(\{z : |z| < 1/n\})$ and $G_n = f^{-1}(\{z : |1-z| < 1/n\})$. Since F_n and G_n are not in ζ, for each n, we can choose a unit vector u_n in $E(F_n)(H)$ and v_n in $E(G_n)(H)$. Let S_n be the operator that interchanges u_n and v_n and is zero on the orthogonal complement of span $\{u_n, v_n\}$. Note that $u_n \perp v_n$ since F_n and G_n are disjoint; hence $\| S_n \| = 1$ for each n. Also $(T-a)u_n \to 0$ and $(T-a)v_n \to 0$. Thus

$$\| S_n T - T S_n \| = \| S_n (T-a) - (T-a) S_n \| \to 0.$$

However, $f(T) u_n \to 0$ and $(f(T)-1) v_n \to 0$. Thus $S_n f(T) - f(T) S_n$ evaluated at the vector u_n has norm that approaches 1, a contradiction to the assumption on $A (= f(T))$. Thus there is a continuous function g such that $f = g$ a.e. (E). Thus $A = g(T)$, and the theorem is proved. □

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