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Leading Singularity of the Scattering Kernel for Moving Obstacles

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1. Introduction

Let $Q \subset \mathbb{R}^{n+1}$, $n \geqq 3$, be an open domain with C^∞ smooth boundary ∂Q. Set

$$\Omega(t) = \{x \in \mathbb{R}^n : (t,\ x) \in Q\},$$
$$\mathscr{K}(t) = \{x \in \mathbb{R}^n : (t,\ x) \notin Q\}.$$

We make the following assumptions:

(H_1) [There exists $\rho > 0$ such that $\mathscr{K}(t) \subset \{x : |x| \leqq \rho\}$ for all $t \in \mathbb{R}$.

(H_2) [If $(v_t,\ v_x)(t,\ x)$ is the unit normal at $(t,\ x) \in \partial Q$, pointing into Q, then for all $(t,\ x) \in \partial Q$ we have $|v_t| < |v_x|$.

The condition (H_1) says that the obstacle $\mathscr{K}(t)$ remains in a fixed ball, while (H_2) means that the boundary ∂Q can move with a speed less than 1.

Let $v^* = (-v_t,\ v_x)$ be the conormal vector field. Introduce the operator

$$\mathscr{B} = \begin{cases} Id \text{ (Dirichlet problem)}, \\ \dfrac{\partial}{\partial v^*} \text{ (Neumann problem)}, \end{cases}$$

and consider the problem

$$\text{(1.1)} \qquad \begin{cases} \Box u = 0 \quad \text{in} \quad Q, \\ \mathscr{B}u = 0 \quad \text{on} \quad \partial Q, \\ u(s, x) = f_1(x), \quad u_t(s,\ x) = f_2(x). \end{cases}$$

For this problem we introduce the energy space $H(t)$ as follows. For $\mathscr{B} = Id$ we define $H(t)$ as the closure of the space $C_0^\infty(\Omega(t)) \times C_0^\infty(\Omega(t))$ with respect to the norm

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$$(1.2)\qquad \|f\|_{H(t)}=\left(\int_{\Omega(t)}(|\nabla f_1|^2+|f_2|^2)\,dx\right)^{1/2},\quad f=(f_1, f_2).$$

For $\mathscr{B}=\partial/\partial v^*$ we take $H(t)$ to be the closure of the space $C^\infty_{(0)}(\overline{\Omega(t)})\times C^\infty_{(0)}(\overline{\Omega(t)})$ with respect to (1.2). Notice that in the latter case $(u(t,.), u_t(t,.))\in H(t)$ does not guarantee that $\partial_{v^*}u=0$ on ∂Q. For Neumann problem we interpret the boundary condition in the sense of distributions. We refer to [2], [5] for the precise definition of the solution of (1.1). In particular, we can introduce the map

$$H(s)\ni f\to U(t, s)f=(u(t, x), u_t(t, x))\in H(t),$$

where $u(t, x)$ is the solution of (1.1).

Next let $U_0(t)$ be the unitary group in the space H_0 related to the Cauchy problem for the wave equation (see [8], [9]). The space H_0 is given as the closure of $C_0^\infty(\mathbf{R}^n)\times C_0^\infty(\mathbf{R}^n)$ with respect to the norm

$$\|f\|_0=\left(\int_{\mathbf{R}^n}(|\nabla f_1|^2+|f_2|^2)dx\right)^{1/2}.$$

We have the estimate

$$\|U(t, s)f\|_{H(t)}\leqq C_A\|f\|_{H(s)}\quad\text{for}\quad |t|\leqq A,\quad |s|\leqq A.$$

In general, the operator $U(t, s)$ is not uniformly bounded for $t, s\in\mathbf{R}$ and the local (global) energy of the solutions of (1.1) can increase as $t\to\infty$. For these reasons we cannot develop a scattering theory and we cannot define the scattering kernel as in the case of stationary obstacles (see [8], [10], [17]). Nevertheless, assuming (H_1), (H_2) and n odd, we can introduce a generalized scattering kernel following the approach of Cooper and Strauss [2], [3]. For even space dimensions we treat this question in section 6 making an additional assumption (H_3). It is important to note that if the scattering operator S exists, then for n odd the generalized scattering kernel $K^\#$ coinsides with the kernel of the operator $\mathscr{R}_n(S-Id)\mathscr{R}_n^{-1}$, $\mathscr{R}_n$ being the translation representation of the group $U_0(t)$ in L^2 $(\mathbf{R}\times S^{n-1})$. For n even, if the scattering operator S exists, then $K^\#$ coinsides with the kernel of the operator $\mathscr{R}_n^+S(\mathscr{R}_n^-)^{-1}-K$, where K is the Hilbert transform with respect to t in L^2 $(\mathbf{R}_t\times S^{n-1})$ and $\mathscr{R}_n^\pm$ are the translation representations of $U_0(t)$ related to the spaces $D_\pm$ (see section 6).

For fixed $s, \theta, \omega, \theta\neq\omega$, $\max_{s'}\operatorname{sing\,supp} K^\#\ (s', \theta; s, \omega)$ is closely related to the geometry of the obstacle. For stationary obstacles $Q=\mathbf{R}\times\Omega$ this problem has been investigated by A. Majda [10], V. Petkov [13] and H. Soga [18]. More precisely, for fixed $\theta\neq\omega$ we have

$$(1.2)\qquad \max_{s'}\operatorname{sing\,supp} K^\#\ (s', \theta;\ s, \omega)=s+\max_{x\in\partial\Omega}\langle x, \theta-\omega\rangle.$$

For moving obstacles the analogue of (1.2) for Dirichlet problem, n odd and regular directions θ-ω has been proved by J. Cooper and W. Strauss [3] (in this work it is treated also the degenerate case for $n=3$). The localization procedure in [3] is not sufficiently rigorous and in [4] J. Cooper and W. Strauss completed their argument by several lemmas. Nevertheless, to cover the case of degenerate directions θ-ω for all dimensions $n \geqq 3$, the approach in [3] seems to be not available. In this paper we deal with this problem proposing a localization procedure which works without change for Neumann and Robin problems (see [1]). Moreover, this localization is essential in the paper of V. Georgiev [6], where Maxwell's equations outside moving obstacle are treated.

After the localization we construct a microlocal parametrix and we use the calculus of pseudodifferential operators in order to obtain an information for lower order terms of the parametrix. This makes it possible to apply the result of H. Soga [18] for oscillatory integrals with degenerate phase functions. Thus we obtain a complete result for Dirichlet problem covering all directions $\theta \neq \omega$ and also the case $n \geqq 4$, n even. Finally, we treat simultaneously Dirichlet and Neumann problem.

From the leading singularity of the generalized scattering kernel $K^{\#}$ we can recover the convex hull of the obstacles $\mathscr{K}(t)$ for all $t \in \mathrm{R}$ (see [3] for more details). In this direction, it is natural to study the following question.

(U) If the (generalized) scattering kernel $K^{\#}$ determines uniquely the obstacles $\mathscr{K}(t)$ for all $t \in \mathrm{R}$?

Recently, P. Stefanov [19] attacked this problem and he proved that if $\mathscr{K}(t)$ is stationary for $|t| \geqq A$ with some $A > 0$, then the answer of (U) is positive. On the other hand, P. Stefanov [19] constructed an example of periodically moving obstacles for which the uniqueness in (U) fails.

The plan of the paper is as follows. In section 2 we collect some preliminary results including some lemmas proved in [4]. In section 3 we localize the problem by using a suitable partition of unity. Section 4 is devoted to the analysis of the generic case when θ-ω is a regular direction. In section 5 we treat the degenerate case, while in section 6 we discuss the modifications needed to cover the case of even space dimensions $n \geqq 4$. A part of our results has been announced in [14], [16].

2. Preliminary results

Throughout this section we assume $n \geqq 3$ odd and we suppose the assumptions (H_1) and (H_2) fulfilled. Consider the function

$$h_1(t) = \begin{cases} t, & t \geqq 0, \\ 0, & t < 0 \end{cases}$$

and introduce the solution $\Gamma^+(t, x; s, \omega)$ of the problem

$$\begin{cases} \Box \Gamma^+ = 0 \text{ in } Q, \\ \mathscr{B}\Gamma^+ = 0 \text{ on } \partial Q, \\ \Gamma^+|_{t < -s-\rho} = h_1(t + s - \langle x, \omega \rangle). \end{cases}$$

Here $s \in \mathsf{R}$, $\omega \in S^{n-1}$ and $\langle , \rangle$ denotes the inner product in R^n. For $t < -s-\rho$ the initial data belong to the space

$$H_0^{\mathrm{loc}} = \{f \in \mathscr{D}'(\mathsf{R}^n) \times \mathscr{D}'(\mathsf{R}^n) : \varphi f \in H_0 \quad \text{for each} \quad \varphi \in C_0^\infty(\mathsf{R}^n)\}.$$

Moreover, these data vanish in a neighbourhood of the obstacle and we can determine Γ^+. Set

$$u^+(t, x; s, \omega) = \Gamma^+(t, x; s, \omega) - h_1(t+s-\langle x, \omega\rangle),$$

$$u_{sc}(t, x; s, \omega) = \partial_s^{(n+1)/2} u^+(t, x; s, \omega).$$

To show that u^+ has an asymptotic wave profile in the sense of Cooper and Strauss [2], introduce a function $\chi(x) \in C_0^\infty(\mathsf{R}^n)$ such that $\chi(x) = 1$ for $|x| \leqq \rho + 1$, $\chi(x) = 0$ for $|x| \geqq \rho + 2$. The function

$$u = u^+(t, x; s, \omega) + \chi h_1(t+s-\langle x, \omega\rangle)$$

satisfies the conditions:

$$\begin{cases} \Box u = \langle 2\nabla\chi, \omega\rangle h_1' - (\Delta\chi) h_1, \\ u|_{t<-s-\rho-2} = 0. \end{cases}$$

According to the results in [2], the function u has an asymptotic wave profile $u_+^\#(s', \theta; s, \omega) \in L^2_{\mathrm{loc}}(\mathsf{R}_{s'} \times S_\theta^{n-1})$ which coinsides with that of u^+.

Now we shall sketch the argument in [2] leading to the so called generalized scattering kernel. Let $\mathscr{R}_n : H_0 \to L^2(\mathsf{R} \times S^{n-1})$ be the translation representation of $U_0(t)$ (see [7]). Given $\varphi \in H_0$ with $\mathscr{R}_n\varphi \in C_0^\infty(\mathsf{R} \times S^{n-1})$, $\mathscr{R}_n\varphi = 0$ for $|s| > a$, we define

$$W_-\varphi = \lim_{t\to\infty} U(0, -t)U_0(-t)\varphi = U(0, -t)U_0(-t)\varphi \qquad \text{for } t > a+\rho.$$

The definition of $W_-\varphi$ does not depend on the choice of t. Introduce

$$(v_0(t,.), \partial_t v_0(t,.)) = U_0(t)\varphi,$$

$$(v(t,.), \partial_t v(t,.)) = U(t, 0)W_-\varphi.$$

If $v_0^\#(s, \omega)$ is the asymptotic wave profile of $v_0(t, x)$, it is easy to see that

$$v(t, x) = v_0(t, x) - (2\pi)^{-(n-1)/2} \iint u^+(t, x; s, \omega).$$

$$\cdot\, \partial_s^{(n+1)/2} v_0^\#(s, \omega)\, \mathrm{d}s\, \mathrm{d}\omega,$$

where the integral is taken in the sense of distributions. Taking the asymptotic wave profiles we obtain

$$v^\#(s', \theta) = v_0^\#(s', \theta) - (2\pi)^{-(n-1)/2} \iint u_+^\#(s', \theta; s, \omega)$$

$$\cdot\, \partial_s^{(n+1)/2} v_0^\#(s, \omega)\ \mathrm{d}s\mathrm{d}\omega.$$

Thus it is natural to make the following

Definition 2.1. The distribution

$$K^\#(s', \theta; s, \omega) = (-2\pi)^{-(n-1)/2} \partial_s^{(n+1)/2} u_+^\#(s', \theta; s, \omega)$$

is called a generalized scattering kernel of the problem (1.1).

In the case that the scattering operator $S: H_0 \to H_0$ exists, $K^\#$ coinsides with the Schwartz's kernel of the operator $\tilde{S}-Id$, where

$$\tilde{S}=\mathcal{R}_n S \mathcal{R}_n^{-1}.$$

Notice that for stationary obstacles the operator S commutes with $U_0(t)$ and $K^\#$ depends on $s'-s$, θ, ω.

Set $w_s(t, x; s, \omega)=\partial_s^2 \Gamma^+(t, x; s, \omega)$.

Repeating the arguments in [3], it is easy to obtain the following

Theorem 2.2. *The generalized scattering kernel $K^\#(s', \theta; s, \omega)$ admits the representation*

$$K^\#(s', \theta; s, \omega)=\frac{1}{2(2\pi)^{n-1}} \int_{\partial Q} [\partial_s^{(n-3)/2} w_s(t, x; s, \omega) \tag{2.1}$$

$$\cdot \partial_{s'}^{(n-1)/2} \partial_{\nu^*} \delta(t+s'-\langle x, \theta\rangle)-\partial_s^{(n-3)/2} \partial_{\nu^*} w_s(t, x; s, \omega)$$

$$\cdot \partial_{s'}^{(n-1)/2} \delta(t+s'-\langle x, \theta\rangle)] dS.$$

Moreover, $K^\#$ is a C^∞ function of s, θ, ω with values in the space of distributions in s'.

For Dirichlet problem the first term in right-hand side of (2.1) vanishes, while for Neumann problem the same is true for the second one. For Dirichlet problem (2.1) was established in [3].

Below we shall discuss some geometric properties of reflecting rays. Fix s, θ, ω, so that $\theta \neq \omega$ and introduce the arrival surface $\Gamma(s, \omega) = \{x \in \mathbf{R}^n : (\langle x, \omega\rangle - s, x) \in \partial Q\}$ and the departure surface $\Gamma(s', \theta) = \{x \in \mathbf{R}^n : (\langle x, \theta\rangle - s', x) \in \partial Q\}$. The vector $N=\nu_x+\nu_t\omega$ is normal to $\Gamma(s, \omega)$, while $N'=\nu_x+\nu_t\theta$ is normal to $\Gamma(s', \theta)$. On the other hand, if a ray with incident direction $(1, \omega)$ hits the boundary ∂Q transversally at $(\hat{t}, \hat{x})$ and its reflecting direction is $(1, \theta)$, then $N(\hat{t}, \hat{x})$ is parallel to $\theta-\omega$. These properties are proved in [2], [3].

Consider the set

$$\Sigma(s, \omega)=\partial Q \cap \{(t, x): t=\langle x, \omega\rangle - s\}.$$

Exploiting the assumption (H_2), it is not hard to obtain the following

Lemma 2.3. *Let (s_0, ω) be fixed and let $(t_0, x_0) \in \Sigma(s_0, \omega)$. Given $s > s_0$, there exists $(t, x) \in \Sigma(s, \omega)$ such that*

$$|x-x_0| < t_0 - t. \tag{2.2}$$

This lemma is due to J. Cooper and W. Strauss [4] (see also [15], Chapter VIII, Lemma 8.2.4).

Next for $\theta \neq \omega$ introduce the function

$$h(s)=\max_{x \in \Gamma(s, \omega)} \langle x, \theta-\omega\rangle.$$

The following lemma is also proved by J. Cooper and W. Strauss [4].

Lemma 2.4. *For fixed $\varepsilon_0>0$ there exists $\delta>0$ such that for every ε, $0\leqq\varepsilon\leqq\delta$, we have*

$$\partial Q\cap\{(t,x): t+s+h(s)\leqq\langle x,\theta\rangle+\varepsilon\}\subset\{(t,x): t+s\leqq\langle x,\omega\rangle+\varepsilon_0\}.$$

Proof. Assume that there exists a sequence $\varepsilon_m\searrow 0$ and points $(t_m, x_m)\in\partial Q$ so that

$$t_m+s+h(s)\leqq\langle x_m,\theta\rangle+\varepsilon_m,\qquad t_m+s>\langle x_m,\omega\rangle+\varepsilon_0.$$

By compactness we can suppose that $(t_m, x_m)\xrightarrow[m\to\infty]{}(\tilde{t},\tilde{x})\in\partial Q$. Then $\tilde{t}+s+h(s)\leqq\langle\tilde{x},\theta\rangle$ and $\tilde{t}+s\geqq\langle\tilde{x},\omega\rangle+\varepsilon_0$ for $(\tilde{t},\tilde{x})\in\Sigma(\tilde{s},\omega)$, $\tilde{s}<s$. By Lemma 2.3 we find a point $(t,x)\in\Sigma(s,\omega)$ such that

$$|x-\tilde{x}|<\tilde{t}-t.$$

Thus

$$\tilde{t}+s+h(s)-\langle\tilde{x},\theta\rangle=\tilde{t}-t+\langle x-\tilde{x},\theta\rangle+h(s)-\langle x,\theta-\omega\rangle\geqq\tilde{t}-t-|x-\tilde{x}|>0$$

and we obtain a contradiction. The proof is complete.

3. Localization of the leading singularity

In this section we use the notations and assumptions of the previous one. For the solution $w_s(t,x;s,\omega)$ we have the following

Lemma 3.1. (i) $\operatorname{supp} w_s\subset\{(t,x):\langle x,\omega\rangle\leqq t+s\}$,
(ii) $\operatorname{supp}\partial_{\nu^*}w_s|_{\partial Q}\cup\operatorname{supp} w|_{\partial Q}\subset\{(t,x):\langle x,\theta\rangle\leqq t+s+h(s)\}$.

Proof. The inclusion (i) follows by the principle of causality for the wave equation. This principle implies that if $(t_0,x_0)\in\operatorname{supp} w_s$, then the cone $C_0=\{(t,x):|x-x_0|<t_0-t\}$ must intersect the support of $\delta(t+s-\langle x,\omega\rangle)$. Consequently, there exists a point (t,x) such that

$$t+s-\langle x,\omega\rangle=0,\qquad |x-x_0|\leqq t_0-t.$$

This yields

$$\langle x_0,\omega\rangle\leqq\langle x,\omega\rangle+|x_0-x|\leqq t+s+t_0-t=t_0+s.$$

To prove (ii), suppose for example that $(t_0,x_0)\in\operatorname{supp}\partial_{\nu^*}w_s|_{\partial Q}$. Choose s_0 so that $(t_0,x_0)\in\Sigma(s_0,\omega)$. By Lemma 2.3 there exists $(t,x)\in\Sigma(s,\omega)$ such that (2.2) holds. By using (i), we get

$$\langle x_0,\theta\rangle\leqq\langle x,\omega\rangle+\langle x,\theta-\omega\rangle+|x_0-x|\leqq t+s+h(s)+t_0-t=t_0+s+h(s).$$

The proof is complete.

The above lemma implies the following

Theorem 3.2. *We have*

$$\operatorname{supp} K^{\#}(s',\theta;s,\omega)\subset\{(s',s): s'\leqq s+h(s)\}. \tag{3.1}$$

Proof. If $(s', \theta; s, w) \in \operatorname{supp} K^{\#}$, then there exists a point $(\hat{t}, \hat{x}; s, \omega)$ $\in \operatorname{supp} \partial_{\nu^*} w_s|_{\partial Q} \cup \operatorname{supp} w_s|_{\partial Q}$ such that $\hat{t}+s'-\langle \hat{x}, \theta\rangle=0$. By (ii) of Lemma 3.1, we deduce $\langle \hat{x}, \theta\rangle \leqq \hat{t}+s+h(s)$. This implies

$$s'-s \leqq \langle \hat{x}, \theta\rangle-\hat{t}+\hat{t}+h(s)-\langle \hat{x}, \theta\rangle=h(s).$$

Remark 3.3. The proof of the above result is the same as that in [3].

In the sequel we shall study the singularity of $K^{\#}$ at $s'=s+h(s)$. We assume s, θ, ω fixed and for the simplicity of notations we shall write h instead of $h(s)$. Choose a function $\Phi(s') \in C_0^\infty$ (R) with the properties

$$\Phi(s')=0 \quad \text{for} \quad |s'-s-h|>\varepsilon,$$
$$\Phi(s')=1 \quad \text{for} \quad |s'-s-h|<\varepsilon/2, \ 0 \leqq \Phi(s') \leqq 1.$$

The function Φ depends on $\varepsilon>0$ but we shall not indicate this in the notations.

According to (2.1) we have

$$\begin{aligned} I(\sigma) &= \langle K^{\#}(s', \theta; s, \omega), \quad \Phi(s')e^{-i\sigma s'}\rangle \\ &= (-1)^{(n-1)/2} d_n^2 \Bigg[\sum_{j=0}^{(n+1)/2} c_j(-i\sigma)^{((n+1)/2-j)} \\ &\quad \cdot \int_{\partial Q} e^{-i\sigma(\langle x,\theta\rangle - t)} (\partial^j \Phi)(\langle x,\theta\rangle - t) \langle N', \theta\rangle \partial_s^{(n-3)/2} w_s dS \\ &\quad - \sum_{j=0}^{(n-1)/2} c'_j(-i\sigma)^{((n-1)/2-j)} \cdot \int_{\partial Q} e^{-i\sigma(\langle x,\theta\rangle - t)} (\partial^j \Phi)(\langle x, \theta\rangle - t) \partial_s^{(n-3)/2} \partial_{\nu^*} w_s dS \Bigg] \end{aligned} \tag{3.2}$$

with $d_n=2^{-1/2}(2\pi)^{(1-n)/2}$. Here c_j, c'_j are some real constants and $c_0=c'_0=1$. Our aim is to show that $I(\sigma)$ is not a decreasing function of σ as $|\sigma| \to \infty$ provided that ε is chosen sufficiently small.

Consider the function

$$v(t, x; s, \omega)=w_s(t, x; s, \omega)-\delta(t+s-\langle x, \omega\rangle).$$

To localize the mixed problem for v, introduce a partition of unity in a neighbourhood of $\Gamma(s, \omega)$ given by functions

$$\{\psi_j(x)\}_{j=1}^L, \quad \psi_j(x) \in C_0^\infty(\mathrm{R}^n), \quad \sum_{j=1}^L \psi_j(x)=1.$$

Let $v_j(t, x; s, \omega)$ be the solution of the problem

$$\begin{cases} \square v_j=0 \text{ in } 0, \\ \mathscr{B} v_j=-\psi_j(x) \mathscr{B} \delta(t+s-\langle x, \omega\rangle) \text{ on } \partial Q, \\ v_j|_{t<-s-\rho}=0. \end{cases} \tag{3.3}$$

Clearly, we must investigate only those v_j for which

$$\operatorname{supp} \psi_j \cap \Gamma(s, \omega) \neq \emptyset.$$

Set

$$V_j=\{(t,\ x)\in\partial Q\colon\ (t,\ x)\in\operatorname{sing\,supp}\partial_{\nu^*}v_j|_{\partial Q}\cup\operatorname{sing\,supp}v_j|_{\partial Q}\},$$

$$t_j=\min_{x\in\operatorname{supp}\psi_j}\langle x,\ \omega\rangle-s.$$

The following lemma is a simple consequence of the principle of causality.

Lemma 3.4. *For each* $(t,\ y)\in V_j$ *we have*

$$\operatorname{dist}(y,\ \operatorname{supp}\psi_j)\leqq t-t_j. \tag{3.4}$$

Proof. Let $u_j(t,\ x;\ s,\ \omega)$ be the solution of the problem

$$\begin{cases}\Box u_j=0 \text{ in } Q,\\ \mathscr{B}u_j=-\psi_j(x)\mathscr{B}h_2(t+s-\langle x,\ \omega\rangle) \text{ on } \partial Q,\\ u_j|_{t<-s-\rho}=0,\end{cases}$$

where

$$h_2(t)=\begin{cases}t^2/2, & t\geqq 0,\\ 0, & t<0.\end{cases}$$

We have $v_j=\partial_s^3 u_j$ and it is sufficient to study the traces on ∂Q of u_j and its conormal derivative $\partial_{\nu^*}u_j$.

Let $(\hat{t},\ \hat{y})\in V_j$ be such that dist $(y,\ \operatorname{supp}\psi_j)>\hat{t}-t_j$. Choosing $\delta>0$ small enough, we can arrange the inequality

$$\operatorname{dist}(\hat{y},\ \operatorname{supp}\psi_j)>\hat{t}-t_j+2\delta.$$

Introduce the cone

$$C_0=\{(t,\ y)\in\mathbb{R}^{n+1}\colon|\hat{y}-y|<\hat{t}-t+\delta,\ t<\hat{t}+\delta\}.$$

It is easy to check that

$$C_0\cap\partial Q\cap\operatorname{supp}(\psi_j(x)\mathscr{B}h_2(t+s-\langle x,\ \omega\rangle))=\emptyset.$$

Applying the principle of causality for u_j, we conclude that $u_j(t,\ x)=0$ for $(t,\ x)\in C_0\cap\bar{Q}$. This implies $(\hat{t},\ \hat{y})\in V_j$ and the proof is complete.

Going back to (3.2), we must study the points of the set

$$W_j=\{(t,\ y)\in\partial Q\colon\Phi(\langle y,\ \theta\rangle-t)\neq 0,\ (t,\ y)\in V_j\}.$$

Let $x\in\operatorname{supp}\psi_j$, $(t,\ y)\in W_j$, $\hat{y}\in\Gamma(s,\ \omega)$, $\langle\hat{y},\ \theta-\omega\rangle=h$, $\operatorname{supp}\psi_j\cap\Gamma(s,\ \omega)\neq\emptyset$.

Consider the equality

$$t-\langle y,\ \theta\rangle+s+h=t-\langle x,\ \omega\rangle+s+\left\langle x-y,\ \frac{\theta+\omega}{2}\right\rangle+\frac{1}{2}\langle\hat{y}-x,\ \theta-\omega\rangle+\frac{1}{2}\langle\hat{y}-y,\ \theta-\omega\rangle. \tag{3.5}$$

From $\Phi(\langle y,\ \theta\rangle-t)\neq 0$ we deduce

$$-\varepsilon\leqq t+s+h-\langle y,\ \theta\rangle\leqq\varepsilon. \tag{3.6}$$

According to Lemma 2.4, we take ε_0 small and choose $0<\varepsilon<\delta(\varepsilon_0)$ so that

$$t+s\leqq\langle y,\ \omega\rangle+\varepsilon_0.$$

Thus

$$\langle y,\ \theta-\omega\rangle=\langle y,\ \theta\rangle-t-s+t+s-\langle y,\ \omega\rangle\leqq h+\varepsilon+\varepsilon_0. \tag{3.7}$$

Here and below ε depends on ε_0 but for simplicity we shall omit this.

Next we choose ε and $\operatorname{supp}\psi_j$ in order to arrange

$$t-\langle x,\ \omega\rangle+s\geqq t-t_j-\varepsilon/2 \text{ for } x\in\operatorname{supp}\psi_j,$$

$$\left\langle x-y,\ \frac{\theta+\omega}{2}\right\rangle\geqq-|x-y|\geqq-\operatorname{dist}(y,\ \operatorname{supp}\psi_j)-\varepsilon/2,$$

$$\langle\hat{y}-x,\ \theta-\omega\rangle=\langle\hat{y}-z,\ \theta-\omega\rangle+\langle z-x,\ \theta-\omega\rangle\geqq-\varepsilon \text{ for } z\in\Gamma\ (s,\ \omega).$$

By Lemma 3.4 we deduce

$$t-\langle x,\ \omega\rangle+s+\left\langle x-y,\ \frac{\theta+\omega}{2}\right\rangle\geqq-\varepsilon.$$

Combining this with (3.5) and (3.6), we get

$$\langle\hat{y}-x,\ \theta-\omega\rangle+\langle\hat{y}-y,\ \theta-\omega\rangle\leqq 4\varepsilon. \tag{3.8}$$

Thus (3.7) and (3.8) imply

$$-\varepsilon\leqq\langle\hat{y}-x,\ \theta-\omega\rangle\leqq 5\varepsilon+\varepsilon_0.$$

Finally, choosing ε_0 and $\delta(\varepsilon_0)$ sufficiently small, we conclude that the point x must lie in a small neighbourhood U_0 of the set

$$\mathscr{R}_0=\{x\in\Gamma(s,\ \omega):\langle x,\ \theta-\omega\rangle=h\}.$$

On the other hand, if $\operatorname{supp}\psi_j\cap U_0=\emptyset$, the distribution v_j does not contribute to the singularity of $K^{\#}$ at $s'=s+h$.

Now we shall show that the wave front set

$$WF(\mathscr{B}\delta(t+s-\langle x,\ \omega\rangle)|_{\partial Q})$$

over $\mathscr{R}_0$ contains only hyperbolic points of the wave operator $\partial_t^2-\Delta_x$. We pass to local coordinates so that ∂Q is given locally by $y_n=l(t,\ y')$, $y'=(y_1,\dots,\ y_{n-1})$, $|l_t|<1$, $\nabla_{y'}l(0,\ 0)=0$.

Changing variables

$$x_j=y_j,\ j=1,\dots,\ n-1,\quad x_n=y_n-l(t,\ y'), \tag{3.9}$$

we transform $\partial_t^2-\Delta_x$ into an operator P with principal symbol

$$p(t,\ x,\ \tau,\ \xi)=b^2\,[(\xi_n-\lambda)^2-\mu], \tag{3.10}$$

where

$$b^2=1+|\nabla_{y'}l|^2-l_t^2>0,$$
$$\lambda(t,\ x,\ \tau,\ \xi')=b^{-2}(\langle l_{y'},\ \xi'\rangle-l_t\tau),$$
$$\mu(t,\ x,\ \tau,\ \xi')=\lambda^2+b^{-2}(\tau^2-|\xi'|^2).$$

Here (τ, ξ', ξ_n) are the variables dual to (t, x', x_n). We need to study the roots of the equation $p=0$ with respect to ξ_n in the case that

(3.11) $\tau=1-l_t\omega_n,\ \xi'=-\omega'-l_{x'}\omega_n,\ \omega'=(\omega_1,\dots,\ \omega_{n-1}),\quad x\in\mathscr{R}_0.$

Without loss of generality we may assume that the unit exterior normal to $\Gamma(s, \omega)$ at the points on $\mathscr{R}_0$ has the form

$$\frac{N}{|N|}=(0,\ 0,\dots,\ 1).$$

Since $N=v_x+v_t\omega$, this yields

(3.12) $$l_{x'}=-l_t\omega',\ (t,\ x',\ 0)\in\partial Q,\ (x',\ 0)\in\mathscr{R}_0.$$

Thus for τ, ξ' given by (3.11) we have

$$\mu\geqq b^{-2}(\tau^2-|\xi'|^2)=b^{-2}(1-l_t\omega_n)^2(1-|\omega'|^2)=b^{-2}(1-l_t\omega_n)^2\omega_n^2.$$

The fact that the function $\langle x,\ \theta-\omega\rangle$ has maximum at $x\in\mathscr{R}_0$ implies

$$\frac{\theta-\omega}{|\theta-\omega|}=\pm(0,\ 0,\dots,0,\ 1).$$

If $\omega_n=0$, then $0=\langle\theta-\omega,\ \omega\rangle=\langle\theta,\ \omega\rangle-1$, that is $\theta=\omega$ which is a contradiction with our assumption. Hence $\omega_n\neq0$ and $|l_t\omega_n|<1$ show that the equation $p=0$ for $(t,\ x,\ \tau,\ \xi')$, determined by (3.11), has two simple real roots.

Now let v_j be such that $\operatorname{supp}\psi_j\cap\mathscr{R}_0\neq\emptyset$. For $\operatorname{supp}\psi_j$ sufficiently small the wave front set of

$$g_j=\psi_j(x)\mathscr{B}\delta\ (t+s-\langle x,\ \omega\rangle)|_{\partial Q}$$

contains only hyperbolic points for the wave operator. Consequently, the singularities of v_j for small t-t_j are propagating along the outgoing generalized bicharacteristics of $\Box$ issued from $WF_b(g_j)$ (see [7]). For ε_0 small enough and $0<\delta_1(\varepsilon_0)\leqq t-t_j<\delta_2(\varepsilon_0)$ these rays leave the set

$$\Pi_0=\{(t,\ x):t+s-\langle x,\ \omega\rangle\leqq\varepsilon_0\}.$$

Moreover, $\delta_1(\varepsilon_0)\geqq\delta_0>0$ as $\varepsilon_0\to0$. We claim that $v_j\in C^\infty$ for $t\geqq t_j+\delta_1(\varepsilon_0)$ and $(t,\ x)\in\Pi_0$. To prove this, choose $t_0\geqq t_j+\delta_1(\varepsilon_0)$ so that $v_j=0$ on $\partial Q\cap\{t\geqq t_0\}$, $v_j\in C^\infty$ on $\Pi_0\cap\{t=t_0\}$. It is easy to see that for sufficiently small $\operatorname{supp}\psi_j$ and $\varepsilon_0\psi$ there exists such t_0. On the other hand, the singularities of v_j are propagating along the outgoing generalized bicharacteristics $\gamma(\sigma)=(t(\sigma),\ x(\sigma),\ \tau(\sigma),\ \xi(\sigma))$ of $\Box$. Recall that $\gamma\ (\sigma)$ is outgoing if $t(\sigma)$ increases as σ increases. If the derivatives $dt/d\sigma=\dot{t}(\sigma)$, $dx/d\sigma=\dot{x}(\sigma)$ exist, then we find

$$\frac{\mathrm{d}}{\mathrm{d}\sigma}(t(\sigma)+s-\langle x(\sigma),\ \omega\rangle)=\dot{t}(\sigma)\left(1-\left\langle\frac{\dot{x}(\sigma)}{\dot{t}(\sigma)},\ \omega\right\rangle\right)\geqq 0.$$

This proves the claim.

The above observation shows that in the analysis of the leading singularity of $K^{\#}$ we can assume $(t,\ x)\in\Pi_0$. Furthermore, we need to examine v_j for $(t,\ x)$ sufficiently close to $\operatorname{supp} g_j$. This will be done in the next section. Remark that by compactness we can choose $\varepsilon_0>0$ uniformly for all j such that

$$\operatorname{supp}\psi_j\cap\mathscr{R}_0\neq\emptyset.$$

Fixing a suitable $\varepsilon_0>0$, we choose $\varepsilon>0$ small enough so that the assertion in Lemma 2.4 holds. After for fixed ε_0, ε we shall examine the singularities of $v_j(t,x;s,\omega)$ for $(t,\ x)\in\Sigma(s,\ \omega)$ and x sufficiently close to $\mathscr{R}_0$.

4. Leading singularity of $K^{\#}$ (generic case)

In this section we assume s, θ, ω fixed and we shall use freely the notations of the previous sections. Our aim is to study the asymptotic of (3.2) as $|\sigma|\to\infty$.

We shall say that $\theta-\omega$ is a regular direction if the set $\mathscr{R}_0$ is formed by a finite number of points $x_j\in\Gamma(s,\ \omega),\ j=1,\dots\ M$, for which $\langle x_j,\ \theta-\omega\rangle=h$ and the Gauss curvature $k(x_j)$ of $\Gamma\ (s,\ \omega)$ at each point x_j is positive. The set of vectors $(\theta-\omega)|\theta-\omega|$ for which $\theta=\omega$ is regular is dense in S^{n-1} (see [3]).

We shall use a technical lemma due to Cooper and Strauss.

Lemma 4.1 [3]. *Assume $g(t,\ x)\in C_0^\infty(R^{n+1})$. Then*

$$\int_{\partial Q} g(t,\ x)\,\mathrm{d}S=\int_{-\infty}^{\infty}\ \int_{\Gamma(\tau,\omega)} g(\langle x,\ \omega\rangle-\tau,\ x)\ \frac{\mathrm{d}\Gamma}{|N|}\ \mathrm{d}\tau, \tag{4.1}$$

where $\mathrm{d}\Gamma$ *is the induced measure on* $\Gamma(\tau,\ \omega)$.

In the sequel we shall study the Neumann problem since the analysis of the Dirichlet problem is similar and simpler. Replacing $w(t,x;\ s,\omega)$ by $v(t,\ x;\ s,\ \omega)+\delta(t+s-\langle x,\ \omega\rangle)$ in the first term in the right-hand side of (3.2), we obtain several terms containing $\partial^j\Phi(\langle x,\theta\rangle-t)$. We shall consider the integrals

$$I_1(\sigma)=\int_{\partial Q}\mathrm{e}^{-i\sigma(<x,\theta>-t)}\langle N',\theta\rangle\Phi(\langle x,\theta\rangle-t)\partial_s^{(n-3)/2}\delta(t+s-\langle x,\omega\rangle)\,\mathrm{d}S.$$

$$I_2(\sigma)=\int_{\partial Q}\mathrm{e}^{-i\sigma(<x,\theta>-t)}\langle N',\theta\rangle\Phi(\langle x,\theta\rangle-t)\partial_s^{(n-3)/2}v\,\mathrm{d}S.$$

The analysis of other integrals is similar.

Approximating $\delta(t+s-\langle x,\omega\rangle)$ by smooth functions and using Lemma 4.1, we deduce

$$I_1(\sigma)=\int_{-\infty}^{\infty}\delta^{(n-3)/2}(s-\tau)\int_{\Gamma(\tau,\omega)}\mathrm{e}^{-i\sigma(<x,\theta-\omega>+\tau)}\langle N',\theta>\Phi(\langle x,\theta-\omega\rangle+\tau)\frac{\mathrm{d}\Gamma}{|N|} \tag{4.2}$$

$$=\sum_{j=0}^{(n-3)/2}(-i\sigma)^{((n-3/2)-j)}\int_{\Gamma(s,\omega)}\mathrm{e}^{-i\sigma(<x,\theta-\omega>+s)}\alpha_j(x,s,\theta,\omega)\,\mathrm{d}\Gamma.$$

Here $\alpha_j(x, s, \theta, \omega)$ are real-valued functions and

$$(4.3)\quad \alpha_0(x, s, \theta, \omega) = \frac{\langle N', \theta\rangle}{|N|}\Phi(\langle x, \theta-\omega\rangle + s)\left(\frac{1-\langle w, \theta\rangle}{1-\langle w, \omega\rangle}\right)^{(n-3)/2}(\langle x, \omega\rangle - s, x).$$

In (4.3) by $w = dx/dt$ we denote the velocity of ∂Q, assuming that ∂Q is parametrized locally by $(t, z) \to (t, x, (t, z))$. We shall prove that at the critical points of the phase in (4.2) the function α_0 does not depend on the choice of coordinates. Thus the form of the function $x(t, z)$ in the parametrization will be not important for our calculus.

For regular directions θ-ω we have $\mathscr{R}_0 = \{x_1, \dots, x_M\}$. Repeating for $\delta(t+s-\langle x, \omega\rangle)$ the localization procedure, exposed in the previous section, we conclude that for the leading singularity of $K^\#$ is essential the integration in (4.2) over the set $\cup_{j=1}^M U_j$, U_j being sufficiently small neighbourhoods of x_j in $\Gamma(s, \omega)$ whose size depends on the support of Φ.

Now we turn to the analysis of $I_2(\sigma)$. Fix $\hat{y} \in \mathscr{R}_0$ and in a small neighbourhood of $(\hat{t}, \hat{y}) \in \Sigma(s, \omega)$ choose local coordinates so that ∂Q is given locally by $y_n = l(t, y')$, $|l_t| < 1$. We use the change of variable (3.9) and obtain operator P with principal symbol (3.10). We extend the coefficients of P preserving the strict hyperbolicity with respect to t. In the coordinates t, x set

$$\hat{x} = (\hat{y}', \hat{y}_n - l(\hat{t}, \hat{y}')),\quad g = -\hat{\psi}(t, x')\langle N, \omega\rangle \delta'(t+s-\langle x', \omega'\rangle - l(t, x')\omega_n).$$

Here $\hat{\psi} \in C_0^\infty(\mathbf{R}^n)$, $\hat{\psi}(\hat{t}, \hat{x}') = 1$ and the wave front set $WF(g)$ is included in the set of hyperbolic points of the operator P.

After the change (3.9) the operator ∂_{v^*} is transformed into

$$B = b^2 b_1^{-1}(\partial_{x_n} - i\lambda(t, x', D_t, D_{x'})).$$

Here $b_1^2 = 1 + |\nabla_{t,x'} l|^2$ and $\lambda(t, x', D_t, D_{x'}) = b^{-2}(\langle l_{y'}, D_{x'}\rangle - l_t D_t)$.

Let $\hat{g}(\tau, \xi')$ be the Fourier transform of g. We shall construct a microlocal parametrix

$$(4.4)\qquad (Gg)(t, x) = (2\pi)^{-n}\int_{\mathbf{R}^n} e^{i\varphi(t,x,\tau,\xi')} a(t, x, \tau, \xi')\, \hat{g}(\tau, \xi')\, d\tau\, d\xi'$$

with the properties

$$(4.5)\qquad \begin{cases} P(Gg) \in C^\infty(\hat{W}), \\ B(Gg)_{x_n=0} - g \in C^\infty(\hat{W} \cap \{x_n = 0\}), \\ Gg\,|_{t<\hat{T}} \in C^\infty. \end{cases}$$

Here $\hat{T} = \min\{t$: there exists x' such that $(t, x') \in \operatorname{supp}\hat{\psi}\}$, while $\hat{W}$ is a small neighbourhood of $(\hat{t}, \hat{x})$. The integral (4.4) is interpreted as an oscillatory integral in the sense of Hörmander [7] with phase φ and amplitude

$$a \sim \sum_{k=1}^{\infty} a_{-k}(t, x, \tau, \xi'),$$

$a_{-k}(t, x\ \tau, \xi')$ being homogeneous of order $(-k)$ with respect to (τ, ξ') for $|\tau|+|\xi'|\neq 0$. The phase φ is chosen as a solution of the eikonal equation

$$(4.6)\qquad \begin{cases} \varphi_{x_n}=\lambda(t, x, \varphi_t, f_{x'})+\varepsilon\sqrt{\mu(t, x', \varphi_t, \varphi_{x'})}, \\ \varphi'|_{x_n=0}=t\tau+\langle x', \xi'\rangle. \end{cases}$$

Here $\varepsilon=\pm 1$ and the choice of ε will guarantee the outgoing condition $Gg_{\hat{t}<\hat{T}}\in C^{\infty}$. Notice that the singularities of Gg are propagating along the generalized rays of P issued from $WF(g)$. A ray $\gamma(\sigma)=(t(\sigma), x(\sigma), \tau(\sigma), \xi(\sigma))$ will be outgoing if

$$\frac{\dot{x}_n(\sigma)}{\dot{t}(\sigma)}=\frac{\partial p}{\partial \xi_n}\left(\frac{\partial p}{\partial \tau}\right)^{-1}>0 \quad \text{for} \quad \xi_n=\lambda+\varepsilon\sqrt{\mu}.$$

This condition is equivalent to

$$(4.7)\qquad \varepsilon\tau-l_t(\varepsilon\lambda+\sqrt{\mu})<0.$$

We must arrange (4.7) for $(t, x', \tau, \xi')\in WF(g)$. Without loss of generality we can assume that at $(\hat{t}, \hat{x})$ the relation (3.12) holds. Therefore for

$$\hat{\tau}=1-l_t(\hat{t}, \hat{x}')\omega_n,\ \hat{\xi}'=(l_t(\hat{t}, \hat{x}')\omega_n-1)\omega'$$

we obtain

$$(-\lambda+\sqrt{\mu})(\hat{t}, \hat{x}, \hat{\tau}, \hat{\xi}')=\omega_n(l_t\omega_n-1)(1+l_t\omega_n)^{-1}$$

since $\omega_n=\langle N/|N|, \omega\rangle<0$ at $(\hat{t}, \hat{x})$. Choosing $\varepsilon=-1$ and observing that $|l_t\omega_n|<1$, we deduce

$$-\hat{\tau}-l_t(\sqrt{\mu}-\lambda)(\hat{t}, \hat{x}, \hat{\tau}, \hat{\xi}')=-\frac{1-l_t\omega_n}{1+l_t\omega_n}<0.$$

Next we solve (4.6) with $\varepsilon=-1$ and determine inductively the symbols a_{-k} in order to arrange

$$P(e^{i\varphi}\sum_{k=1}^{\infty} a_{-k})\sim 0\ \left((|\tau|+|\xi'|)^{-m}\right) \text{ for all } m\in \mathbf{N}.$$

The asymptotic expansion formula for pseudodifferential operators yields the transport equations

$$(4.8)\qquad \sum_{j=0}^{n} \frac{\partial p}{\partial \xi_j}(x, d_x\varphi)\frac{\partial a_{-k}}{\partial x_j}+A_0(x, d_x\varphi)a_{-k}=-iP(a_{-k+1}),\ k\geqq 1.$$

Here for simplicity of notations we make $x_0=t$, $x=(x_0, x', x_n)$, $\tau=\xi_0$, $\xi=(\xi_0, \xi', \xi_n)$ and set $a_0=0$. Moreover,

$$A_0(x, d_x\varphi)=ip_1(x, d_x\varphi)+\sum_{|\alpha|=2}\frac{\partial^{\alpha}p}{\partial \xi^{\alpha}}(x, d_x\varphi)\partial_x^{\alpha}\varphi,$$

where p_1 is the symbol of P of first order. Remark that $A_0(x, d_x\varphi)$ is a real-valued function.

We solve the equations (4.8) with initial conditions

(4.9) $$a_{-1}|_{x_n=0}=i\tilde{p}^{-1}|_{x_n=0},$$

(4.10) $$a_{-k}|_{x_n=0}=-i\tilde{p}^{-1}L(a_{-k+1})|_{x_n=0},\quad k=2,\ 3,\dots$$

Here $\tilde{p}_1=b^2b_1^{-1}\sqrt{\mu}$, while L is a first order differential operator having the form

$$L(a)=b^2b_1^{-1}\left(\frac{\partial a}{\partial x_n}-\lambda(t,\ x,\ a_t,\ a_{x'})\right).$$

We can solve (4.8) since $\partial p/\partial\xi_n=2b(\xi_n-\lambda)=-2b\sqrt{\mu}\neq 0$ for $\xi_n=\lambda-\sqrt{\mu}$ and $(t,\ x,\ \tau,\ \xi')\in WF(g)$. This completes the construction of Gg.

To determine the trace $Gg|_{x_n=0}$, we introduce a pseudodifferential operator Π with principal symbol $q_1=i\tilde{p}^{-1}$. Next we replace $\partial_s^{(n-3)/2}v|_{\partial Q}$ in $I_2(\sigma)$ near $(\hat{t},\ \hat{x})$ by $\partial_s^{(n-3)/2}\Pi g$ modulo terms with lower order singularities. Putting

$$\psi=\langle x',\ \theta'\rangle+l(t,\ x')\theta_n-t,$$

we obtain mod $O(|\sigma|^{-2})$ the equality

$$\int\limits_{\partial Q\cap\hat{W}} e^{-i\sigma(\langle x,\theta\rangle-t)}\Phi(\langle x,\ \theta\rangle-t)\langle N',\ \theta\rangle\partial_s^{(n-3)/2}v\ dS$$

$$=\int\limits_D e^{-i\sigma\psi}\Phi(\psi)\langle N',\ \theta\rangle\Pi\partial_s^{(n-3)/2}g\,b_1 dx'dt$$

(4.11)
$$=\int\limits_D \Pi^*[\Phi(\psi)\langle N',\ \theta\rangle e^{-i\sigma\psi}]\partial_s^{(n-3)/2}g\,b_1\ dx'dt$$

$$=(i\sigma)^{-1}\int\limits_D e^{-i\sigma\psi}\Phi(\psi)\tilde{p}_1^{-1}(t,\ x')\langle N',\ \theta\rangle\partial_s^{(n-3)/2}g\ b_1\ dx'dt+O(|\sigma|^{-2}).$$

Here $D\subset\mathbb{R}^n$ is some open domain, Π^* is the operator adjoint to Π and

$$\tilde{p}_1(t,\ x')=b^2b_1^{-1}\sqrt{\mu}(t,x',0,\psi_t,\psi_{x'}).$$

It is important to note that the value of $\tilde{p}_1$ at $(\hat{t},\ \hat{x})$ does not depend on the choice of the local coordinates. To prove this, notice that the vector $\theta-\omega$ is parallel to N at $(\hat{t},\ \hat{x})$. Thus we obtain $\theta'=\omega'$ and the equality (3.12) holds at $(\hat{t},\ \hat{x})$. This implies

$$\tilde{p}_1(t,\ x)=b_1^{-1}[(1-l_t\theta_n)^2\theta_n^2]^{1/2}=b_1^{-1}(1-l_t\theta_n)\theta_n$$

since $1-l_t\theta_n>0$ and $\theta_n=\langle N/|N|,\ \theta\rangle>0$. On the other hand,

$$\tilde{p}_1(\hat{t},\ \hat{x})=\frac{\langle N,\theta\rangle}{b_1|N|}\left(\frac{1-l_t\theta_n}{1-l_t\omega_n}\right)(1-l_t\omega_n)=\langle N,\ \theta\rangle\left(\frac{1-\langle \mathrm{v},\theta\rangle}{1-\langle \mathrm{v},\omega\rangle}\right)(\hat{t},\ \hat{x}),$$

where $\mathrm{v}=-v_tv_x/|v_x|^2$ is the normal speed. The last equality follows from the observation that at $(\hat{t},\ \hat{x})$ we have

$$\frac{1-\langle \mathrm{v},\theta\rangle}{1-\langle \mathrm{v},\omega\rangle}=\frac{1-l_t\theta_n}{1-l_t\omega_n},\quad b_1^2|N|^2=b_1^2|v_x+v_t\omega|^2=(1-l_t\omega_n)^2.$$

By using Lemma 4.1, we write the leading term in (4.11) in the form

$$(4.12)\qquad (i\sigma)^{-1}\int_{-\infty}^{\infty}\partial_s^{(n-1)/2}\delta(s-\tau)\int_{\Gamma(\tau,\omega)\cap\hat{U}} e^{-i\sigma(\langle x,\theta-\omega\rangle+\tau)}$$

$$\cdot\hat{\psi}(x)\Phi(\langle x,\ \theta-\omega\rangle+\tau)\ \langle N',\ \theta\rangle\langle N,\ \omega\rangle\tilde{p}_1^{-1}(\langle x,\ \omega\rangle-\tau,\ x)\frac{d\Gamma}{|N|}d\tau$$

$$=-(-i\sigma)^{(n-3)/2}\int_{\Gamma(s,\omega)\cap\hat{U}} e^{-i\sigma(\langle x,\theta-\omega\rangle+s)}\hat{\psi}(x)\Phi(\langle x,\ \theta-\omega\rangle+s)$$

$$\cdot\langle N',\ \theta\rangle\frac{\langle N,\omega\rangle}{|N|}\tilde{p}_1^{-1}(\langle x,\ \omega\rangle-s,\ x)\left(\frac{1-\langle \mathrm{w},\theta\rangle}{1-\langle \mathrm{w},\omega\rangle}\right)^{(n-1)/2}d\Gamma$$

+ terms involving lower order powers of σ. Here $\hat{U}$ is a small neighbourhood of $\hat{x}$, $\hat{\psi}(x)\in C_0^\infty(\mathbf{R}^n)$ and has sufficiently close support to $\hat{x}$ and $\hat{\psi}(\hat{x})=1$.

Taking together the leading terms in (4.2) and (4.12), we must examine the integral

$$(4.13)\qquad (-i\sigma)^{(n-3)/2}\int_{\Gamma(s,\omega)\cap\hat{U}} e_s^{-i\sigma\varphi_s}\,\hat{\psi}(x)\Phi(\langle x,\ \theta-\omega\rangle+s)$$

$$\cdot\frac{\langle N',\theta\rangle}{|N'|}\frac{|N'|}{|N|}\left(\frac{1-\langle \mathrm{w},\theta\rangle}{1-\langle \mathrm{w},\omega\rangle}\right)^{(n-3)/2}\left[1-\langle N,\omega\rangle\tilde{p}_1^{-1}(\langle x,\omega\rangle-s,x)\left(\frac{1-\langle \mathrm{w},\ \theta\rangle}{1-\langle \mathrm{w},\omega\rangle}\right)\right]d\Gamma$$

with phase function $\varphi_s=\langle x,\ \theta-\omega\rangle+s$ and velocity $w=\partial x/\partial t$ introduced in (4.3). For sufficiently small $\hat{U}$ the phase φ_s has only one non-degenerate critical point $\hat{x}$ and

$$\frac{|N'|}{|N|}(\hat{t},\ \hat{x})=\left(\frac{1-\langle \mathrm{v},\theta\rangle}{1-\langle \mathrm{v},\omega\rangle}\right)(\hat{t},\ \hat{x}),$$

$$1-\langle N,\ \omega\rangle\tilde{p}_1^{-1}(\hat{t},\ \hat{x})\left(\frac{1-\langle \mathrm{v},\theta\rangle}{1-\langle \mathrm{v},\omega\rangle}\right)(\hat{t},\ \hat{x})=2,$$

because $\langle N(\hat{t},\ \hat{x}),\ \omega+\theta\rangle=0$. We apply the stationary phase method for the asymptotic of (4.13) with respect to σ. The phase $-\ \varphi_s$ has a minimum at $\hat{x}$ and $|\det \mathrm{Hess}\ \varphi_s(\hat{x})|=k(\hat{x})\,|\theta-\omega|^{n-1}$, where $k(\hat{x})$ is the Gauss curvature of $\Gamma(s,\ \omega)$ at $\hat{x}$. On the other hand,

$$\frac{N'}{|N'|}(\hat{t},\ \hat{x})=\frac{\theta-\omega}{|\theta-\omega|},\quad \frac{2\langle N',\theta\rangle}{|N'|}(\hat{t},\ \hat{x})=\frac{2(1-\langle\omega,\theta\rangle)}{2-2\langle\omega,\theta\rangle}=1$$

and

$$\frac{1-\langle \mathrm{w},\theta\rangle}{1-\langle \mathrm{w},\omega\rangle}(\hat{t},\hat{x})=\frac{1-\langle \mathrm{v},\theta\rangle}{1-\langle \mathrm{v},\omega\rangle}(\hat{t},\ \hat{x}).$$

Thus the leading term of (4.13) becomes

$$(2\pi)^{(n-1)/2}(-i\sigma)^{-1}|\theta-\omega|^{(3-n)/2}\left(\frac{1-\langle \mathrm{v},\theta\rangle}{1-\langle \mathrm{v},\omega\rangle}\right)^{(n-1)/2}$$
$$\cdot k(\hat{x})^{-1/2}e^{i\sigma(s+h)},$$

where v is the normal velocity of ∂Q at $(\hat{t},\ \hat{x})$.

The analysis near each point $x_j, j=1,\dots, M$ of $\mathcal{R}_0$ is completely analogous. Summing the contributions from all points $x_j \in \mathcal{R}_0$, we obtain the following leading term in (3.2)

$$I(\sigma)=\frac{1}{2}\left(\frac{i\sigma}{2\pi}\right)^{(n-1)/2}|\theta-\omega|^{(3-n)/2}e^{-i\sigma(s+h)} \tag{4.14}$$
$$\cdot\sum_{j=1}^{M}\left(\frac{1-\langle \mathrm{v}_j,\theta\rangle}{1-\langle \mathrm{v}_j,\omega\rangle}\right)^{(n-1)/2}k(x_j)^{-1/2}+O(|\sigma|^{(n-3)/2}).$$

Here v_j is the normal velocity of ∂Q at $(\langle x_j,\ \omega\rangle-s,\ x_j)$. Hence $K^{\#}$ is singular at $s'=s+h$.

For Dirichlet problem we repeat the above argument. For this purpose we construct a microlocal parametrix for the problem

$$\begin{cases} P(Gg)\in C^{\infty}(\hat{W}),\\ Gg|_{x_n=0}-g\in C^{\infty}(\hat{W}\cap\{x_n=0\}),\\ Gg|_{t<T}\in C^{\infty}.\end{cases}$$

We take Gg in the form (4.4) with the same phase φ and $a\sim\Sigma_{k=0}^{\infty}a_{-k}$. We solve transport equations (4.8) with initial conditions

$$a_0|_{x_n=0}=1,\quad a_{-k}|_{x_n=0}=0,\quad k=1,\ 2,\dots$$

Applying the operator B to Gg and taking the trace $x_n=0$, we get

$$B(Gg)|_{x_n=0}=\Pi_1 g \ \bmod\ C^{\infty},$$

where Π_1 is a first order pseudodifferential operator with principal symbol $-i\tilde{p}_1$ and

$$g=-\hat{\psi}(t,\ x')\,\delta(t+s-\langle x',\ \omega'\rangle-l(t,\ x')\omega_n).$$

Next we replace in (3.2) $\partial_s^{(n-3)/2}\partial_{\nu}\mathrm{v}|_{\partial Q}$ near $(\hat{t},\ \hat{x})$ by $\partial_s^{(n-3)/2}\Pi_1 g$ modulo lower order terms. By using Lemma 4.1 and a stationary phase argument, we deduce

$$I(\sigma)=-\frac{1}{2}\left(\frac{i\sigma}{2\pi}\right)^{(n-1)/2}|\theta-\omega|^{(3-n)/2}e^{-i\sigma(s+h)}$$
$$\cdot\sum_{j=1}^{M}\left(\frac{1-\langle \mathrm{v}_j,\theta\rangle}{1-\langle \mathrm{v}_j,\omega\rangle}\right)^{(n-1)/2}k(x_j)^{-1/2}+O(|\sigma|^{(n-3)/2}).$$

Notice that the factors $1-\langle v_j, \theta\rangle/1-\langle v_j, \omega\rangle$ are related to the Doppler factor (see [3]).

Finally, we obtain the following

Theorem 4.2. *Assume $n \geqq 3$, n odd, $\theta \neq \omega$ and (H_1) and (H_2) fulfilled. Then for regular directions $\theta-\omega$ the generalized scattering kernel $K^{\#}$ for s' sufficiently close to $s+h(s)$ has the form*

$$K^{\#} = \pm \frac{1}{2}(2\pi)^{(1-n)/2} \, |\theta-\omega|^{(3-n)/2} \tag{4.15}$$

$$\cdot \sum_{j=1}^{M} \left(\frac{1-\langle v_j, \theta\rangle}{1-\langle v_j, \omega\rangle}\right)^{(n-1)/2} k(x_j)^{-1/2} \delta^{(n-1)/2}(s'-s-h(s))$$

$$+ \textit{smoother terms.}$$

Here we take the sign $(-)$ for Dirichlet problem and the sign $(+)$ for Neumann problem.

For Dirichlet problem Theorem 4.2 has been obtained in [3].

5. Leading singularity of $K^{\#}$ (degenerate case)

In this section we deal with the degenerate case when the set $\mathscr{R}_0$ is not formed by a finite numbers of isolated points. The localization procedure exposed in section 3 does not depend on the form of $\mathscr{R}_0$ and we can find a finite number of functions $\psi_j \in C_0^{\infty}(R^n)$ such that

$$\operatorname{supp} \psi_j \cap \mathscr{R}_0 \neq \emptyset, \quad \sum_j \psi_j = 1 \text{ on } \mathscr{R}_0.$$

For Neumann problem we repeat the construction of a microlocal parametrix and we obtain locally

$$v|_{\partial Q \cap W_j} = \Pi_j(\tilde{\psi}_j \gamma) \mod C^{\infty}.$$

Here

$$\tilde{\psi}_j = \psi_j(x)|_{x \in \Gamma(s,\omega)}, \quad \gamma = -\langle N, \omega\rangle \delta'(t+s-\langle x, \omega\rangle)|_{\partial Q}.$$

Π_j is a first order pseudodifferential operator with non-vanishing symbol and W_j is a space-time neighbourhood of $\operatorname{supp} \psi_j$. Clearly for $(t, x) \in W_j \cap W_k \neq \emptyset$ we have

$$\Pi_j \tilde{\psi}_j \gamma = \Pi_k \tilde{\psi}_k \gamma.$$

Thus we obtain a global pseudodifferential operator Π on ∂Q such that

$$v|_{\partial Q} = \Pi \gamma \mod C^{\infty}.$$

Lemma 5.1. *In the local coordinates (t, x', x_n), where ∂Q is given locally by $x_n = 0$, the symbol q of Π admits an asymptotic expansion*

$$q(t, x', \tau, \xi') \sim \sum_{k=1}^{\infty} q_{-k}(t, x', \tau, \xi'),$$

where $q_{-k}(t, x', \tau, \xi')$ *are homogeneous of order* $(-k)$ *with respect to* (τ, ξ') *for* $|\tau|+|\xi'|\neq 0$ *and* q_{-k} *are real-valued for k even and purely imaginary-valued for k odd.*

Proof. In the local coordinates (t, x', x_n, τ, ξ') we use the construction of a microlocal parametrix Gg discussed in the previous section. The initial condition (4.9) shows that $a_{-1}\restriction_{x_n=0}$ and $\partial a_{-1}/\partial x_j\restriction_{x_n=0}$ are purely imaginery. Since $\partial p/\partial\xi_n(x, d_x\varphi)\neq 0$ the transport equation for a_{-1} implies that $\partial a_{-1}/\partial x_n\restriction_{x_n=0}$ and hence $L(a_{-1})\restriction_{x_n=0}$ are purely imaginary. This guarantees that $a_{-2}\restriction_{x_n=0}$ and $\partial a_{-2}/\partial x_j\restriction_{x_n=0}$ are real-valued. Thus inductively we obtain that $L(a_{-k})\restriction_{x_n=0}$ are real-valued for k even and purely imaginary for k odd. This completes the proof.

Now setting $W_0=\cap_j W_j$ we have

$$\begin{aligned}
&\int_{\partial Q} e^{-i\sigma(\langle x,\theta\rangle - t)}(\partial^j\Phi)(\langle x, \theta\rangle - t)\partial_s^{(n-3)}v \, dS \\
(5.1)\quad &= \int_{\partial Q\cap W_0} \Pi^*(e^{-i\sigma(\langle x,\theta\rangle - t)}(\partial^j\Phi)(\langle x, \theta\rangle - t))\partial_s^{(n-3)/2}v \, dS \\
&\sim \sum_{k=0}^{\infty} (-i\sigma)^{((n-3)/2)-k} \int_{\Gamma(s,\omega)\cap U_0} e^{-i\sigma(\langle x,\theta-\omega\rangle + s)}\tilde{\alpha}_{k,j}(x, s; \theta, \omega)\, d\Gamma
\end{aligned}$$

mod $0(|\sigma|^{-m})$ for all $m\in\mathbf{N}$. Here U_0 is a sufficiently small neighbourhood of $\mathscr{R}_0$ and $\tilde{\alpha}_{k,j}$ are real-valued functions. In the last equality we have applied the asymptotic expansion formula for the operator Π^* adjoint to Π. Using the argument of the previous section, it is easy to see that for $x\in\mathscr{R}_0$ we have

$$\tilde{\alpha}_{o,o}(x, s; \theta, \omega)=\frac{\langle N', \theta\rangle}{|N'|}\left(\frac{1-\langle \nu, \theta\rangle}{1-\langle \nu, \omega\rangle}\right)^{(n-1)/2}(\langle x, \omega\rangle - s, x).$$

Combining (3.2), (4.2) and (5.1), we conclude that

$$(5.2)\quad I(\sigma)\sim \sum_{j=0}^{\infty} (i\sigma)^{n-1-j} \int_{\Gamma(s,\omega)\cap U_0} e^{-i\sigma(\langle x,\theta-\omega\rangle + s)}\beta_j(x, s; \theta, \omega)\, d\Gamma,$$

where β_j are real-valued and supp $\beta_j\subset U_0$. Moreover, taking U_0 sufficiently small, we can arrange $\beta_0\geqq 0$. The form of $\tilde{\alpha}_{o,o}$ for $x\in\mathscr{R}_0$ and the fact that $N'/|N'|(t, x)=(\theta-\omega)|\theta-\omega|$ for $(t, x)\in\Sigma(s, \omega)$ and $x\in\mathscr{R}_0$ imply $\beta_0>0$ for $x\in\mathscr{R}_0$.

For the oscillatory integral (5.2) we can apply the result of H. Soga [18]. The phase $-\varphi_s=-\langle x, \theta-\omega\rangle - s$ has minimum only for the points $x\in\mathscr{R}_0$. The assumptions of Theorem 2 in [18] are satisfied and we conclude that $I(\sigma)$ is not a rapidly decreasing function as $|\sigma|\to\infty$. Consequently, $K^{\#}(s', \theta; s, \omega)$ is singular at $s'=s+h(s)$. For Dirichlet problem we use the same argument and we show that

$$\partial_{\nu^*}v\restriction_{\partial Q}=-\Pi_1\,\delta(t+s-\langle x, \omega\rangle)\restriction_{\partial Q},$$

where Π_1 is a first order pseudodifferential operator whose symbol r admits an asymptotic expansion

$$r \sim \sum_{k=-1}^{\infty} r_{-k}(t, x', \tau, \xi')$$

with r_{-k} real-valued for k even and purely imaginary-valued for k odd. Thus we have proved the following

Theorem 5.2. *Assume $n \geqq 3$, n odd, $\theta \neq \omega$ and* (H_1) *and* (H_2) *fulfilled. Then for fixed s, θ, ω we have*

$$s+h(s) \in \operatorname*{sing\,supp}_{s'} K^{\#}(s', \theta; s, \omega). \tag{5.3}$$

Remark 5.3. For all directions $\theta \neq \omega$ from the leading singularity of $K^{\#}$ we can determine the function $h(s)$ for all $s \in \mathrm{R}$. According to [3], the leading singularity of $K^{\#}(s', -\omega; s, \omega)$ determines the convex hull of $\mathscr{K}(t)$ for all $t \in \mathrm{R}$.

Remark 5.4. If the obstacle is stationary and $Q = \mathrm{R} \times \Omega$, then the set $\Gamma(s, \omega)$ does not depend on s and ω and $h = \max_{x \in \partial\Omega} \langle x, \theta - \omega \rangle$.

6. Leading singularity of $K^{\#}$ for even space dimensions

Throughout this section we assume $n \geqq 4$, n even, and the conditions (H_1) and (H_2) fulfilled. First we shall recall the translation representations of the group $U_0(t)$ related to the spaces

$$D^a_{\pm} = \{f \in H_0 : U_0(t)f = 0 \quad \text{for} \quad |x| \leqq \pm t + a, \pm t \geqq 0\}.$$

For $f = (f_1, f_2) \in \mathscr{S}(\mathrm{R}^n) \times \mathscr{S}(\mathrm{R}^n)$ denote by $\tilde{f}_i(s, \omega)$ the Radon transform of f_i, $i = 1, 2$, and set $\mathscr{R}f = \partial_s \tilde{f}_1(s, \omega) - \tilde{f}_2(s, \omega)$. Let $p_{\pm}(s) \in \mathscr{S}'(\mathrm{R})$ be tempered distribution with Fourier transform

$$\hat{p}_{\pm}(\sigma) = \begin{cases} \sigma^{(n-1)/2}, & \sigma \geqq 0, \\ -|\sigma|^{(n-1)/2} e^{\pm i((n-3)/2)\pi}, & \sigma < 0. \end{cases}$$

Therefore translation representations $\mathscr{R}_n^{\pm} : \mathscr{S}(\mathrm{R}^n) \times \mathscr{S}(\mathrm{R}^n) \to L^2(\mathrm{R} \times S^{n-1})$ have the form

$$\mathscr{R}_n^{\pm} f = d_n p_{\pm} * \mathscr{R}f, \quad d_n = 2^{-n/2} \pi^{-(n-1)/2}.$$

These maps can be extended as unitary maps from H_0 into $L^2(\mathrm{R} \times S^{n-1})$ (see [9], [11]).

Now let $k(s, \omega)$, $l(s, \omega) \in C_0^{\infty}(\mathrm{R} \times S^{n-1})$ be such that

$$k(s, \omega) = 0 \quad \text{for} \quad |s| \geqq a, \quad l(s, \omega) = 0 \quad \text{for} \quad |s| \geqq b.$$

Set

$$\varphi = (\mathscr{R}_n^-)^{-1} k, \quad (v_0(t,.), \quad \partial_t v_0(t,.)) = U_0(t)\varphi,$$

$$\psi = (\mathscr{R}_n^+)^{-1} l, \quad (\alpha(t,.), \quad \partial_t \alpha(t,.)) = U_0(t)\psi.$$

Clearly, $U_0(-t)\varphi \in D^{\rho}_{-}$ for $t > \rho + a$ and we determine

$$W_- \varphi = \lim_{t \to \infty} U(0, -t) U_0(-t) \varphi.$$

Next we set $(v(t,.),\ \partial_t v(t,.)) = U(t,\ 0)W_-\varphi$. To guarantee the existence of an asymptotic wave profile of v, we make the following assumption

$$\text{(H3)} \qquad \left[\begin{array}{l}\text{For each } k \in C_0^\infty(\mathrm{R}\times S^{n-1}) \text{ there exists the limit} \\ \quad S_1 k = \lim_{t\to\infty} U_0(-t)U(t,0)W_-(\mathscr{R}_n^-)^{-1}k.\end{array}\right.$$

Assuming (H_3), we set $\tilde{S}_1 k = \mathscr{R}_n^+ S_1 k$. It is not hard to see that the asymptotic wave profiles of $v(t,\ x)$ and $v_0(t,\ x)$ as $t\to\infty$ exist and these profiles coinside respectively with $(\tilde{S}_1 k)\ (s,\ \theta)$ and $(Kk)\ (s,\ \theta)$, where K is the Hilbert transform with respect to s (see [15] for more details). Notice that

$$\mathscr{R}_n^+ \varphi = K\mathscr{R}_n^- \varphi = Kk.$$

Next we need the following

Lemma 6.1. *For* $\tau > \rho + \max(\mathrm{a},\ \mathrm{b})$ *we have*

$$(6.1) \qquad \int_{\Omega(\tau)} [\langle \nabla_x(v-v_0),\ \nabla_x\alpha\rangle + (v-v_0)_t\alpha_t]\mathrm{d}x = \int_{\partial Q}\left(v\frac{\partial^2\alpha}{\partial v^*\partial t} - \frac{\partial v}{\partial v^*}\frac{\partial\alpha}{\partial t}\right)\mathrm{d}S.$$

The proof is exactly the same as that of Lemma 4 in [3] and we omit it. A finite speed of propagation argument shows that

$$v - v_0 = 0 \quad \text{for} \quad |x| > t + 2\rho + a.$$

On the other hand, $\alpha = 0$ for $|x| < t - b$. Thus the integration in (6.1) is over the set $\{x \in \mathrm{R}^n :\ -b + t < |x| < 2\rho + a + t\}$. Taking the limit $\tau\to\infty$ and exploiting the L^2 convergence of the asymptotic wave profiles of v and v_0, we deduce

$$(6.2) \qquad \int_{-\infty}^{\infty}\int_{S^{n-1}} ((\tilde{S}_1 - K)k)(s',\ \theta)l(s',\ \theta)\mathrm{d}s'\mathrm{d}\theta$$

$$= (-1)^{(n+1)/2}\int_{\partial Q}\left(\frac{\partial v}{\partial v^*}\frac{\partial\alpha}{\partial t} - v\frac{\partial^2\alpha}{\partial v^*\partial t}\right)\mathrm{d}S.$$

It is easy to see that

$$\frac{\partial\alpha}{\partial t}(t,\ x) = d_n\left\langle D_s^{\frac{n}{2}-1}\delta(t+s'-\langle x,\ \theta\rangle),\ \ (D_{s'} - iO)^{1/2}l(s',\ \theta)\right\rangle.$$

$$v(t,x) = -id_n\langle D_s^{\frac{n}{2}-1}w(t,x;\ s,\omega),\ \ (D_s + iO)^{1/2}k(s,\omega)\rangle.$$

Here $w_s(t,\ x;\ s,\ \omega) = \partial_s^2\Gamma^+\ (t,\ x;\ s,\ \omega)$, where Γ^+ is determined as in section 2. On the other hand, in the second equality we take $(D_s + iO)^{1/2}$ since $v|_{t<-s-\rho} = v_0$ and v_0 is expressed by $k = \mathscr{R}_n^-\varphi$.

Replacing in (6.2) v, $\partial v/\partial v^*$, $\partial\alpha/\partial t$, $\partial^2\alpha/\partial v^*\partial t$ by their representations, we arrive at the following

Theorem 6.2. *Assume* $n \geqq 4$ *even and the conditions* (H_1), (H_2), (H_3) *fulfilled. The kernel of the operator* $\tilde{S}_1 - K$ *has the representation*

$$(6.3)\qquad K^{\#}(s', \theta; s, \omega)=\frac{i}{2(2\pi)^{(n-1)}}(D_{s'}+iO)^{1/2}(D_s-iO)^{1/2}$$

$$\cdot\int_{\partial Q}\Big[\partial_s^{\frac{n}{2}-2}w_s(t, x; s, \omega)\,\partial_{s'}^{\frac{n}{2}-1}\partial_{\nu}\cdot\delta(t+s'-\langle x, \theta\rangle)$$

$$-\partial_s^{\frac{n}{2}-2}\partial_{\nu}\cdot w_s(t, x; s, \omega)\,\partial_{s'}^{\frac{n}{2}-1}\delta(t+s'-\langle x, \theta\rangle)\Big]dS.$$

Moreover, $K^{\#}$ is a C^{∞} function of s, θ, ω with values in the space of distributions in s'.

Definition 6.3. For n even we call $K^{\#}$ generalized scattering kernel.

In the case that the scattering operator S exists, we have

$$\tilde{S}_1=\mathscr{R}_n^+\circ S\circ(\mathscr{R}_n^-)^{-1}$$

and the assumption (H_3) is trivially satisfied.

Our purpose is to show that the assertions of Theorems 3.2 and 5.2 remain true for $n\geqq 4$ even. To establish (3.1), notice that by (ii) of Lemma 3.1 we have

$$\Big(\operatorname{supp}(\partial_s^{\frac{n}{2}-1}\partial_{\nu}\cdot w_s|_{\partial Q})\cup\operatorname{supp}(\partial_s^{\frac{n}{2}-1}w_s|_{\partial Q})\Big)$$

$$\subset\{(t, x; s, \omega): s\geqq\langle x, \theta\rangle-t-h(s)\}.$$

On the other hand, the operator $(D_s-iO)^{1/2}$ preserves the supports in the half lines $\{s: s\geqq s_0\}$. Thus we conclude that

$$\operatorname{supp}(D_s-iO)^{1/2}\int_{\partial Q}(\ldots)\,dS\subset\{(s', s): s'\leqq s+h(s)\}$$

by using the argument exposed in section 3. Next the operator $(D_{s'}+iO)^{1/2}$ preserves the supports in the half lines $\{s': s'\leqq s'_0\}$. This completes the proof of (3.1).

To examine the singularity of $K^{\#}$ at $s'=s+h(s)$, we set

$$(6.4)\qquad K_1^{\#}(s', \theta; s, \omega)=id_n^2\int_{\partial Q}[\partial_s^{\frac{n}{2}-2}w_s(t, x; s, \omega)\,\partial_{s'}^{\frac{n}{2}-1}\partial_{\nu}\cdot\delta(t+s'-\langle x, \theta\rangle)$$

$$-\partial_s^{\frac{n}{2}-1}\partial_{\nu}\cdot w_s(t, x; s, \omega)\,\partial_{s'}^{\frac{n}{2}-1}\delta(t+s'-\langle x, \theta\rangle)]\,dS.$$

We repeat the localization argument exposed in section 3 for $K_1^{\#}$ .The analysis of the singularity of $K_1^{\#}$ at $s'=s+h(s)$ is exactly the same as that of $K^{\#}$ for n odd. We claim that for fixed $\theta\neq\omega$

$$(6.5)\qquad \operatorname*{sing\,supp}_{s,s'}K^{\#}=\operatorname*{sing\,supp}_{s,s'}K^{\#}.$$

To prove this, consider the set $\mathscr{T}_0=\{(s', \sigma', s, \sigma): \sigma'\sigma=0\}$, where σ', σ are the variables dual respectively to s' and s. It is easy to see that

$$WF((\mathscr{R}_n^-\varphi)(s, \omega))\cap\{\sigma=0\}=\emptyset,\quad \varphi\in H_0,$$

$$WF((\mathscr{R}_n^+\psi)(s', 0))\cap\{\sigma'=0\}=\emptyset,\quad \psi\in H_0.$$

Since $(\tilde{S}_1 - K)k = \mathcal{R}_n^+ S_1 \varphi - K\mathcal{R}_n^- \varphi$ with $k = \mathcal{R}_n^- \varphi$, we obtain

$$WF(K^\# (s', \theta; s, \omega)) \cap \mathcal{T}_0 = \emptyset.$$

Moreover, if $WF(f) \cap \mathcal{T}_0 = \emptyset$, then the action of the operator $(D_s - iO)^{-1/2}(D_{s'} + iO)^{-1/2}$ on f is equivalent to that of a pseudo-differential operator. Therefore, the above claim follows from the equality

$$K_1^\# (s', \theta; s, \omega) = (D_s - iO)^{-1/2}(D_{s'} + iO)^{-1/2} K^\# (s', \theta; s, \omega).$$

Finally, we deduce (5.3) and the leading singularity of $K^\# (s', \theta; s, \omega)$ is $s + h(s)$.

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