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Orthomorphisms on Riesz Spaces of Riesz Space-Valued Functions

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Let E be an Archimedean Riesz space, X a nonempty set, and $\mathscr{F}(X, E)$ the Riesz space of all functions $X \to E$. Suppose that L is a Riesz subspace of $\mathscr{F}(X, E)$ satisfying $\{f(x): f \in L\} = E$ for all $x \in X$. It is shown that the f-algebra Orth (L) can be embedded via a function Φ into the f-algebra $\mathscr{F}(X$, Orth $(E))$, and that Φ embeds the center $Z(L)$ with its uniform norm isometrically into $\mathscr{F}_b(X, Z(E))$ equipped with the supremum norm. If E is a Banach lattice, X a locally compact Hausdorff topological space, and L a Riesz subspace of $\mathscr{C}(X, E)$ satisfying some additional conditions, then Φ maps Orth (L) (resp. $Z(L)$) onto the f-algebra $\mathscr{C}^s(X, Z(E))$ (resp. $\mathscr{C}^s_b(X, Z(E))$) of all strongly continuous (and bounded) $Z(E)$—valued functions on X.

Introduction

There is by now a large amount of literature devoted to the study of orthomorphisms on Archimedean Riesz spaces. See for example [1], [2], [3], [5], [8] and [9]. In some special cases the orthomorphisms on a Riesz space L are described directly in terms of L. This holds in particular when L is an Archimedean unital f-algebra [8, Theorem 3], and also in the following cases:
(1) L is a Riesz space that consists of real functions defined on some point set X [8, Theorem 4].
(2) L is a Riesz space of all real continuous with compact carrier defined on a locally compact Hausdorff topological space X [8, Theorem 5].

Let E be an Archimedean Riesz space and let $\mathscr{F}(X, E)$ be the Riesz space of all functions $X \to E$ (with pointwise defined operations). The main purpose of this note is to generalize (1) and (2) on Riesz subspaces L of $\mathscr{F}(X, E)$. Orthomorphisms on L are represented by certain functions $X \to \mathrm{Orth}(E)$ what is done by embedding Orth (L) as a Riesz subalgebra into the f-algebra $\mathscr{F}(X$, Orth$(E))$ with pointwise defined operations. The established embedding Φ gives also a more detailed description of the center $Z(L)$. Namely, it is shown that Φ maps $Z(L)$ equipped with the I_L-uniform norm isometrically into the f-algebra $\mathscr{F}_b(X, Z(L))$ of all bounded functions $X \to Z(L)$ normed by the supremum norm. It is proved also that in many cases Orth (L) can be completely described by recognizing the range of Φ.

Let E be a Banach lattice and X a locally compact Hausdorff topological space. Denote by $\mathscr{C}(X, E)$ the Riesz space of all continuous functions $X \to E$ (pointwise defined operations), and by $\mathscr{C}_c(X, E)$ its subspace of all functions with compact carrier. Furthermore, let $\mathscr{C}^s(X, Z(E))$ denote the Riesz algebra of all functions $X \to Z(E)$ which are continuous for the strong operator topology on $Z(E)$, and let $\mathscr{C}^s_b(X, Z(E))$ be its subalgebra of bounded functions. It is shown that if a Riesz subspace L of $\mathscr{C}(X, E)$ satisfies some additional condition, then orthomorphisms on L can be represented by functions of $\mathscr{C}^s(X, Z(E))$. The surjectivity of the embeddings Orth $(L) \to \mathscr{C}^s(X, Z(E))$, $Z(L) \to \mathscr{C}^s_b(X, Z(E))$ is investigated and confirmed in particular for spaces $L = \mathscr{C}(X, E), \mathscr{C}_c(X, E)$.

For the theory of Riesz spaces and orthomorphisms we refer the reader to [4], [9], [6] and [2]. The terminology and notation will be standard. If E is a Riesz space, Orth (E) denotes the f-algebra of all orthomorphisms on E and $Z(E)$ the center of E which consists of all $T \in$ Orth (E) dominated by a multiple of the identity I_E of E. The I_E-uniform norm on $Z(E)$ is defined by

$$\|T\|_{Z(E)} = \inf\{\lambda > 0 : |T| \leqq \lambda I_E\}.$$

It is well known that if E is a Banach lattice then Orth $(E) = Z(E)$, and the I_E-uniform norm on $Z(E)$ coincides with the operator norm.

Results

Let E be an Archimedean Riesz space, X a nonempty set, and L a Riesz subspace of $\mathscr{F}(X, E)$. We shall start with a simple but useful lemma.

Lemma 1. *Let $T \in$ Orth (L), $f, g \in L$, and $x \in X$. Then $f(x) = g(x)$ implies $T(f)(x) = T(g)(x)$.*

P r o o f. Put $h = |f - g|$, assume without loss of generality that $T \geqq 0$, and observe that it suffices to show that $T(h)x = 0$. To this end set

$$T_n = T - T \wedge nI_L, \quad n = 1, 2, \ldots,$$

note that $T_n(h)(x) = T(h)(x)$ (when $f(x) = g(x)$), and use the well known inequality $0 \leqq T_n \leqq (1/n)T^2$ to get

$$0 \leqq T(h)(x) = T_n(h)(x) \leqq (1/n)T^2(h)(x), \quad n = 1, 2, \ldots$$

Since E is Archimedean this yields the desired equality $T(h)(x) = 0$.

The following result generalizes [8, Theorem 4].

Theorem 1. *Let L be a Riesz subspace of $\mathscr{F}(X, E)$ such that $\{f(x) : f \in L\} = E$ for all $x \in X$. Then* Orth (L) *is embedded as a Riesz subalgebra into the f-algebra $\mathscr{F}(X$,* Orth $(E))$ *via the function Φ :* Orth $(L) \to \mathscr{F}(X$, Orth $(E))$ *defined by*

$$\Phi(T)(x)(u) = T(f)(x); \quad f \in L,\ f(x) = u \in E.$$

The restriction $\Phi_b = \Phi|_{Z(L)}$ embeds $Z(L)$ isometrically into $\mathscr{F}_b(X, Z(E))$.

Proof. Let $T\in\mathrm{Orth}\,(L)$ and $x\in X$. By Lemma 1 $\Phi(T)(x)$ is a well defined linear mapping on E which is also regular (hence order bounded) since $\Phi(T)(x)=\Phi(T^+)(x)-\Phi(T^-)(x)$ and the maps $\Phi(T^+)(x)$, $\Phi(T^-)(x)$ are positive. To see that $\Phi(T)(x)$ is an orthomorphism on E we may therefore assume without loss of generality that $T\geqq 0$.

Suppose $u, v\in E$ satisfy $u\wedge v=0$. Take $f, g\in L$ such that $f(x)=u$, $g(x)=v$, and put

$$f_1=|f|-|f|\wedge|g|, \quad g_1=|g|-|f|\wedge|g|.$$

Since $f_1\wedge g_1=0$ and $T\in\mathrm{Orth}\,(L)$, it follows that

$$\Phi(T)(x)(u)\wedge v=T(f_1)(x)\wedge g_1(x)=(T(f_1)\wedge g_1)(x)=0,$$

thus $\Phi(T)$ is band preserving as claimed.

It can be easily verified that Φ is an injective homomorphism of Riesz algebras, hence Φ embeds Orth (L) into $\mathscr{F}(X, \mathrm{Orth}\,(E))$.

In order to prove the remaining part of the theorem suppose that $T\in Z(L)$ and observe that

$$|T|\leqq\lambda I_L\,(0<\lambda\in\mathrm{R})\leftrightarrow|\Phi(T)|(z)\leqq\lambda I_E \text{ for all } z\in X.$$

It follows that Φ_b embeds $Z(L)$ into $\mathscr{F}_b(X, Z(E))$ and

$$\|T\|_{Z(L)}=\sup_{z\in X}\|\Phi(T)(z)\|_{Z(E)}=\|\Phi(T)\|_\infty,$$

which completes the proof.

If $L=\mathscr{F}(X, E)$, then Φ is an isomorphism of Orth (L) onto $\mathscr{F}(X, \mathrm{Orth}\,(E))$. Its inverse $\Psi=\Phi^{-1}$ satisfies

$$\Psi(\rho)(f)(x)=\rho(x)(f(x))$$

for all $\rho\in\mathscr{F}(X, \mathrm{Orth}\,(E))$, $f\in\mathscr{F}(X, E)$, $x\in X$.

Let A be a Riesz subalgebra of $\mathscr{F}(X, \mathrm{Orth}\,(E))$. A Riesz subspace L of $\mathscr{F}(X, E)$ is said to be A-invariant whenever L is invariant for all $\Psi(\rho)$, $\rho\in A$. Note that if L is A-invariant, then

$$\Psi_A(\rho)(f)=\Psi(\rho)(f); \quad \rho\in A, \quad f\in L$$

defines a Riesz algebra homomorphism $\Psi_A: A\to\mathrm{Orth}\,(L)$. It is trivial to verify that $A\subset\mathscr{F}_b(X, Z(E))$ implies that the range of Ψ_A is contained in $Z(L)$. Note that $a=\Phi(\Psi_A(a))$ for all $a\in A$, so the next result follows easily.

Corollary 1. *Let A be a Riesz subalgebra of $\mathscr{F}(X$, Orth $(E))$ and let L be an A-invariant Riesz subspace of $\mathscr{F}(X, E)$ such that $\{f(x): f\in L\}=E$ for all $x\in X$. If A contains the range of Φ, then Φ maps* Orth (L) *isomorphically onto A. If in addition A is a subset of $\mathscr{F}_b(X, Z(E))$, then Φ_b is an isometric isomorphism of $Z(L)$ onto A.*

Let from now on E be a Banach lattice, X a locally compact Hausdorff topological space, and let L be a Riesz subspace of $\mathscr{C}(X, E)$, the Riesz space of all continuous E-valued functions on X under the pointwise ordering. Denote by

$\mathscr{C}^s(X, Z(E))$ the vector subspace of $\mathscr{F}(X, Z(E))$ consisting of all functions which are continuous for the strong operator topology $\mathscr{C}$ on $Z(E)$. Observing that σ is locally solid it is easy to see that $\mathscr{C}^s(X, Z(E))$ is a Riesz subspace of $\mathscr{F}(X, Z(E))$. By the uniform boundedness principle every $\rho\in\mathscr{C}^s(X, Z(E))$ is bounded on each compact subset of X, hence locally bounded. It follows by standard arguments that $\mathscr{C}^s(X, Z(E))$ is closed for pointwise defined multiplication and therefore a Riesz subalgebra of $\mathscr{F}(X, Z(E))$. Let $\mathscr{C}^s_b(X, Z(E))$ denote the subalgebra of $\mathscr{C}^s(X, Z(E))$ consisting of all bounded functions. It is routine to verify that under the supremum norm

$$\|\rho\|_\infty = \sup_{x\in X} \|\rho(x)\|_{Z(E)}, \quad \rho\in\mathscr{C}^s_b(X, Z(E))$$

$\mathscr{C}^s_b(X, Z(E))$ becomes a Banach lattice subalgebra of $\mathscr{F}_b(X, Z(E))$.

Theorem 2. *Let L be a Riesz subspace of $\mathscr{C}(X, E)$ satisfying the following condition. For each $x_0\in X$ and $u\in E$ there exists a neighborhood U of x_0 and a function $f\in L$ such that $f(x)=u$ for all $x\in U$.*

If L is $\mathscr{C}^s(X, Z(E))$-invariant, then Orth (L) *is isomorphic to the f-algebra* $\mathscr{C}^s(X, Z(E))$. If L is $\mathscr{C}^s_b/(X, Z(E))$*-invariant, then $Z(L)$ is isometrically isomorphic to the Banach lattice algebra* $\varphi^s_b(X, Z(E))$.

Proof. According to Corollary 1 it suffices to show that $A=\mathscr{C}^s(X, Z(E))$ (resp. $A=\mathscr{C}^s_b(X, Z(E))$) contains the range of Φ. To this end let $T\in$Orth (L) (resp. $T\in Z(L)$), fix any $x_0\in X$, $u\in E$, and choose an element $f\in L$ such that $f(x)=u$ for all x from some neighborhood U of x_0. Then

$$\Phi(T)(x)(u)-\Phi(T)(x_0)(u)=T(f)(x)-T(f)(x_0)$$

holds for all $x\in U$. It follows that $\Phi(T)\in A$, so the proof is complete.

Our next result provides us with some $\mathscr{C}^s(X, Z(E))$-invariant and some $\mathscr{C}^s_b(X, Z(E))$-invariant Riesz subspaces of $\mathscr{C}(X, E)$.

Lemma 2. *The space* $\mathscr{C}(X, E)$ as well as every order ideal of $\mathscr{C}_c(X, E)$ is $\mathscr{C}^s(X, Z(E))$-invariant. Every order ideal of $\mathscr{C}(X, E)$ is $\mathscr{C}^s_b(X, Z(E))$-invariant.

Proof. Let $\rho\in\mathscr{C}^s(X, Z(E))$ and $f\in\mathscr{C}(X, E)$. Fix some $x_0\in X$, choose a compact neighborhood K of x_0 and observe that

$$\|\rho\|_K=\sup\{\|\rho(x)\| : x\in K\}<\infty.$$

The estimate

$$\|\rho(x)f(x)-\rho(x_0)f(x_0)\|\leqq\|\rho(x)\|\;\|f(x)-f(x_0)\|+\|(\rho(x)-\rho(x_0))f(x_0)\|$$

shows that the function $x\mapsto\rho(x)(f(x))$ is continuous at x_0, hence $\mathscr{C}(X, E)$ is $\mathscr{C}^s(X, Z(E))$-invariant.

If f has compact carrier K, then

$$|\rho(x)(f(x))|\leqq\|\rho\|_K|f(x)| \quad \text{for all } x\in X,$$

therefore every order ideal of $\mathscr{C}_c(X, E)$ is $\mathscr{C}^s(X, Z(E))$-invariant.

If $\rho\in\mathscr{C}^s_b(X, E)$, then

$$|\rho(x)(f(x))| \leqq \|\rho\| \, |f(x)| \quad \text{for all } x \in X,$$

and the remaining conclusion of lemma follows easily.

Denote by $\mathscr{C}_b(X, E)$ the Banach lattice of all bounded E-valued continuous functions on X with the supremum norm, and by $\mathscr{C}_0(X, E)$ its Banach sublattice of functions vanishing at infinity. Observe that $\mathscr{C}(X, E)$, $\mathscr{C}_b(X, E)$, $\mathscr{C}_0(X, E)$ and $\mathscr{C}_c(X, E)$ satisfy the conditions of Theorem 2, so the next generalization of some classical results follows.

Corollary 2. *Let X be a locally compact Hausdorff topological space and let E be a Banach lattice. Then the f-algebras* Orth $(\mathscr{C}(X, E))$ *and* Orth $(\mathscr{C}_c(X, E))$ *are isomorphic to the f-algebra* $\mathscr{C}^s(X, Z(E))$, while $Z(\mathscr{C}(X, E))$, $Z(\mathscr{C}_b(X, E))$, $Z(\mathscr{C}_0(X, E))$ and $Z(\mathscr{C}_c(X, E))$ are isometrically isomorphic to $\mathscr{C}_b^s(X, Z(E))$.

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