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Quadrature Formulae for Entire Functions with 2-Periodic Data

Dimiter P. Dryanov

Presented by P. Kenderov

Quadrature formulae with equidistant nodes involving 2-periodic data of not necessarily consecutive derivatives are considered in the present paper. Theorem 4.1 answers the question of existence and uniqueness of such formulae which have the highest degree of precision with respect to entire functions of exponential type. The quadrature formulae of the highest degree of precision are obtained without knowing the corresponding interpolation process.

1. Introduction

Quadrature formulae with equidistant nodes involving 2-periodic data of not necessarily consecutive derivatives are considered in the present paper. Theorem 4.1 answers the question of existence and uniqueness of such formulae which have highest degree of precision with respect to entire functions of exponential type. The quadrature formulae of highest degree of precision are obtained without knowing the corresponding interpolation process. Former results of R. P. Boas [2], R. Kress [5], P. Olivier and Q. I. Rahman [6] and the result given in [3] are particular cases of our result. A representation of the remainder for functions belonging to a certain Sobolev space is given.

2. Problem formulation

Let R be the real axis and denote by $B_{\gamma,1}$ the set of all entire functions of exponential type γ which belong to $L_1(R)$. Further, let $\vec{k}=(k_0, k_1,\dots,k_{m-1})$ and $\vec{k}'=(k'_0, k'_1,\dots,k'_{m_1-1})$, $0=k_0<k_1<k_2<\dots<k_{m-1}$ and $0\leqq k'_0<k'_1<k'_2<\dots<k'_{m_1-1}$ $(m\geqq 1,\ m_1\geqq 1)$ where k_s, $s=1, 2,\dots,m-1$ and k'_j, $j=1, 2,\dots,m_1-1$ are integers. We shall denote by S the class of all complex valued functions which are defined on the real axis R.

Suppose that for a function $f\in S$ we are given the following 2-periodic information:

$$(2.1)\qquad f^{(k_s)}\left(\frac{2\nu\pi}{\sigma}\right),\quad \nu=0,\ \pm 1,\ \pm 2,\dots,s=0,\ 1,\ 2,\dots,m-1,$$

$$f^{(k'_j)}\left(\frac{2\nu+1}{\sigma}\pi\right),\quad \nu=0,\ \pm 1,\ \pm 2,\ldots,\ j=0,\ 1,\ 2,\ldots,m_1-1.$$

Let us determine the following expressions based on the data (2.1):

$$f^{(k_s)}_{\sigma,e}=\sum_{\nu=-\infty}^{\infty} f^{(k_s)}\left(\frac{2\nu\pi}{\sigma}\right),\quad s=0,\ 1,\ldots,m-1 \tag{2.2}$$

and

$$f^{(k'_j)}_{\sigma,0}=\sum_{\nu=-\infty}^{\infty} f^{(k'_j)}\left(\frac{2\nu+1}{\sigma}\pi\right),\quad j=0,\ 1,\ldots,m_1-1.$$

The problem is to find a quadrature formula, determined by the 2-periodic information (2.1) of the form

$$\int_{-\infty}^{\infty} f(x)\mathrm{d}x\approx\frac{2\pi}{\sigma}\sum_{s=0}^{m-1}\frac{c_s}{\sigma^{k_s}}f^{(k_s)}_{\sigma,e}+\frac{2\pi}{\sigma}\sum_{j=0}^{m_1-1}\frac{d_j}{\sigma^{k'_j}}f^{(k'_j)}_{\sigma,0} \tag{2.3}$$

c_s, d_j are complex numbers,
so that this quadrature formula possesses highest degree of precision.

Definition 2.1. A quadrature formula of the form (2.3) has entire degree of precision $\gamma>0$, when it is precise for every function $f\in B_{\gamma,1}$ and for every $\delta>0$ there exists a function $g\in B_{\gamma+\delta,1}$ for which the quadrature formula is not precise.

We shall use HEDP for "highest entire degree of precision", EDP for "entire degree of precision" and Q. F. for "quadrature formula".

The solution of this problem will depend on the number of even integers in the vectors $\vec{k}$ and $\vec{k}'$. Thus let ω_e and ω'_e denote the number of even integers in the sets $\{0=k_0<k_1<\ldots<k_{m-1}\}$ and $\{0\leq k'_0<k'_1<\ldots<k'_{m_1-1}\}$, respectively. Let ω_0 and ω'_0 denote the number of the odd integers in the above sets, so that $\omega_e+\omega_0=m$ and $\omega'_e+\omega'_0=m_1$.

3. Auxiliary results

Let $\hat{g}$ denote the Fourier transform of a function g. Further, let $k^*=\max(k_{m-1}, k'_{m_1-1})$ and $W^{k^*}_1\,BV$ be the usual Sobolev space with $f^{(k^*)}\in BV(R)$. Let $\omega\geq 0$ be integer and $\vec{c}=(c_0, c_1,\ldots,c_{m-1})$ and $\vec{d}=(d_0, d_1,\ldots,d_{m_1-1})$.

Lemma 3.1 (Poisson summation formula), [7]. *If $f\in S\cap L_1\cap BV$ then for every $a\in R$*

$$\sum_{k=-\infty}^{\infty} f\left(a+\frac{2k\pi}{\sigma}\right)=\frac{\sigma e^{ik\sigma a}}{2\pi}\sum_{k=-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)e^{-ik\sigma x}\mathrm{d}x \tag{3.1}$$

where $f(x)=\dfrac{f(x+0)+f(x-0)}{2}$ for $x\in R$.

Lemma 3.2 [4]. *Let $0<t_1<t_2<\ldots<t_q$ and $m_1<m_2<\ldots<m_q$ be distinct real numbers. Then*

$$(3.2) \qquad \det \begin{vmatrix} t_1^{m_1} & t_1^{m_2} & \dots & t_1^{m_q} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ t_q^{m_1} & t_q^{m_2} & \dots & t_q^{m_q} \end{vmatrix} > 0.$$

Lemma 3.3. *Let* $0<t_1<t_2<t_3<\dots<t_{q+q_1}$, $0<b_1<b_2<b_3<\dots<b_{q+q_1}$, $t_0>0$, $b_0\geqq 0$, $m_1<m_2<m_3<\dots<m_{q+1}$ *and* $n_1<n_2<n_3<\dots<n_{q_1}$, *are real numbers. Then*

$$\operatorname{sign}\left(\det \begin{vmatrix} t_1^{m_1} & t_1^{m_2} & \dots & t_1^{m_q} & -b_1^{n_1} & -b_1^{n_2}\dots & -b_1^{n_{q_1}} \\ t_2^{m_1} & t_2^{m_2} & \dots & t_2^{m_q} & b_2^{n_1} & b_2^{n_2}\dots & b_2^{n_{q_1}} \\ t_3^{m_1} & t_3^{m_2} & \dots & t_3^{m_q} & -b_3^{n_1} & -b_3^{n_2}\ . & -b_3^{n_{q_1}} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ t_{q+q_1}^{m_1} & t_{q+q_1}^{m_2} & \dots & t_{q+q_1}^{m_q} & (-1)^{q+q_1} b_{q+q_1}^{n_1} & (-1)^{q+q_1} b_{q+q_1}^{n_2} \dots & (-1)^{q+q_1} b_{q+q_1}^{n_{q_1}} \end{vmatrix}\right)$$

$$=\operatorname{sign}\left(\det \begin{vmatrix} t_0 & 0 & \dots 0 & b_0 & 0 & \dots & 0 \\ t_1^{m_1} & t_1^{m_2} & \dots t_1^{m_{q+1}} & -b_1^{n_1} & -b_1^{n_2} & \dots & -b_1^{n_{q_1}} \\ t_2^{m_1} & t_2^{m_2} & \dots t_2^{m_{q+1}} & b_2^{n_1} & b_2^{n_2} & \dots & b_2^{n_{q_1}} \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ t_{q+q_1}^{m_1} & t_{q+q_1}^{m_2} & \dots t_{q+q_1}^{m_{q+1}} & (-1)^{q+q_1} b_{q+q_1}^{n_1} & (-1)^{q+q_1} b_{q+q_1}^{n_2} & \dots & (-1)^{q+q_1} b_{q+q_1}^{n_{q_1}} \end{vmatrix}\right)=$$

$$=(-1)^{qq_1+q_1(q_1+1)/2}.$$

Proof of Lemma 3.3. Let us denote the first determinant by D_1 and the second by D_2. By Laplace's rule on determinants we get for the first determinant:

$$D_1= \sum_{1\leq i_1<i_2<\dots<i_q\leq q+q_1} (-1)^{\frac{q(q+1)}{2}}(-1)^{i_1+i_2+\dots+i_q} D\binom{i_1 i_2 \dots i_q}{1\ 2 \dots q} D\binom{i'_1 i'_2 \dots i'_{q_1}}{1\ 2 \dots q_1}.$$

By Lemma 3 we obtain:

$$\operatorname{sign} D\binom{i_1 i_2 \dots i_q}{1\ 2 \dots q}=+1 \text{ and } \operatorname{sign} D\binom{i'_1 i'_2 \dots i'_{q_1}}{1\ 2 \dots q_1}=(-1)^{i'_1+i'_2+\dots+i'_{q_1}} \text{ and}$$

because of $\sum_{i=1}^{q} i_j+\sum_{j=1}^{q_1} i'_j=\frac{(q+q_1)(q+q_1+1)}{2}$ we obtain

sign $D_1=(-1)^{qq_1+q_1(q_1+1)/2}$. The sign D_2 can be obtained using the same arguments.

Lemma 3.4. *For the existence and the uniqueness of a formula* $A(f, \vec{k}, \vec{k}', \sigma, \omega, \vec{c}, \vec{d})$, $\vec{c}=(c_0, c_1, c_2,\dots,c_{m-1})$, $\vec{d}=(d_0, d_1,\dots,d_{m_1-1})$ *of the form*

$$\frac{2\pi}{\sigma}\sum_{s=0}^{m-1}\frac{c_s}{\sigma^{k_s}} f_{\sigma,e}^{(k_s)}+\frac{2\pi}{\sigma}\sum_{j=0}^{m_1-1}\frac{d_j}{\sigma^{k'_j}} f_{\sigma,0}^{(k'_j)}-\int_{-\infty}^{\infty} f(x)\mathrm{d}x$$

$$(3.3)\qquad =\sqrt{2\pi}\sum_{s=0}^{m-1}\frac{c_s}{\sigma^{k_s}}\sum_{|\nu|\geqq\omega+1}\hat{f}^{(k_s)}(\sigma\nu)$$

$$+\sqrt{2\pi}\sum_{j=0}^{m_1-1}\frac{d_j}{\sigma^{k'_j}}\sum_{|\nu|\geqq\omega+1}(-1)^\nu\hat{f}^{(k'_j)}(\sigma\nu),$$

c_s, d_j are complex numbers, for some integer $\omega\geqq 0$ and for every function $f\in W_1^{k^*}BV$ there are three possibilities:

1) *If $\omega<\omega_e+\omega'_e-1$ then the formula of the form* (3.3) *is not unique.*

2) *If $\omega>\omega_e+\omega'_e-1$ then such formula does not exist.*

3) *If $\omega=\omega_e+\omega'_e-1$ according to $\omega_0+\omega'_0$ two subcases exist:*

a) *if $\omega_0+\omega'_0\leqq\omega_e+\omega'_e-1$ then this formula is uniquely determined and for $\vec{c}$ and $\vec{d}$ we have the following:*

– $c_0+\varepsilon(k'_0)d_0=1$, $\varepsilon(0)=1$, $\varepsilon(k'_0)=0$ *for* $k'_0>0$,

– $c_s=0$ *for* k_s *odd and* $d_j=0$ *for* k'_j *odd,*

– *c_s and d_j are real numbers for k_s and k'_j even and they are uniquely determined by the system* (3.11) *for $\omega=\omega_e+\omega'_e-1$.*

b) *if $\omega_0+\omega'_0>\omega_e+\omega'_e-1$ then there are many formulae of the form* (3.3) *and for $\vec{c}$ and $\vec{d}$ of these formulae we have:*

– *c_s for k_s even and d_j for k'_j even are real and uniquely determined by the system* (3.11),

– *a necessary and sufficient condition for c_s, d_j for k_s, k'_j odd to be coefficients of the formula* (3.3) *is c_s and d_j for k_s, k'_j odd to satisfy the systems* (3.13) *and* (3.14).

Proof of Lemma 3.4. By Lemma 3.1 with $a=0$ and $f^{(k_s)}$ for f one gets

$$(3.4)\qquad \frac{2\pi}{\sigma}f^{(k_s)}_{\sigma,e}=\sqrt{2\pi}\sum_{\nu=-\omega}^{\omega}(\nu\sigma i)^{k_s}\hat{f}(\sigma\nu)+\sqrt{2\pi}\sum_{|\nu|\geqq\omega+1}\hat{f}^{(k_s)}(\sigma\nu).$$

By Lemma 3.1 with $a=\frac{\pi}{\sigma}$ and $f^{(k'_j)}$ for f one gets

$$(3.5)\qquad \frac{2\pi}{\sigma}f^{(k'_j)}_{\sigma,0}=\sqrt{2\pi}\sum_{\nu=-\omega}^{\omega}(-1)^\nu(\nu\sigma i)^{k'_j}\hat{f}(\sigma\nu)+\sqrt{2\pi}\sum_{|\nu|\geqq\omega+1}\hat{f}^{(k'_j)}(\sigma\nu).$$

Multiplying (3.4) by c_s and (3.5) by d_j and summing them in s from 0 to $m-1$ and in j from 0 to m_1-1 we obtain the formula

$$\frac{2\pi}{\sigma}\sum_{s=0}^{m-1}\frac{c_s}{\sigma^{k_s}}f^{(k_s)}_{\sigma,e}+\frac{2\pi}{\sigma}\sum_{j=0}^{m_1-1}\frac{d_j}{\sigma^{k'_j}}f^{(k'_j)}_{\sigma,0}=\sqrt{2\pi}\sum_{s=0}^{m-1}c_s\sum_{\nu=-\omega}^{\omega}(i\nu)^{k_s}\hat{f}(\sigma\nu)$$

$$+\sqrt{2\pi}\sum_{j=0}^{m_1-1} d_j \sum_{\nu=-\omega}^{\omega} (-1)^\nu (i\nu)^{k'_j} \hat{f}(\sigma\nu) + \sqrt{2\pi}\sum_{s=0}^{m-1} \frac{c_s}{\sigma^{k_s}} \sum_{|\nu|\geqq\omega+1} \hat{f}^{(k_s)}(\sigma\nu)$$

$$+\sqrt{2\pi}\sum_{j=0}^{m_1-1} \frac{d_j}{\sigma^{k'_j}} \sum_{|\nu|\geqq\omega+1} (-1)^\nu \hat{f}^{(k'_j)}(\sigma\nu). \tag{3.6}$$

By the formula (3.6) the following necessary and sufficient condition for the existence of $A(f, \vec{k}, \vec{k}', \sigma, \omega, \vec{c}, \vec{d})$ is obtained:

The equalities

$$\int_{-\infty}^{\infty} f(x)\,dx = \sqrt{2\pi}\sum_{s=0}^{m-1} c_s \sum_{\nu=-\omega}^{\omega} (i\nu)^{k_s} \hat{f}(\sigma\nu) + \sqrt{2\pi}\sum_{j=0}^{m_1-1} d_j \sum_{\nu=-\omega}^{\omega} (-1)^\nu (i\nu)^{k'_j} \hat{f}(\sigma\nu)$$

$$=\sqrt{2\pi}\sum_{\nu=-\omega}^{\omega} \hat{f}(\sigma\nu)\Big(\sum_{s=0}^{m-1} c_s (i\nu)^{k_s} + (-1)^\nu \sum_{j=0}^{m_1-1} d_j (i\nu)^{k'_j}\Big) \tag{3.7}$$

hold for every function $f \in W_1^{k^*} BV$.

Since $B_{\sigma(\omega+1),1} \subset W_1^{k^*} BV$ we can choose trial functions only from $B_{\sigma(\omega+1),1}$ and this will be important for the next results. Let us choose the functions $f_n(x)$ such that $\hat{f}_n \in C_0^\infty[(n-1)\sigma, (n+1)\sigma]$, $\hat{f}(n\sigma) \neq 0$ and $f = f_n$ for $n = 0, \pm 1, \pm 2, \dots, \pm\omega$. Because of $f_n \in B_{\sigma(\omega+1),1}$ (by a modification of Paley-Wiener's theorem, (see [1]) after the substitutions $f = f_n$, $n = 0, \pm 1, \pm 2, \dots, \pm\omega$ we obtain the following necessary and sufficient condition for the existence of the formula $A(f, \vec{k}, \vec{k}', \sigma, \omega, \vec{c}, \vec{d})$:

The vectors $\vec{c}$ and $\vec{d}$ have to satisfy the system:

$$\left|\begin{array}{l} c_0 + \varepsilon(k'_0) d_0 = 1 \\ \sum_{s=0}^{m-1} c_s (i\nu)^{k_s} + (-1)^\nu \sum_{j=0}^{m_1-1} (i\nu)^{k'_j} d_j = 0 \\ \nu = \pm 1, \pm 2, \dots, \pm\omega. \\ \varepsilon(0) = 1, \ \varepsilon(k'_0) = 0 \text{ if } k'_0 > 0. \end{array}\right. \tag{3.8}$$

Let $c_s = c'_s + ic''_s$, $s = 0, 1, 2, \dots, m-1$ and $d_j = d'_j + id''_j$, $j = 0, 1, 2, \dots, m_1 - 1$. Then the system (3.8) can be splitted to the following two systems:

$$
(3.9)\quad \left|\begin{array}{l}
c_0' + \varepsilon(k_0')d_0' = 1 \\[2mm]
\displaystyle\sum_{\substack{s=0\\ k_s\ \text{even}}}^{m-1} c_s'(-1)^{\frac{k_s}{2}} v^{k_s} + \sum_{\substack{s=1\\ k_s\ \text{odd}}}^{m-1} c_s''(-1)^{\frac{k_s+1}{2}} v^{k_s} \\[2mm]
\displaystyle +(-1)^v \sum_{\substack{j=0\\ k_j'\ \text{even}}}^{m_1-1} d_j'(-1)^{\frac{k_j'}{2}} v^{k_j'} + (-1)^v \sum_{\substack{j=0\\ k_j'\ \text{odd}}}^{m_1-1} d_j''(-1)^{\frac{k_j'+1}{2}} v^{k_j'} = 0 \\[2mm]
v = \pm 1,\ \pm 2,\ \pm 3, \ldots, \pm\omega
\end{array}\right.
$$

and

$$
(3.10)\quad \left|\begin{array}{l}
c_0'' + \varepsilon(k_0')d_0'' = 0 \\[2mm]
\displaystyle\sum_{\substack{s=0\\ k_s\ \text{even}}}^{m-1} c_s''(-1)^{\frac{k_s}{2}} v^{k_s} + \sum_{\substack{s=1\\ k_s\ \text{odd}}}^{m-1} c_s'(-1)^{\frac{k_s-1}{2}} v^{k_s} \\[2mm]
\displaystyle +(-1)^v \sum_{\substack{j=0\\ k_j'\ \text{even}}}^{m_1-1} d_j''(-1)^{\frac{k_j'}{2}} v^{k_j'} + (-1)^v \sum_{\substack{j=0\\ k_j'\ \text{odd}}}^{m_1-1} d_j'(-1)^{\frac{k_j'-1}{2}} v^{k_j'} = 0 \\[2mm]
v = \pm 1,\ \pm 2,, \ldots, \pm\omega.
\end{array}\right.
$$

The systems (3.9) and (3.10) can be splitted to the following four systems which are the necessary and sufficient condition for the existence of the formula (3.3):

$$
(3.11)\quad \left|\begin{array}{l}
c_0' + \varepsilon(k_0')d_0' = 1 \\[2mm]
\displaystyle\sum_{\substack{s=0\\ k_s\ \text{even}}}^{m-1} c_s'(-1)^{\frac{k_s}{2}} v^{k_s} + (-1)^v \sum_{\substack{j=0\\ k_j'\ \text{even}}}^{m_1-1} d_j'(-1)^{\frac{k_j'}{2}} v^{k_j'} = 0 \\[2mm]
v = 1,\ 2, \ldots, \omega
\end{array}\right.
$$

$$
(3.12)\quad \left|\begin{array}{l}
c_0'' + \varepsilon(k_0')d_0'' = 0 \\[2mm]
\displaystyle\sum_{\substack{s=0\\ k_s\ \text{even}}}^{m-1} c_s''(-1)^{\frac{k_s}{2}} v^{k_s} + (-1)^v \sum_{\substack{j=0\\ k_j'\ \text{even}}}^{m_1-1} d_j''(-1)^{\frac{k_j'}{2}} v^{k_j'} = 0 \\[2mm]
v = 1,\ 2, \ldots, \omega
\end{array}\right.
$$

$$(3.13)\qquad \left|\begin{array}{l} \displaystyle\sum_{\substack{s=1\\ k_s \text{ odd}}}^{m-1} c'_s(-1)^{\frac{k_s-1}{2}}\nu^{k_s}+(-1)^{\nu}\sum_{\substack{j=0\\ k'_j \text{ odd}}}^{m_1-1} d'_j(-1)^{\frac{k'_j-1}{2}}\nu^{k'_j}=0\\[2ex] \nu=1,\ 2,\dots,\omega \end{array}\right.$$

$$(3.14)\qquad \left|\begin{array}{l} \displaystyle\sum_{\substack{s=1\\ k_s \text{ odd}}}^{m-1} c''_s(-1)^{\frac{k_s+1}{2}}\nu^{k_s}+(-1)^{\nu}\sum_{\substack{j=0\\ k'_j \text{ odd}}}^{m_1-1} d''_j(-1)^{\frac{k'_j-1}{2}}\nu^{k'_j}=0\\[2ex] \nu=1,\ 2,,\dots,\omega \end{array}\right.$$

The system (3.11) and (3.12) have the following matrix:

$$(3.15)\qquad \begin{pmatrix} 1 & 0 & \cdots & 0 & \varepsilon(k'_0) & 0 & \cdots & 0\\ 1 & 1 & \cdots & 1 & -1 & -1 & \cdots & -1\\ 1 & 2^{k_{i_2}} & \cdots & 2^{k_{i_{\omega_e}}} & 2^{k'_{i_1}} & 2^{k'_{i_2}} & \cdots & 2^{k'_{i_{\omega'_e}}}\\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots\\ 1 & \omega^{k_{i_2}} & \cdots & \omega^{k_{i_{\omega_e}}} & (-1)^{\omega}\omega^{k'_{i_1}} & (-1)^{\omega}\omega^{k'_{i_2}} & \cdots & (-1)^{\omega}\omega^{k'_{i_{\omega'_e}}} \end{pmatrix}.$$

$$\underbrace{\qquad\qquad\qquad}_{\omega_e}\qquad\qquad\qquad\underbrace{\qquad\qquad\qquad}_{\omega'_e}$$

The systems (3.13) and (3.14) have the matrix:

$$(3.16)\qquad \begin{pmatrix} 1 & 1 & \cdots & 1 & -1 & -1 & \cdots & -1\\ 2^{k_{j_1}} & 2^{k_{j_2}} & \cdots & 2^{k_{j_{\omega_0}}} & 2^{k'_{j_1}} & 2^{k'_{j_2}} & \cdots & 2^{k'_{j_{\omega'_0}}}\\ 3^{k_{j_1}} & 3^{k_{j_2}} & \cdots & 3^{k_{j_{\omega_0}}} & -3^{k'_{j_1}} & -3^{k'_{j_2}} & \cdots & -3^{k'_{j_{\omega'_0}}}\\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots\\ \omega^{k_{j_1}} & \omega^{k_{j_2}} & \cdots & \omega^{k_{j_{\omega_0}}} & (-1)^{\omega}\omega^{k'_{j_1}} & (-1)^{\omega}\omega^{k'_{j_2}} & \cdots & (-1)^{\omega}\omega^{k'_{j_{\omega'_0}}} \end{pmatrix}.$$

$$\underbrace{\qquad\qquad\qquad\qquad}_{\omega_0}\qquad\qquad\qquad\underbrace{\qquad\qquad\qquad}_{\omega'_0}$$

By Lemma 3.3 using the systems (3.11)—(3.14) we obtain 1) and 3). It remains to show that if $\omega>\omega_e+\omega'_e-1$ the formula of the form (3.3) for every $f\in W_1^{k^*}BV$ does not exist. This follows easily because if $\omega>\omega_e+\omega'_e-1$ then the system of equations (3.11) with $\nu=1, 2, 3,\dots,\omega_e+\omega'_e$ is a homogeneous system with $\omega_e+\omega'_e$ unknowns and (by Lemma 3.3) whose determinant is non-zero. Hence $c'_0=d'_0=0$ which contradicts the first equation $c'_0+\varepsilon(k'_0)d'_0=1$. This ends the proof.

Corollary 3.1. *If we are given* $\vec{k}, \vec{k}', \sigma>0$ *then* $\max\{\omega : \omega$ *integer*, $A(f, \vec{k}, \vec{k}', \sigma, \omega, \vec{c}, \vec{d})$ *exists for every* $f \in W_1^{k^*} BV\} = \omega_e + \omega_e' - 1$.

Remark 3.1. Lemma 3.4 gives the error estimate for the Q. F. of the form (2.3) in the space $W_1^{k^*} BV$.

4. Existence and uniqueness for Q. F. of HEDP

The solution of our problem will depend on the number of even integers in the vectors $\vec{k}, \vec{k}'$ respectively ω_e and ω_e'.

Our main result is:

Theorem 4.1. *A. Q. F. of the form* (2.3) *has HEDP equal to* $(\omega_e+\omega_e')\sigma$. *More precisely, we have the following situation:*

1) *If* $\omega_0+\omega_0' \leqq \omega_e+\omega_e'-1$ *then a Q. F. of the form* (2.3) *of HEDP exists and is unique. The coefficients* $\vec{c}$ *and* $\vec{d}$ *are uniquely determined by the conditions:*

a) $\vec{c}$ *and* $\vec{d}$ *are real vectors.*

b) $c_s = d_j = 0$ *for* k_s, k_j' *odd.*

c) c_s *and* d_j *for* k_s, k_j' *even are determined by the system:*

$$(4.1)\qquad \left|\begin{array}{l} c_0+\varepsilon(k_0')d_0=1,\ \varepsilon(0)=1,\ \varepsilon(k_0')=0 \text{ for } k_0'>0 \\[2ex] \displaystyle\sum_{\substack{s=1\\ k_s \text{ even}}}^{m-1} c_s(-1)^{\frac{k_s}{2}} \nu^{k_s} + (-1)^{\nu} \sum_{\substack{j=0\\ k_j' \text{ even}}}^{m_1-1} d_j(-1)^{\frac{k_j'}{2}} \nu^{k_j'} = 0 \\[2ex] \nu = 1, 2, 3, \ldots, \omega_e+\omega_e'-1. \end{array}\right.$$

2) *If* $\omega_0+\omega_0' > \omega_e+\omega_e'-1$ *there are many Q. F. of HEDP of the form* (2.3). *The vectors* $\vec{c}$ *and* $\vec{d}$ *are coefficients of such Q. F. iff they satisfy the following conditions:*

a) c_s *and* d_j *for* k_s, k_j' *even are real and uniquely determined by the system* (4.1).

b) c_s *and* d_j *for* k_s, k_j' *odd satisfy the systems* (3.13), (3.14) *for* $\omega=\omega_e+\omega_e'-1$.

Remark 4.1. It is easy to see that in the case $2/c_s=d_j=0$ for k_s, k_j' odd satisfy (3.13) and (3.14). The coefficients c_s and d_j for k_s, k_j' odd in the case 2) can be imaginary.

Corollary 4.1. *If in the Q. F.* (2.3) *we are free to choose the* $m-1$ *numbers* $0<k_1<k_2<k_3<\ldots<k_{m-1}$ *and the* m_1 *numbers* $0 \leqq k_0'<k_1'<k_2'<\ldots<k_{m_1-1}'$ *then the Q. F. with HEDP is obtained when* $\omega_e=m$ *and* $\omega_e'=m_1$, *and in this case HEDP is* $(m+m_1)\sigma$. *Thus all elements in* $\vec{k}$ *and* $\vec{k}'$ *have to be even.*

Proof of Theorem 4.1. Let the Q. F. of the form (2.3) be precise for every $f \in B_{\sigma(\omega_e+\omega_e'),1}$. Then for the given $\vec{k}, \vec{k}', \sigma>0, \vec{c}, \vec{d}$ by (3.6) and Paley-Wiener's theorem [1] the condition (3.7) holds for every $f \in B_{\sigma(\omega_e+\omega_e'),1}$ and $\omega=\omega_e+\omega_e'-1$. Thus $\vec{c}$ and $\vec{d}$ satisfy the systems (3.11)—(3.14) with $\omega=\omega_e+\omega_e'-1$ and this is a necessary and sufficient condition for the existence of $A(f, k, k', \sigma, \omega_e+\omega'_e-1, \vec{c}, \vec{d})$. Thus the systems (3.11)—(3.14) are a necessary and sufficient condition for the Q. F. to be precise for every $f \in B_{\sigma(\omega_e+\omega'_e),1}$.

Let us assume the Q. F. possesses EDP equal to $\sigma(\omega_e+\omega'_e)+\gamma$, $\gamma>0$. We choose the function f_* with the following properties:

a) $f_* \in B_{\sigma(\omega_e+\omega'_e)+\delta,1}$, $0<\delta<\min(\gamma, \sigma)$.

b) $\hat{f}_*(\sigma(\omega_e+\omega'_e))=\hat{f}_*(-\sigma(\omega_e+\omega'_e))\neq 0$.

But $f_* \in W_1^{k*} BV$ and by Lemma 3.4 the formula (3.3) is precise for f_*, hence by Paley-Wiener's theorem we obtain:

$$0=\frac{2\pi}{\sigma}\sum_{s=0}^{m-1}\frac{c_s}{\sigma^{k_s}} f^{(k_s)}_{*\sigma,e}+\frac{2\pi}{\sigma}\sum_{j=0}^{m_1-1}\frac{d_j}{\sigma^{k'_j}} f^{(k'_j)}_{*\sigma,0}-\int_{-\infty}^{\infty} f_*(x)\,dx$$

$$(4.2)\qquad =\sqrt{2\pi}\sum_{s=0}^{m-1}\frac{c_s}{\sigma^{k_s}}[\hat{f}^{(k_s)}_*(\sigma(\omega_e+\omega'_e))+\hat{f}^{(k_s)}_*(-\sigma(\omega_e+\omega'_e))]$$

$$+\sqrt{2\pi}\sum_{j=0}^{m_1-1}\frac{d_j}{\sigma^{k'_j}}(-1)^{\omega_e+\omega'_e}[\hat{f}^{(k'_j)}_*(\sigma(\omega_e+\omega'_e))+\hat{f}^{(k'_j)}_*(-\sigma(\omega_e+\omega'_e))]$$

$$=2\sqrt{2\pi}\,\hat{f}(\sigma(\omega_e+\omega'_e))\Big(\sum_{\substack{s=0\\ k_s \text{ even}}}^{m-1} c_s(\omega_e+\omega'_e)^{k_s}(-1)^{\frac{k_s}{2}}$$

$$+\sum_{\substack{j=0\\ k'_j \text{ even}}}^{m_1-1} d_j(-1)^{\omega_e+\omega'_e}(\omega_e+\omega'_e)^{k'_j}(-1)^{\frac{k'_j}{2}}\Big).$$

The system (3.12) for $\omega=\omega_e+\omega'_e-1$ gives $c''_s=d''_j=0$ for k_s even and k'_j even. Thus from (3.11) for $\omega=\omega'_e+\omega_e-1$ and (4.2) the vectors $\vec{c}$ and $\vec{d}$ have to satisfy the system:

$$(4.3)\qquad \left|\begin{array}{l} c'_0+\varepsilon(k'_0)d'_0=1 \\ \displaystyle\sum_{\substack{s=0\\ k_s \text{ even}}}^{m-1} c'_s(-1)^{\frac{k_s}{2}}\nu^{k_s}+(-1)^{\nu}\sum_{\substack{j=0\\ k_j \text{ even}}}^{m_1-1} d'_j(-1)^{\frac{k'_j}{2}}\nu^{k'_j}=0. \\ \nu=1, 2,\dots,\omega_e+\omega'_e-1,\ \omega_e+\omega'_e. \end{array}\right.$$

From the last $\omega_e+\omega'_e$ equations and Lemma 3.3 follows $c_0=d_0=0$ which contradicts the first equation $c_0+\varepsilon(k'_0)d_0=1$. Hence the HEDP of the Q. F. of the form (2.3) is $(\omega_e+\omega'_e)\sigma$.

The conditions 1), a), b), c) and 2), a), b) follow by the systems (3.11)—(3.14) and Lemma 3.3. This completes the proof.

5. Examples

Example 1. Let $\vec{k}'\equiv\vec{k}$. Then $\omega'_e=\omega_e$ and $\omega'_0=\omega_0$, $m_1=m$. The system (4.1) can be written in the form:

$$(5.1)\qquad \begin{cases} c_0+d_0=1 \\ \sum\limits_{\substack{s=0\\ k_s \text{ even}}}^{m-1} (c_s-d_s)(-1)^{\frac{k_s}{2}}\, \nu^{k_s}=0, \quad \nu=1,\ 3,\dots,2\omega_e-1 \\ \sum\limits_{\substack{s=0\\ k_s \text{ even}}}^{m-1} (c_s+d_s)(-1)^{\frac{k_s}{2}}\, \nu^{k_s}=0, \quad \nu=2,\ 4,\dots,2\omega_e-2. \end{cases}$$

Let $2\omega_0 \leqq 2\omega_e-1$. By Theorem 4.1 the Q. F. of the form (2.3) of HEDP equal to $2\omega_e\sigma$ exists and is unique and $c_s=d_s=0$ for k_s odd. From the system (5.1) and Lemma 3.2 we obtain $c_s=d_s$ for k_s even and the system (5.1) can be written in the form:

$$(5.2)\qquad \begin{cases} c_0=\frac{1}{2} \\ \sum\limits_{\substack{s=0\\ k_s \text{ even}}}^{m-1} (c_s(-1)^{\frac{k_s}{2}}\, \nu^{k_s}=0, \quad \nu=2,\ 4,\dots,2\omega_e-2. \end{cases}$$

The condition $2\omega_0 \leqq 2\omega_e-1$ is equivalent to the condition $\omega_0 \leqq \omega_e-1$. If we substitute $2^{k_s+1} c_s=b_s$ then the system (5.2) will be the system:

$$(5.3)\qquad \begin{cases} b_0=1 \\ \sum\limits_{\substack{s=0\\ k_s \text{ even}}}^{m-1} b_s(-1)^{\frac{k_s}{2}}\, \nu^{k_s}=0 \\ \nu=1,\ 2,\dots,\ \omega_e-1 \end{cases}$$

and the Q. F. of HEDP can be written in the form

$$(5.4)\qquad \frac{\pi}{\sigma}\sum_{s=0}^{m-1} \frac{b_s}{(2\sigma)^{k_s}}(f^{(k_s)}_{\sigma,e}+f^{(k_s)}_{\sigma,0}) \approx \int_{-\infty}^{\infty} f(x)dx$$

and this is the result in [3]. In the paper [3] there are many examples which are particular cases of Example 1.

Example 2. Let p be even and $k=(0,\ p,\ 2p,\dots,(m-1)p)$, $m_1=m-1$, $k'=(p,\ 2p,\dots,(m-1)p)$ thus $k'_i=k_{i+1}$, $i=0,\ 1,\ 2,\dots,m-2$; $\omega_e=m$, $\omega'_e=m-1$, $\omega_0=\omega'_0=0$. From Theorem 4.1 the Q. F. of HEDP $(2m-1)\sigma$ exists and is unique. The system 4.1 can be written in the form:

$$(5.5)\qquad \begin{cases} c_0=1 \\ \sum\limits_{s=1}^{m-1} (c_s+d_s)(-1)^{\frac{sp}{2}}\, (2\nu)^{sp}+c_0=0, \\ \sum\limits_{s=1}^{m-1} (c_s-d_s)(-1)^{\frac{sp}{2}}\, (2\nu-1)^{sp}+c_0=0, \\ \nu=1,\ 2,\ 3,\dots,m-1 \end{cases}$$

because $\omega_e+\omega'_e-1=2m-2$. We will give a convenient way to compute the coefficients $(c_0, c_1, c_2, \ldots, c_{m-1})$ and $(d_1, d_2, \ldots, d_{m-1})$.

Let us consider the polynomials:

$$p_e(x)=\prod_{s=1}^{m-1}\left(1-\frac{x^p}{(2s)^p}\right)=1+\sum_{s=1}^{m-1}\frac{(-1)^s|p_e^{(sp)}(0)|}{(sp)!}x^{sp}, \tag{5.6}$$

$$p_0(x)=\prod_{s=1}^{m-1}\left(1-\frac{x^p}{(2s-1)^p}\right)=1+\sum_{s=1}^{m-1}\frac{(-1)^s|p_0^{(sp)}(0)|}{(sp)!}x^{sp}. \tag{5.7}$$

By (5.5), (5.6) and (5.7) one may write the following equalities:

$$\begin{aligned}&(-1)^{\frac{sp}{2}}(c_s+d_s)=(-1)^s\frac{|p_e^{(sp)}(0)|}{(sp)!}, \quad s=1, 2, \ldots, m-1,\\ &(-1)^{\frac{sp}{2}}(c_s-d_s)=(-1)^s\frac{|p_0^{(sp)}(0)|}{(sp)!}, \quad s=1, 2, \ldots, m-1.\end{aligned} \tag{5.8}$$

From (5.8) it is trivial to see that

$$\begin{aligned}&c_s=(-1)^{s(1+p/2)}\frac{|p_e^{(sp)}(0)|+|p_0^{(sp)}(0)|}{2(sp)!}, \quad s=1, 2, \ldots, m-1,\\ &d_s=(-1)^{s(1+p/2)}\frac{|p_e^{(sp)}(0)|-|p_0^{(sp)}(0)|}{2(sp)!}, \quad s=1, 2, \ldots, m-1.\end{aligned} \tag{5.9}$$

For $m=2$ by (5.9) one gets:

$$c_1=(-1)^{1+p/2}\frac{1}{2}\left(\frac{1}{2^p}+1\right), \quad d_1=(-1)^{1+p/2}\frac{1}{2}\left(\frac{1}{2^p}-1\right)$$

and the Q. F. of HEDP$=3\sigma$ will be the following:

$$\int_{-\infty}^{\infty} f(x)dx\approx\frac{2\pi}{\sigma}\left(f_{\sigma,e}+\frac{(-1)^{1+p/2}}{2\sigma^p}\left(\frac{1}{2^p}+1\right)f_{\sigma,e}^{(p)}+\frac{(-1)^{1+p/2}}{2\sigma^p}\left(\frac{1}{2^p}-1\right)f_{\sigma,0}^{(p)}\right). \tag{5.10}$$

Example 3. Let p be even number, $\vec{k}=(0, p, 2p, \ldots, (m-1)p)$, $m_1=m$, $\vec{k}'=(p, 2p, 3p, \ldots, (m-1)p, mp)$ thus $k'_i=k_i+p$, $i=0, 1, 2, \ldots, m-1$. Then $\omega_e=\omega'_e=m$, $\omega_0=\omega'_0=0$ and by Theorem 4.1 the Q. F. of HEDP equal to $2m\sigma$ exists and is unique. We will give a way to compute the coefficients $(c_0, c_1, c_2, \ldots, c_{m-1})$, $(d_1, d_2, d_3, \ldots, d_m)$. The system (4.1) can be written in the form $(\omega_e+\omega'_e-1=2m-1)$:

$$\left|\begin{aligned}&c_0=1\\ &\sum_{s=1}^{m-1}(c_s+d_s)(-1)^{\frac{ps}{2}}(2v)^{sp}+d_m(-1)^{\frac{mp}{2}}(2v)^{mp}+c_0=0\\ &v=1, 2, \ldots, m-1\end{aligned}\right. \tag{5.11}$$

$$\left| \begin{array}{l} \sum_{s=1}^{m-1} (c_s-d_s)(-1)^{\frac{sp}{2}} (2v-1)^{sp}-d_m(-1)^{\frac{mp}{2}} (2v-1)^{mp}+c_0=0 \\ v=1,\ 2,\dots,m-1,\ m \end{array} \right.$$

Let us consider the polynomials:

$$\Pi_0(x)=\prod_{s=1}^{m} (1-\frac{x^p}{(2s-1)^p})=1+\sum_{s=1}^{m} (-1)^s \frac{|\Pi_0^{(sp)}(0)|}{(sp)!} x^{sp},$$

(5.12)

$$r_e(x)=\prod_{s=1}^{m-1} (1-\frac{x^p}{(2s)^p}),$$

$$\Pi_e(x)=r_e(x)(1+(-1)^{m-1+\frac{mp}{2}}[(2m-2)!!]^p d_m x^p)$$

$$=1+\sum_{s=1}^{m-1} [(-1)^s \frac{|r_e^{(sp)}(0)|}{(sp)!}+(-1)^{s+m+\frac{mp}{2}}[(2m-2)!!]^p \frac{|r_e^{((s-1)p)}(0)|}{((s-1)p)!}] x^{sp}+d_m(-1)^{\frac{mp}{2}} x^{mp}.$$

It's easy to see $\Pi_0(2v-1)=0$, $v=1, 2,\dots,m$ and $\Pi_e(2v)=0$, $v=1, 2,\dots,m-1$. From (5.11) and (5.12) one may write the following expressions for $\vec{c}$ and $\vec{d}$:

$$d_m=(-1)^{m-1+mp/2} \frac{|\Pi_0^{(mp)}(0)|}{(mp)!}=(-1)^{m-1+mp/2} \frac{1}{[(2m-1)!!]^p},$$

$$c_s+d_s=(-1)^{s(1+p/2)} (\frac{|r_e^{(sp)}(0)|}{(sp)!}-\frac{|r_e^{((s-1)p)}(0)|}{[(s-1)p]!}[\frac{(2m-2)!!}{(2m-1)!!}]^p),$$

(5.13)

$$s=1,\ 2,\dots,m-1$$

$$c_s-d_s=(-1)^{s(1+p/2)} \frac{|\Pi_0^{(sp)}(0)|}{(sp)!}, \quad s=1,\ 2,\dots,m-1.$$

Let us take the case $m=2$, $\vec{k}=(0,\ p)$, $\vec{k}'=(p,\ 2p)$. Then

$$\Pi_0(x)=(1-x^p)(1-\frac{x^p}{3^p})=1-(1+\frac{1}{3^p})x^p+\frac{1}{3^p}x^{2p} \text{ and } r_e(x)=1-\frac{x^p}{2^p}.$$

For the coefficients we obtain by (5.13):

$$d_2=(-1)^{p+1}\frac{1}{3^p}, \quad c_1+d_1=(-1)^{1+p/2}(\frac{1}{2^p}-(\frac{2}{3})^p),$$

(5.14)

$$c_1-d_1=(-1)^{1+p/2}(1+\frac{1}{3^p}).$$

Hence for $(c_0,\ c_1)$ and $(d_1,\ d_2)$ one may write the following:

$$c_0=1, \quad c_1=(-1)^{1+p/2}\frac{1}{2}(1+\frac{1}{2^p}+\frac{1}{3^p}-(\frac{2}{3})^p),$$

$$d_1=(-1)^{1+p/2}\frac{1}{2}(\frac{1}{2^p}-1-\frac{1}{3^p}-(\frac{2}{3})^p),\quad d_2=(-1)^{p+1}\frac{1}{3^p}.$$

For $p=2$ we will obtain

$c_0=1$, $c_1=\frac{11}{24}$, $d_1=-\frac{47}{72}$, $d_2=-\frac{1}{9}$ and the Q. F. of HEDP 4σ will be

$$(5.16)\qquad \int_{-\infty}^{\infty} f(x)dx\approx\frac{2\pi}{\sigma}(f_{\sigma,e}+\frac{11}{24\sigma^2}f^{(2)}_{\sigma,e}-\frac{47}{72\sigma^2}f^{(2)}_{\sigma,0}-\frac{1}{9\sigma^4}f^{(4)}_{\sigma,0}).$$

Example 4. Let $m=1$, $m_1=1$, $\vec{k}=(0)$, $\vec{k}'=(21)$, $\omega_e=\omega'_e=1$, $\omega_0=\omega'_0=0$. Then by using Example 3. for $m=1$ or the system (4.1) we obtain for the coefficients c_0 and d_0:

$$(5.17)\qquad \left|\begin{array}{l} c_0=1 \\ c_0-d_0(-1)^1=0, \quad 1\geqq 1 \end{array}\right.$$

and the Q. F. of HEDP equal 2σ will be the following:

$$(5.18)\qquad \int_{-\infty}^{\infty} f(x)dx\approx\frac{2\pi}{\sigma}(f_{\sigma,e}+\frac{(-1)^1}{\sigma^{21}}f^{(21)}_{\sigma,0}).$$

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Bulgarian Academy of Sciences
Institute of Mathematics
P. O. B. 373, 1090 Sofia
BULGARIA

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