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Controlled Wiener Process by External Interventions Maximizing the Average Time until Reaching a Level

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1. Statement of the problem

Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let $W=(W(t), t\in[0, \infty))$ be a standard one-dimensional Wiener process and ξ — an independent of W non-negative random variable in $(\Omega, \mathcal{F}, P)$.

Definition 1. The diffusion process $Y=(Y(t), t\in[0, \infty))$, defined by the equation:

$$Y(t)=b+\mu t+W(t),\ t\in[0, \infty),\ b\in R,\ \mu\in R,$$

is said to be a Wiener process with initial state b and linear drift with parameter μ.

Definition 2. The random process $Z=(Z(t), t\in[0, \infty))$, defined by the equation:

$Z(t)=b+\mu.(t-\xi).I\{\xi<t\}+W(t)$, $t\in[0, \infty)$, b, $\mu\in R$, is said to be a Wiener process with initial state b, which acquires a linear drift with parameter μ at a moment of time ξ.

Let the process W acquires a linear drift with parameter μ, $\mu>0$, at a random moment of time ξ. If no external interventions be realized, as a result we will get the random process $X=(X(t), t\in[0, \infty))$, $X(t)=W(t)+\mu.(t-\xi)I\{\xi<t\}$, which be considered up to its first reaching a fixed level A. Here A is a positive real number, and $I\{\cdot\}$ is an indicator of the set $\{\cdot\}$. If we introduce an external intervention before the process X reaches the level A, for example, if we eliminate the drift appeared, then the process X and the moment of first reaching of level A will change.

Our purpose is to find a rule to control the process by correcting it (by means of external interventions) so as to maximize the average time until the process reaches the level A.

At first we need some definitions and clarifications. Let b be a real non-negative number, and $W^b=(W^b(t), t\in[0, \infty))$ be an one-dimensional Wiener

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process with an absorbing screen in the state b (c.f. [2], § 11). We define the following moments of first reaching the level b:

$$\tau(b)=\inf\{t\geqq 0:\ W(t)=b\};$$

$$\tau'(b,\ \xi)=\inf\{t\geqq 0:\ W'(t)=b,\ W'(t)=W^b(\xi)+W(t)+\mu t\};$$

$$T(b,\ \xi)=\inf\{t\geqq 0:\ W(t)+\mu.(t-\xi)I\{\xi<t\}=b\}.$$

Since W is a Markov process and ξ is an independent of W random variable, we have:

$$T(b,\ \xi)=\tau(b).I\{\sup_{0\leq t\leq \xi} W(t)\geqq b\}+(\xi+\tau'(b,\ \xi)).I\{\sup_{0\leq t\leq \xi} W(t)<b\};$$

$$(1)\qquad T(b,\ \xi)=\tau(b).I\{\tau(b)\leqq \xi\}+(\xi+\tau'(b,\ \xi)).I\{\tau(b)>\xi\}.$$

Definition 3. Let b be a real number, $b\in[0,\ A)$, and $T(b,\ \xi)$ be the moment of first reaching the level b by the random process X. Control by an external intervention of type (A) of the process X (or control by means of an external intervention of type (A) of the process X) is said to be the following act:

1) When the process X reaches the level b, i. e. at the moment $T(b,\ \xi)$, the drift of the process is eliminated (provided the drift is appeared);

2) Then the process behaves as a Wiener process with initial state b and it can acquire a linear drift with parameter μ after a random time interval ξ', beginning from the moment $T(b,\ \xi)$. The random variable ξ' is independent of the behavior of the process discussed and of the random variable ξ. Thus, after the moment $T(b,\ \xi)$ a new random process $X'=(X'(t),\ t\in[0,\ \infty))$, $X'(t)=b+W(t)+\mu.(t-\xi')I\{\xi'<t\}$ holds.

The level b is said to be an level of external intervention, or briefly, an intervention level.

Remark. The control of the process X by one external intervention of type (A) is determined by choosing of a value of the intervention level b.

Consider the case when $n-1$ interventions of type (A) are assumed to be realized. Let b_i, $i=1,\dots,n-1$, be the intervention levels. Clearly, the intervention levels form a nondecreasing sequence: $0\leqq b_1\leqq b_2\leqq\dots\leqq b_{n-1}<A$. Denote by ξ_1 the random moment of time at which the discussed process can acquire a drift if no interventions are realized and by ξ_{i+1} — the length of the random time interval after which expiry the process can acquire a drift after i-th intervention, $i=1,2,\dots,n-1$. Consider the processes: $X_i=(X_i(t),\ t\in[0,\infty))$, $X_i(t)=b_{i-1}+W(t)+\mu(t-\xi_i)I\{\xi_i<t\}$, $i=1,\dots,n$, $b_0=0$. Let $X^n=(X^n(t),\ t\in[0,\ \infty))$ be the process having control by $n-1$ interventions of type (A). Before realization of the first intervention the behavior of X^n is described by the random process X_1. At the moment of first reaching of the level b the drift of the process is eliminated (provided the drift is appeared). Then the controlled process is represented as a process X_2 which is a Wiener process with initial state b_1 and can acquire a linear drift with parameter μ after random time interval ξ_2, beginning from the moment $T(b_1,\ \xi_1)$. The random variable ξ_2 is independent of the behaviour of the process discussed and of the random variable ξ_1. Then, at the moment of first

reaching of the level b_2 the drift of the process is eliminated again (provided the drift is appeared) and the process keeps on behaving in the described manner as long as the random process X^n reaches a beforehand fixed level A.

Introduce the designations; $a_1=b_1$, $a_2=b_2-b_1,\dots$, $a_k=b_k-b_{k-1},\dots$, $a_n=A-b_{n-1}$. Then the controlled process X^n can be represented in the form:

$$X^n(t)=\sum_{i=1}^{n} X_i(t-\sum_{j=1}^{i-1} I(a_j,\ \xi_j)).I\{\sum_{j=1}^{i-1} T(a_j,\ \xi_j)\leqq t<\sum_{j=1}^{i} T(a_j,\ \xi_j)\}$$

where $\sum_{j=1}^{0} T(a_j,\ \xi_j)=0$.

Remark that the moment of first reaching of level A by the random process X^n is $\sum_{i=1}^{n} T(a_i,\ \xi_i)$ because of the strong Markov property of the Wiener process (c. f. [1], § 1.6).

Definition 4. Control of the process X by means of $n-1$ external interventions of type (A) is determined by choosing of values of the intervention levels $b_1, b_2,\dots,b_{n-1}$ to satisfy the conditions: $0\leqq b_1\leqq b_2\leqq\dots\leqq b_{n-1}<A$.

Definition 5. Optimal control by means of at most $n-1$ external interventions of type (A) is said to be a control maximizing the average time for first reaching of level A by the controlled process.

2. General form of the optimal control problem of Wiener process by external interventions of type (A)

In order to realize an optimal control of a random process X by means of external interventions of type (A) it is necessary to determine the number and concrete values of the intervention levels to maximize the average time for which the controlled process reaches the level A. We assume that at most $n-1$ interventions, $n\geqq 1$, can be realized. Because of the intrinsic restriction that the intervention levels $b_1, b_2,\dots,b_{n-1}$ form a nondecreasing sequence it is sufficient to determine the distances $a_i=b_i-b_{i-1}$, $i=1,\dots,n$, $b_n=A$, $b_0=0$, among them.

Let $n\geqq 1$ be a fixed integer number. The problem of optimal control of Wiener process by realizing no more than $n-1$ external interventions of type (A), can be defined in the following way:

Problem (A). Determine the numbers $a_1, a_2,\dots,a_n$ so that

(2) $\min E(-\sum_{i=1}^{n} T(a_i,\ \xi_i))$ is reached under the restrictions:

(3) $$\sum_{i=1}^{n} a_i=A;\quad a_i\geqq 0,\quad i=1,\dots,n.$$

The relation between the numbers $a_1, a_2,\dots,a_n$ and the intervention levels $b_1, b_2,\dots,b_{n-1}$ is obviously:
$b_i=a_1+a_2+\dots+a_i$, $i=1, 2,\dots,n-1$. Remark in addition that the last

intervention level can not coincide with the level A. It means that the optimal number of interventions of type (A) for optimal control by at most $n-1$ interventions is possible to be less than preassigned number of interventions. Obviously, it will be less by one than the subscript of the last non-zero a_i in sequence of optimal values of numbers $a_1, a_2, \dots, a_n$.

If we define $m=\sup\{i: a_i \neq 0,\ i=1,\dots,n\}$, then the optimal number of interventions will be $m-1$, $m \leqq n$. Denote by $M(n, A)$ the set of all sequences $\{a_i\}$, $i \geqq 1$, which satisfy the conditions: $\Sigma_{i=1}^{n} a_i = A$; $a_i \geqq 0$ for $i=1,\dots,n$ and $a_i=0$ for $i>n$. Denote by

$$F(n) = \max_{\{a_i\}_1^\infty \in M(n, A)} \{E(\sum_{i=1}^{n} T(a_i, \xi_i))\}, \quad n \geqq 1.$$

Since $F(n+1) \geqq \max\limits_{\{a_i\}_1^\infty \in M(n, A)} \{E(\Sigma_{i=1}^{n+1} T(a_i, \xi_i))\} = F(n)$, the sequence $\{F(n)\}$, $n \geqq 1$, is non-decreasing. Obviously, the optimal number of interventions $(m-1)$ depends on the preassigned number of interventions. In general case m will increase with n but $m \leqq n$ for every $n \geqq 1$.

Further we find the form of the object function and prove that under some conditions the Problem (A) is a convex optimization problem which has an unique solution.

Since $ET(b, \xi) = EE(T(b, \xi)/\xi)$, we determine $E(T(b, \xi)/\xi)$ at first and then we determine $ET(b, \xi)$ by randomization according the given distribution of ξ.

Theorem 1. *The conditional expectation $E(T(b, \xi)/\xi)$ of the moment of first reaching $T(b, \xi)$ of level b by a standard one-dimensional Wiener process, which acquire a linear drift with parameter μ at random moment ξ, independent of the behavior of the process, has the form: with probability* 1):

$$(4) \qquad E(T(b, \xi)/\xi) = -b^2 \operatorname{erfc}(b/\sqrt{2\xi}) + b\sqrt{2\xi/\pi}\exp(-b^2/(2\xi)) + \xi \operatorname{erf}(b/\sqrt{2\xi}) + b/\mu.$$

Proof. It follows from (1) that

$$E(T(b, \xi)/\xi) = E(\tau(b) . I\{\tau(b) \leqq \xi\} \mid \xi) + E(\xi I\{\tau(b) > \xi\} \mid \xi) + E(\tau'(b, \xi) . I\{\tau(b) > \xi\}/\xi).$$

The probability density of the random variable $\tau(b)$ is known (cf. [1], § 1.7 or [2], § 26):

$$f_{\tau(a)}(t) = b/\sqrt{2\pi t^3} . \exp(-b^2/(2t)).$$

Using well-known formulae of the integral calculus techniques (cf. [4], ch. 1, § 1.3, ch. 2, § 2.3 and [3], ch. 7), we get:

$$E(\tau(b) . I\{\tau(b) \leqq \xi\} \mid \xi) = b/\sqrt{2\pi} \int_0^\xi t^{-1/2} \exp(-b^2/(2t))\, dt$$

$$=\sqrt{2/\pi}\int_{1/\sqrt{\xi}}^{\infty} x^{-2}\exp(-b^2x^2/2)\,dx.$$

(5) $E(\tau(b).I\{\tau(b)\leqq\xi\}\mid\xi)=-b^2+b\sqrt{2\xi/\pi}.\exp(-b^2/(2\xi))+b^2\operatorname{erf}(b/\sqrt{2\xi})$

(6) $E(\xi.I\{\tau(b)>\xi\}\mid\xi)=\xi P(\tau(b)>\xi)=\xi\operatorname{erf}(b/\sqrt{2\xi}).$

The random variable $\tau'(b,\ \xi).I\{\tau(b)>\xi\}$ can be considered as a moment of first reaching of the level b by the random process $W^b(\xi)+W(t)+\mu t$. Therefore, the following relations are fulfilled:

$$I\{\tau(b)>\xi\}=I\{W^b(\xi)<b\};$$
$$P(\tau'(b,\ \xi)=0)=P(W^b(\xi)=b)=P(\tau(b)\leqq\xi)=\operatorname{erfc}(b/\sqrt{2\xi});$$
$$E(\tau'(b,\ \xi).I\{\tau(b)>\xi\})=E(\tau'(b,\ \xi));$$
$$E(\tau'(b,\ \xi).I\{\tau(b)>\xi\}\mid\xi)=E\{E[\tau'(b,\ \xi)/W^b(\xi),\ \xi]\mid\xi\}.$$

The probability density of the random variable $\tau'(b,\ \xi)$ under the condition $W^b(\xi)=u$ has the form (cf. [2], § 26):

$$f(s\mid u)=(b-u)\exp(-(b-u-\mu s)^2/(2s))/\sqrt{2\pi s^3},\quad s>0,\quad u\in(-\infty,\ b).$$
$$P(\tau'(b,\ \xi)=0\mid W^b(\xi)=u,\ \xi)=\begin{cases}1, & u=b;\\ 0, & u<b.\end{cases}$$

Then we have:

$$E(\tau'(b,\ \xi)|W^b(\xi)=u,\ \xi)=\int_0^{+\infty} sf(s|u)\,ds=(b-u)|\mu,\quad u\in(-\infty,\ b].$$

The probability density of the random variable $W^b(\xi)$ is given by the expression (cf. [5]):

$$g(u)=[\exp(-u^2/(2\xi))-\exp(-(2b-u)^2/(2\xi))]/\sqrt{2\pi\xi},\quad u\in(-\infty,\ b);$$
$$P(W^b(\xi)=b)=P(\tau(b)\leqq\xi)=\operatorname{erfc}(b/\sqrt{2\xi}).$$

We obtain:

$$E(\tau'(b,\ \xi)\mid\xi)=E[(b-W^b(\xi))\ /\mu|\xi]$$
$$=\int_{-\infty}^{b}(b-u)[\exp(-u^2/(2\xi))-\exp(-(2b-u)^2/(2\xi))]/(\sqrt{2\pi\xi}\,\mu)\,du.$$

After some not complicated calculations using the integral calculus techniques (cf. [4], ch. 1, § 1.3, ch. 2, § 2.3), we get:

(7) $E(\tau'(b).I\{\tau(b)>\xi\}\mid\xi)=b|\mu.$

It follows from (5), (6) and (7) that (4) is true. The proof is completed.

Corollary. *Let $\xi_1, \xi_2, \ldots, \xi_n$ be the non-negative, independent random variables with $E\sqrt{\xi_i}<\infty$, $i=1,\ldots,n$. Then the Problem (A) is a convex optimization problem with unique solution. Its object function has the following form:*

(8) $$E\Big(-\sum_{i=1}^{n} T(a_i,\ \xi_i)\Big)=-\sum_{i=1}^{n}\int_{0}^{+\infty} G_i(x)\,\mathrm{d}F_i(x),$$

where $G_i(x)=E(T(a_i,\ \xi_i)\,|\,\xi_i=x)$ *is given by the expression:*

$$G_i(x)=-a_i^2\,\mathrm{erfc}\,(a_i/\sqrt{2x})+a_i\sqrt{2x/\pi}\,\exp\,(-a_i^2/(2x))+x\,\mathrm{erf}\,(a_i/\sqrt{2x})+a_i/\mu$$

and $F_i(x)$ *is the distribution function of the random variable* ξ_i, $i=1,\dots,n$.

Proof. The representation (8) is obviously clear. The form of $G_i(x)$, $i=1,\dots,n$, follows from the representation (4). Moreover, we have

$$\int_{0}^{+\infty} |G_i(x)|\,\mathrm{d}F_i(x)<\infty \text{ for every } i=1,\dots,n$$

because of the following inequalities:

$$\int_{0}^{+\infty} \sqrt{y}\,\mathrm{d}F_i(y)<\infty;$$

$$\mathrm{erfc}\,(a_i/\sqrt{2x})\leqq 1;$$

$x\,.\,\mathrm{erf}\,(a_i/\sqrt{2x})<a_i\sqrt{2x/\pi}$ (cf. [3], ch. 7, 7.1.6), which hold for every $x\in(0,\ \infty)$ and $i=1,\dots,n$.

Using results from [3], Ch. 7, we obtain:

$$\mathrm{d}G_i(x)/\mathrm{d}a_i=-2a_i\,.\,\mathrm{erfc}\,(a_i/\sqrt{2x})+2\sqrt{2x/\pi}\,.\,\exp\,(-a_i^2\,/(2x))+1/\mu;$$

$\mathrm{d}^2G_i(x)/d^2\,a_i=-2\,.\,\mathrm{erfc}\,(a_i/\sqrt{2x})<0$ for every $x\in(0,\ \infty)$ and for every $a_i\in[0,\ A]$, $i=1,\dots,n$.

Moreover, we have

$$\int_{0}^{+\infty} |\mathrm{d}G_i(x)/\mathrm{d}a_i|\,\mathrm{d}F_i(x)<\infty,\qquad \int_{0}^{+\infty} |\mathrm{d}^2\,G_i(x)/\mathrm{d}^2\,a_i|\,\mathrm{d}F_i(x)<\infty.$$

Let us denote by C the matrix

$$\Big\{\partial^2\Big[-\sum_{k=1}^{n} ET(a_k,\ \xi_k)\Big]/(\partial a_i\ \partial a_j)\Big\},\quad i,\ j=1,\dots,n. \text{ We have:}$$

$$\partial^2\Big[-\sum_{k=1}^{n} ET(a_k,\ \xi_k)\Big]/\partial^2\,a_i=-\int_{0}^{+\infty} [\mathrm{d}^2\,G_i(x)/\mathrm{d}^2\,a_i]\,\mathrm{d}F_i(x)>0,$$

for every $a_i\in[0,\ A]$, $i=1,\dots,n$; and

$$\partial^2\Big[-\sum_{k=1}^{n} ET(a_k,\ \xi_k)\Big]/(\partial a_i,\ \partial a_j)=0,\quad i\neq j,$$

for every $i,\ j=1,\dots,n$. Therefore, according to the Silvester criterion (see [7], § 13.6), the quadratic form $y'\,Cy$, $y\in R^n$, is a positive definite one for every vector $(a_1,\ a_2,\dots,a_n)$ with components $a_i\in[0,\ A]$, $i=1,\dots,n$. As a consequence we have that the object function $E(-\Sigma_{k=1}^{n} T(a_k, \xi_k))$ is a strictly convex function on the domain defined by the conditions (3) (see [6], § 6.2). As the domain defined by the conditions (3) is also convex, Problem (A) is a convex optimization problem with unique solution.

The proof is completed.

3. Solution of the optimal control problem in case of exponentially distributed moments of acquiring a drift

Let the random variables ξ_i be exponentially distributed with parameters $\lambda_i > 0$, $i = 1, 2, \dots, n$. In this case the explicit form of $ET(b, \xi)$ is given by the following statement:

Theorem 2. *The mathematical expectation $ET(b, \xi)$ of the moment of first reaching $T(b, \xi)$ of level b by a standard one-dimensional Wiener process, which acquires a linear drift with parameter μ after random time interval ξ, where ξ is an independent of the behavior of the process and exponentially distributed with parameters $\lambda > 0$ random variable, has the form:*

$$ET(b, \xi) = 1/\lambda \ (1 - \exp(-b\sqrt{2\lambda})) + b/\mu. \tag{9}$$

P r o o f. The representation (9) is obtained from (4) after some not complicated calculations applying the formula for the total mathematical expectation, i. e. by randomization with respect to ξ.

Theorem 3. *Optimal control by means of at most $n-1$ external interventions of type (A) in the behavior of the Wiener process, which acquires a linear drift with parameter μ after a random time interval exponentially distributed, is realized by a choice of intervention levels b_i in the form:*

$$b_i = a_1 + a_2 + \dots + a_i, \quad i = 1, 2, \dots, m-1,$$

where $m-1$, $m \leqq n$, is the optimal number of interventions which should be accomplished. The values of a and m are determined by the following algorithm:

1) The parameters λ_i, $i = 1, 2, \dots, n$, are arranged in a non-decreasing sequence: $\lambda_{l_1} \leqq \lambda_{l_2} \leqq \dots \leqq \lambda_{l_i} \leqq \dots \leqq \lambda_{l_n}$. Substitute $\tilde{\lambda}_i = \lambda_{l_i}$, $\tilde{a}_i = a_{l_i}$;

2) The sums $S_i = \Sigma_{j=1}^{i} 1/(2\sqrt{2\tilde{\lambda}_j}) \ln(\tilde{\lambda}_i/\tilde{\lambda}_j)$, $i = 1, \dots, n$, are formed and an index k is defined by the relation: $k = \max\{i = 1, \dots, n : S_i < A\}$;

3) Put $a_{l_i} = \tilde{a}_i$, $i = 1, \dots, n$, where:

$$\tilde{a}_i = (A - \sum_{j=1}^{k} 1/(2\sqrt{2\tilde{\lambda}_j}) \ln(\tilde{\lambda}_i/\tilde{\lambda}_j)) / (\sum_{j=1}^{k} \sqrt{\tilde{\lambda}_i/\tilde{\lambda}_j}), \quad 1 \leqq i \leqq k;$$

$\tilde{a}_i = 0$ for $k < i \leqq n$ when $k < n$;

4) The optimal number of interventions is $(m-1)$ where m is determined by the expression: $m = \max\{i = 1, \dots, n : a_i \neq 0\}$.

Proof. We show that the Problem (A) is a convex optimization problem with respect to arbitrary distribution of non-negative random variables ξ_i with $E\sqrt{\xi_i}<\infty$, $i=1, 2,\dots,n$. When ξ_i are exponentially distributed random variables with parameter λ_i correspondingly, the condition (2) has the form:

$$\min\left(-\sum_{i=1}^{n}[(1-\exp(-a_i\sqrt{2\lambda_i}))/\lambda_i+a_i/\mu]\right). \tag{10}$$

The Problem (10) under the restrictions (3) can be solved by using the Kuhn-Tucker theorem (c. f. [6], § 7). In the case considered the Kuhn-Tucker local conditions have the form:

$$\sqrt{2/\lambda_i}\,.\exp(-a_i\sqrt{2\lambda_i})\leqq v,\quad i=1, 2,\dots,n; \tag{11}$$

$$\sum_{i=1}^{n} a_i=A; \tag{12}$$

$$v>0;\quad a_i\geqq 0,\quad i=1, 2,\dots,n; \tag{13}$$

$$a_i[\sqrt{2/\lambda_i}\,.\exp(-a_i\sqrt{2\lambda_i})-v]=0,\quad i=1, 2,\dots,n. \tag{14}$$

Arrange the parameters λ_i, $i=1, 2,\dots,n$, in a non-decreasing sequence: $\lambda_{l_1}\leqq\lambda_{l_2}\leqq\dots\leqq\lambda_{l_i}\leqq\dots\leqq\lambda_{l_n}$ and put $\tilde{\lambda}_i=\lambda_{l_i}$, $\tilde{a}_i=a_{l_i}$. It follows from (14) that if for some given $i=1, 2,\dots,n$ we have $\sqrt{2/\lambda_i}\leqq v$ then for the same i we imply $a_i=0$. Let the integer k be defined by the relation: $k=\max\{i=1, 2,\dots,n:\sqrt{2/\tilde{\lambda}_i}>v\}$, $1\leqq k\leqq n$. If $k<n$ we have $\tilde{a}_i=0$ for every $i>k$, $i\leqq n$; if $i<k$ the inequality $\sqrt{2/\tilde{\lambda}_i}>v$ holds and (11) implies that $\tilde{a}_i>0$. It follows from (14) that for every $i<k$ we get $\sqrt{2/\tilde{\lambda}_i}\,.\exp(-a_i\sqrt{2\tilde{\lambda}_i})=v$, i.e.:

$$\tilde{a}_i=-\ln(v\sqrt{\tilde{\lambda}_i/2})/\sqrt{2\tilde{\lambda}_i}. \tag{15}$$

The condition (12) implies:

$$\ln v=-\{A+\sum_{j=1}^{k}[\ln(\tilde{\lambda}_j/2)]/(2\sqrt{2\tilde{\lambda}_j})\}/(\sum_{j=1}^{k}1/\sqrt{2\tilde{\lambda}_j}). \tag{16}$$

The inequality $\sqrt{2/\tilde{\lambda}_k}>v$ is equivalent to the inequality $\ln(\sqrt{2/\tilde{\lambda}_k})>\ln v$. Taking into account (16), we obtain:

$$A>\sum_{j=1}^{k}1/(2\sqrt{2\tilde{\lambda}_j})\ln(\tilde{\lambda}_k/\tilde{\lambda}_j).$$

Referring to part 2) of Theorem 3 we form the sums S_i. The sequence $\{S_i\}$, $i=1, 2,\dots,n$, is increasing, so that the index k can be determined also by the following way: $k=\max\{i=1,\dots,n:S_i<A\}$.

The statement in the part 3) of Theorem 3 follows from (15) and (16).

The optimal number of interventions for preassigned n is determined by the requirement that the last intervention level has not to coincide with the level A.

The proof is completed.

4. Some special cases

Let the random variables ξ_i be exponentially distributed with parameters $\lambda_i > 0$, $i = 1, 2, \ldots$. We shall consider some special cases, in which the average times for acquiring a drift by the controlled process after each intervention form an non-decreasing sequence, i. e. we suppose $\lambda_i \leqq \lambda_{i+1}$ for every $i \geqq 1$. Under this requirement the sums S defined in part 2) of Theorem 3 do not depend on preassigned maximal number $(n-1)$ interventions. Then for every $n \geqq 1$ they have the form:

$$S_i = \sum_{j=1}^{i} 1/(2\sqrt{2\lambda_j}) \ln (\lambda_i/\lambda_j), \quad i = 1, 2, \ldots$$

Obviously, the sums S_i, $i \geqq 1$, form an increasing sequence. Define $S = \lim_{n\to\infty} S_n$, $0 \leqq S \leqq \infty$. According to Theorem 3 the relations between S and A determine the form of the optimal solution.

Special case 1. Let $\lambda_i = \lambda$ for every $i \geqq 1$. In this case we have $S = 0$. The solution of Problem (A) is the following: $a_i = A/n$, $1 \leqq i \leqq n$. The optimal number of interventions coincides with the preassigned maximal number interventions, i. e. $m = n$. The sequence $\{F(n)\}$, $n \geqq 1$, is strictly increasing and $F(n)$ has the form:

$$F(n) = n/\lambda(1 - \exp(-A\sqrt{2\lambda}/n)) + A/\mu.$$

Special case 2. Let $S < A$. Thus, $S_i < A$ for every $i \geqq 1$. Therefore, the optimal number of interventions coincides with the preassigned maximal number of interventions, i. e. $m = n$ and one has:

$$a_j = (A - \sum_{j=1}^{n} 1/(2\sqrt{2\lambda_j}) \ln (\lambda_i/\lambda_j))/(\sum_{j=1}^{n} \sqrt{\lambda_i/\lambda_j}), \quad 1 \leqq i \leqq n.$$

In this case the sequence $\{F(n)\}$, $n \geqq 1$, is strictly increasing in addition to Case 1.

Special case 3. Let $S > A$. Define $k = \max\{i \geqq 1 : S_i < A\}$. As $S_1 = 0 < A$ so $k < \infty$. Thus, for $n < k$ the optimal number of interventions is $m - 1 = n - 1$, and for $n \geqq k$ one has $m = k$. In this case the members of the sequence $\{F(n)\}$, $n \geqq 1$, satisfy the relations: $F(n) < F(n+1)$ for $n < k$ and $F(n) = F(n+1)$ for $n \geqq k$.

Special case 4. Let $S = A$. If $S_i < A$ for every $i \geqq 1$ we have the same situation as in the Case 2. Since the sequence $\{S_i\}$, $i \geqq 1$, is increasing and $S_1 = 0$, if $i > 1$ with $S_i = A$ exists, then the results in Case 4 coincides with that in the Case 3.

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