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Distribution of the Ratio of Geometric Mean to Arithmetic Mean in a Sample from a Two-Piece Double Exponential Distribution

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This paper deals with the distribution of the ratio of the geometric mean to arithmetic in a sample drawn from a two-piece double exponential distribution, where two pieces have different scale parameters. For this purpose, hypergeometric functions are utilized and this distribution is expressed in terms of Meijer's G-function.

1. Introduction

This paper deals with two concepts, such as, two-piece distributions and the distribution of the ratio of geometric mean to arithmetic mean. Regarding the former concept, many works are available. S. John (1982) and A. C. Kimber (1985) both deal with two-piece normal distribution. G. S. Lingappaiah (1987), obtains two-piece chi-square and F-type distributions starting from a two-piece normal distribution. G. S. Lingappaiah (1988) takes up two-piece double exponential distribution and obtains the distribution of the sum and also that of ratios, from a sample from such a two-piece double exponential distribution. With respect to the latter concept, E. Ronald Glaser (1976a) obtains the distribution of this ratio of geometric mean to aritmetic mean in a sample from a gamma population and in E. Ronald Glaser (1976b), uses this distribution in reference to Bartlett's test for homogeniety.

What is being done here is to obtain this distribution of the ratio of geometric mean to arithmetic mean, when a sample is drawn from a two-piece double exponential distribution, where the scale parameter is θ_1 if $x<0$ and θ_2 if $x\geqq 0$. For this purpose, $E(z^s)$, (with $z=(\tilde{x}/\bar{x})^n$, where $\tilde{x}$ the geometric mean and $\bar{x}$ the arithmetic mean), is put in terms of hypergeometric functions. By inverting, density of z is expressed in terms of Meijer's G-function.

The motivation for this paper is that the double exponential distribution given in (la), also known as First Law of Laplace, plays an important role in statistical theory. This distribution is also called by many authors by many names such as bilateral exponential, Poisson's First Law of error. There is a vast amount of literature on this double exponential. As such parameters θ_1 and θ_2 in our distribution, given in (1) is very important since, if $\theta_1 = \theta_2$, then we have the double exponential. Also it is to be noted that the exponential distribution θ exp $(-x\theta)$, is actually the folded double exponential, "folded at θ" and $1/\theta$ represents the average life in a life test based on the exponential distribution. In these situations $\theta_1 = \theta_2$ against $\theta_1 \neq \theta_2$ is of importance. That is, one tests whether it is double exponential or piece-wise double exponential. One can carry these tests using the result of this paper. That is, one could compute the power of the test by using (12) and also the first kind of error, under the condition $\theta_1 = \theta_2$. Some more reasons for motivation are given at the end of the paper before the Reference Section. Another reason for this paper is the following. Here we are dealing with two statistics in conjunction. One could carry above tests by using the distributions of $\bar{x}$ or $\tilde{x}$ along separately. However, both statistics used together, as done here, give better results. Further, the distribution of $\tilde{x}$ alone may not be as elegant as the expressions here are.

2. Distribution of $z = (\tilde{x}/\bar{x})^n$

Two-piece double exponential distribution is given by

$$f(x) = \begin{cases} A \exp(\theta_1 x), & x < 0 \\ A \exp(-\theta_2 x), & x \geqq 0 \end{cases} \tag{1}$$

where $A = \theta_1 \theta_2 / (\theta_1 + \theta_2)$. If $\theta_1 = \theta_2$, we have the simple double exponential distribution

$$f(x) = (\theta/2) \underset{-\infty < x < \infty}{\exp} (-\theta |x|). \tag{1a}$$

Now consider the ratio $\tilde{x}/\bar{x}$ where $\tilde{x}$ is the geometric mean $(x_1^{1/n} \dots x_n^{1/n})$ and $\bar{x}$ the arithmetic mean $(x_1 + \dots + x_n)/n$, in a sample of size n drawn from a population denoted by (1). Let

$$z = (\tilde{x}/\bar{x})^n = (x_1 \dots x_n)\, n^n / (x_1 + \dots + x_n)^n. \tag{2}$$

Also, let among $x_1, \dots, x_n$, r of them $x_{i_1}, \dots, x_{i_r}$ be <0 and remaining $(n-r)$, $x_{i_{r+1}}, \dots, x_{i_n}$ be $\geqq 0$. Then

$$E(z^s) = A^n \int_0^\infty \left[\frac{n^{ns} t^{ns-1}}{\Gamma(ns)} \sum_{r=0}^{n} \binom{n}{r} e^{\left(t \sum_{j=1}^{r} x_{i_j}\right) - e\left(t \sum_{j=r+1}^{n} x_{i_j}\right)} \right.$$

$$\left. \cdot \left(\prod_{j=1}^{r} \int_{-\infty}^{0} (-1)^s x_{i_j}^s e^{\theta_1 x_{i_j}} dx_{i_j} \right) \left(\prod_{j=r+1}^{n} \int_0^\infty x_{i_j}^s e^{-\theta_2 x_{i_j}} dx_{i_j} \right) \right] dt \tag{3}$$

where $i_j = 1, 2, \dots, n$ and $j = 1, 2, \dots, n$.
Letting $x_{i_j} = -x_{i_j}$ for $j = 1, 2, \dots, r$, we get (3) as,

$$E(z^s)=A^n \sum_{r=0}^{n} \binom{n}{r} \int_0^\infty \frac{t^{ns-1} n^{ns}}{\Gamma(ns)} \left[\left(\prod_{i=1}^{r} \int_0^\infty e^{-x_{i_j}(t+\theta_1)} x_{i_j}^s \mathrm{d}x_{i_j} \right) \cdot \left(\prod_{i=r+1}^{n} \int_0^\infty e^{-x_{i_j}(t+\theta_2)} x_{i_j}^s \mathrm{d}x_{i_j} \right) \right] \mathrm{d}t \tag{4}$$

$$=A^n \sum_{r=0}^{n} \binom{n}{r} \int_0^\infty \frac{n^{ns} t^{ns-1}}{\Gamma(ns)} \frac{\Gamma^n(s+1)\,\mathrm{d}t}{(t+\theta_1)^{(r)(s+1)}(t+\theta_2)^{(n-r)(s+1)}} \tag{5}$$

From A. Erdelyi et al. (1954), we have

$$\int_0^\infty \frac{x^{\delta-1}\,\mathrm{d}x}{(1+\alpha x^h)^\mu (1+\beta x^h)^\lambda} = [(1/h)(1/\alpha^{\delta/h})\beta(\delta/h,\ \mu+\lambda-\delta/h)]^2 \cdot [F_1(\lambda,\ \delta/h;\ \lambda+\mu;\ 1-\beta/\alpha)]. \tag{6}$$

In (5), we have $\delta=ns$, $\alpha=1/\theta_1$, $\beta=1/\theta_2$, $h=1$, $\mu=r(s+1)$, $\lambda=(n-r)(s+1)$ and hence now (5) is [with $u=(\theta_2/\theta_1)^{n-r}.z$],

$$E(u^s)=A^n \sum_{r=0}^{n} \binom{n}{r} (\theta_2/\theta_1)^r (1/\theta_2^n)\Gamma^n(s+1) n^{ns} \cdot [B(ns,\ n)/\Gamma(ns)]_2 F_1[(n-r)(s+1), ns; n(s+1); (1-\theta_1/\theta_2)] \tag{7}$$

$$=A_0^n \sum_r \binom{n}{r} (\theta_2/\theta_1)^r [I^n(s+1)\Gamma(n)n^{ns}] \cdot \sum_{j=0}^{\infty} \left[\left(1-\frac{\theta_1}{\theta_2}\right)^j \frac{1}{j!} \right] \left[\frac{\Gamma\{(n-r)(s+1)+j\}\Gamma(ns+j)}{\Gamma\{(n-r)(s+1)\}\Gamma(ns)\Gamma(ns+n+j)} \right] \tag{8}$$

where $A_0=(A/\theta_2)=\theta_1/(\theta_1+\theta_2)$

$$\text{But } \Gamma(ns)=(2\Pi)^{(1-n)/2} n^{ns-1/2} \left[\prod_{i=0}^{n-1} \Gamma(s+i/n) \right] \cdot \tag{9}$$

Using (9), we get (9) as

$$E(u^s)=A_0^n \sum_r \binom{n}{r} (\theta_2/\theta_1)^r \left[\sum_{j=0}^{\infty} (1-\theta_1/\theta_2)^j \frac{(n-r)^j}{j!} \frac{\Gamma(n)n^{1/2}}{n^n(2\Pi)^{(1-n)/2}} \right] \left[\frac{\Gamma^n(s+1) \prod_{i=0}^{n-1} \Gamma\left(s+\frac{i+j}{n}\right) \prod_{i=0}^{n-r-1} \Gamma\left(s+1+\frac{i+j}{n-r}\right)}{\prod_{i=0}^{n-1} \Gamma(s+i/n) \prod_{i=0}^{n-1} \Gamma\left(s+1+\frac{i+j}{n}\right) \prod_{i=0}^{n-r-1} \Gamma\left(s+1+\frac{i}{n-r}\right)} \right]. \tag{10}$$

Products involving Γ functions in (10) can be put as

(10a)
$$\prod_{i=1}^{3n-r} \Gamma(s+a_i) \Big/ \prod_{i=1}^{3n-r} \Gamma(s+b_i)$$

where $a_i=1,\ i=1,\dots,n;\ a_i=\frac{i+j-n-1}{n},\ i=n+1,\dots,2n$

$$a_i=\frac{i+j-2n-1}{n-r},\quad i=2n+1,\dots,3n-r$$

(10b) $b_i=\frac{i-1}{n},\quad i=1,\dots,n;\quad b_i=1+\frac{i+j-n-1}{n},\quad i=n+1,\dots,2n$

$$b_i=1+\frac{i-2n-1}{n-r},\quad i=2n+1,\dots,3n-r.$$

To obtain the density of u, we have

(11)
$$f(u)=\int_{c'-i\infty}^{c'+i\infty} E(u^s)u^{-(s+1)}\,ds.$$

Now, using (10a) in (10), we have from (11)

(12)
$$f(u)=\left[\begin{array}{l} A_0^n \sum_r \binom{n}{r}(\theta_2/\theta_1)^r \sum_j \left(\frac{B^j}{j!}\right)\left[\frac{\Gamma(n)\,n^{1/2}}{n^n(2\Pi)^{(1-n)/2}}\right] \\ \cdot\left(\frac{1}{u}\right) G_{3n-r,\,3n-r}^{3n-r,0}\left(u\,\middle|\,\begin{matrix} b_1,\dots,b_n;b_{n+1},\dots,b_{2n};b_{2n+1},\dots,b_{3n-r} \\ a_1,\dots,a_n;a_{n+1},\dots,a_{2n};a_{2n+1},\dots,a_{3n-r}\end{matrix}\right) \\ -\infty<u<\infty \end{array}\right.$$

where $B=(n-r)(1-\theta_1/\theta_2)$. G-form in (12) is,

(12a)
$$G\left(u\,\middle|\,\begin{matrix} 0,\frac{1}{n},\dots\frac{n-1}{n};\ 1+\frac{j}{n},\ 1+\frac{j+1}{n},\dots,1+\frac{j+n-1}{n-r};\ 1,\ 1+\frac{1}{n-r},\dots 1+\frac{n-r-1}{n-r} \\ 1,\ 1,\dots 1;\ \frac{j}{n},\ \frac{j+1}{n},\dots,\frac{j+n-1}{n};\ 1+\frac{j}{n-r},\ 1+\frac{j+1}{n-r},\dots,1+\frac{j+n-r-1}{n-r}. \end{matrix}\right)$$

Comments

As mentioned before at the end of Introduction, the main result here can be used to test several hypotheses. Actually, the result can be much more generalized. For example, suppose the variables have different parameters θ_{1i} and θ_{2i}, $i=1, 2,\dots,n$. Then one could test whether $\theta_{1i}=\theta_i$ and $\theta_{2i}=\theta_2$, $i=1, 2,\dots,n$ or $\theta_{1i}=\theta_{2i}=\theta$. Of course, now the form of (12) would be different when one deals with θ_{1i} and θ_{2i}, $i=1, 2,\dots,n$. Yet the method of development remains the same. When one tests $\theta_{1i}=\theta_1$, $\theta_{2i}=\theta_2$ against $\theta_{1i}\neq\theta_1$, $\theta_{2i}\neq\theta_2$, one is testing whether the variables are identically distributed or not. Similarly, testing $\theta_{1i}=\theta_{2i}=\theta$, one tests non-identically distributed variables against double exponential. As such various tests can be carried out by getting the distribution similar to (12) under non-identical variables case.

3. References

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