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Some Results on the π_1-Semigroups

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A finite semigroup S is called a π_n-semigroup if the order of the set $\pi(S)$ is equal to n, where $\pi(S)=\{|T|>1/2|S|: T$ forms a proper subsemigroup of $S\}$. The characterization of π_0-semigroups was established in [1]. Some questions about the π_1-semigroups will be discussed in the present paper.

1. Preliminaries

The order of a subgroup of a finite group is one of the most basic concepts of the theory of groups, and there are many research work on it. In the theory of semigroups, the similar questions were considered; in fact, at the first All-Union symposium of the theory of semigroups in 1969 in Sverdlovsk Schein [12] raised the question of finding the orders of subsemigroups of the symmetrical semigroup $\mathscr{I}_n$, and partial results have been obtained by K. Todorov [2-4], as well by K. H. Kim, F. W. Rouch [5-6]; at the same time, the question of determining the finite semigroups in which the orders of its subsemigroups satisfy some conditions was raised. And this paper belongs to the latter.

In the theory of finite groups, Lagrange's Theorem shows that the order of a subgroup of a finite group is a divisor of the order of the group. This theorem, in general, doesn't hold for finite semigroups. The question to determine the finite semigroups in which the theorem holds was raised in the paper [7], and such semigroups were called Lagrange semigroups (L-semigroups). Two years later, the same question was raised and all of L-semigroups was found in [1]. The types of local L-semigroups were determined in [8]. An L-semigroup of course contains no proper subsemigroup of order greater than the half of the order of the semigroup; for the converse, the characterization of L-semigroup, which was established in [1], follows that a finite semigroup S must be an L-semigroup if S contains no proper subsemigroup of order greater than $1/2|S|$. The present paper will discuss the finite semigroup which admits at least one proper subsemigroup of order greater than the half of its order and such subsemigroups have the same order.

We adopt the notations and terminologies of J. M. Howie's book [9]; in addition, for a semigroup S and $e\in E(S)$ we denote $\pi(S)$ as the set $\{|T|>1/2|S|: T$ forms a proper subsemigroup of $S\}$ and Tor(e) s the set of element $x\in S$ satisfying $e=x^n$ for some $n\in N$, and we say the subset A of S is a minimal set of generators of S if $S=\langle A\rangle$ and $S\neq\langle B\rangle$ for any proper subset B of A.

Definition 1.1. The finite semigroup S is called a π_n-semigroup if the order of the set $\pi(S)$ is equal to the number n.

In essence the object of the paper [1] and [7] was π_0-semigroups, and in what follows we shall determine the π_1-semigroups under certain conditions.

2. Some preparations

At first, we prove a conclusion about the periodic semigroup:

Theorem 2.1. *Let S be a periodic semigroup with a finite set X of right identities, then, for any subset Y of X, there are the following:*

(1) $\bigcup_{y\in Y}\mathrm{Tor}(y)$ *forms a subsemigroup of S and is isomorphic to* $\mathscr{M}[\mathrm{Tor}(e); I, J]$, $I=\{1, 2,\ldots, |Y|\}$, $J=\{1\}$, $e\in X$;

(2) $S-\bigcup_{y\in Y}\mathrm{Tor}(y)$ *forms a right ideal of S;*

(3) $Y=X$ *if and only if* $S-\bigcup_{y\in Y}\mathrm{Tor}(y)$ *forms a proper ideal of S.*

Proof. (1) Let $a\in\mathrm{Tor}(e)$ and $b\in\mathrm{Tor}(f)$ for any $e,f\in Y$, then there exist $m,n\in N$ such that $a^m=e$, $b^n=f$. Since $Sab\supseteq Sa^2b\supseteq\ldots\supseteq Sa^mb=Seb=Sb\supseteq Sb^2\supseteq\ldots\supseteq Sb^n=Sf=S$, $S(ab)^r=S$ for any number $r\in N$ and so there exist $h\in X$ and $k\in N$ such that $(ab)^k=h$ by the assumption. For the element h, on the one hand, $eh=e$ by $Sh=S$, on the other hand, $eh=e(ab)^k=(ea)b(ab)^{k-1}=(a^ma)b(ab)^{k-1}=(aa^m)b(ab)^{k-1}=(ae)b(ab)^{k-1}=ab(ab)^{k-1}=(ab)^k=h$, hence $e=h$. This shows $ab\in\mathrm{Tor}(e)$, and so $\bigcup_{y\in Y}\mathrm{Tor}(y)$ forms a subsemigroup of S. Moreover, $a=af(\text{since } f\in X)=ab^n=abb^{n-1}\in(\bigcup_{y\in Y}\mathrm{Tor}(y))b(\bigcup_{y\in Y}\mathrm{Tor}(y))$, this shows $\bigcup_{y\in Y}\mathrm{Tor}(y)$ is a simple subsemigroup. It's easy to verify that $\bigcup_{y\in Y}\mathrm{Tor}(y)$ is isomorphic to $\mathscr{M}[\mathrm{Tor}(e); I, J]$, $I=\{1, 2,\ldots, |Y|\}$, $J=\{1\}$.

(2) Clearly, $Sa=S$ for any $a\in\bigcup_{x\in X}\mathrm{Tor}(x)$ and $(\bigcup_{y\in Y}\mathrm{Tor}(y))a=\bigcup_{y\in Y}Tor(y)$ by the assumption and the preceding conclusion, hence $(S-\bigcup_{y\in Y}\mathrm{Tor}(y))(\bigcup_{x\in X}\mathrm{Tor}(x))=S-\bigcup_{y\in Y}\mathrm{Tor}(y)$; moreover, by the assumption $Sa=S$ if and only if $a\in\bigcup_{x\in X}\mathrm{Tor}(x)$ and so $S(S-\bigcup_{x\in X}\mathrm{Tor}(x))=S-\bigcup_{x\in X}\mathrm{Tor}(x)$. So $(S-\bigcup_{y\in Y}\mathrm{Tor}(y))S=S-\bigcup_{y\in Y}\mathrm{Tor}(y)$, that is, $S-\bigcup_{y\in Y}\mathrm{Tor}(y)$ forms a right ideal of S.

(3) By the assumption it's easily verified that $St=S$ for $t\in S$ if and only if $t\in\bigcup_{x\in X}\mathrm{Tor}(x)$, hence $S(S-\bigcup_{x\in X}\mathrm{Tor}(x))=S-\bigcup_{x\in X}\mathrm{Tor}(x)$. From (2) it is proved $S-\bigcup_{x\in X}\mathrm{Tor}(x)$ forms an ideal of S. The conclusion obviously holds.

Now some kinds of semigroups will be presented and they are very important in our discussion:

Definition 2.1. For a semigroup S and its isomorphism θ into another one, we denote $R(S,\theta)$ (resp. $L(S,\theta)$) as the semigroup $S\cup\theta(S)$, where the operation is defined by declaring $x\theta(y)=xy$ and $\theta(x)y=\theta(xy)$ (resp. $x\theta(y)=\theta(xy)$ and $\theta(x(y)=xy)$ for any two elements $x, y\in S$, and keeping products as they are otherwise.

Definition 2.2. For a periodic monoid S and two natural numbers m and n, if $m\times n$ matrix $P=(p_{ji})$ over the subgroup Tor(e) where e is the identity of S satisfies the condition:

$$(p_{1i}p_{ji}^{-1}p_{j1})a=a(p_{1i}p_{ji}^{-1}p_{j1})=a$$

for any $a\in S-\mathrm{Tor}(e)$, then $U(S, P, m, n)$ denote the semigroup

$$\mathcal{M}[\mathrm{Tor}(e);\ I,\ J,\ P]\cup(S-\mathrm{Tor}(e)),$$

where $I=\{1, 2,\dots, n\}$ and $J=\{1, 2,\dots, m\}$, the operation is defined by the rule that $(g, i, j)a=p_{1i}gp_{j1}a$ and $a(g, i, j)=ap_{1i}gp_{j1}$ for every $i\in I$, $j\in J$, $g\in\mathrm{Tor}(e)$, and $a\in S-\mathrm{Tor}(e)$, and keeping products as they are otherwise.

Definition 2.3. The small monoid (See [10]) $S=G\cup A$ where A is a completely simple ideal of S and G is the group of units of S, is called a G-monoid if G is maximal in S yet.

Clearly, for every group G there at least exists a G-monoid (in fact, G^0 is a G-monoid); and for a completely simple semigroup A, it's not necessary to exist a group G such that $S=G\cup A$ forms a G-monoid.

Example 2.1. Let $A=\mathcal{M}[D;\ I,\ J,\ P]$ where $I=J=\{1, 2, 3\}$, D is the dihedral group of order 10 presented by $D=\langle a, b;\ a^5=b^2=(ba)^2=e\rangle$ and P is the following matrix:

$$\begin{pmatrix} e & e & e \\ e & a & a^2 \\ e & a^4 & b \end{pmatrix}$$

then the standard group of units of A is trivial and so, by Proposition 5.10 of Ch. 3 of [10], the small monoid $S=G\cup A$ must have the products: $xg=gx=x$ for any $x\in A$, $g\in G$. Evidently, G can not be a maximal subsemigroup of S. This shows that there exists no G-monoid in the form of $G\cup A$.

About G-monoid we have the following:

Theorem 2.2. *Let $A=\mathcal{M}[H;\ I,\ J,\ P]$, then A can be seen as an ideal of certain G-monoid if and only if there exists a group G containing two subgroups K and L, and a map φ of G into H such that*

(1) $H=\langle\varphi(G)\rangle$; (2) $|I|=[G:L]$, $J=[G:K]$;

(3) $\varphi(kgl)=\varphi(k)\varphi(g)\varphi(l)$ *for any* $k\in K$, $l\in L$ *and* $g\in G$;

(4) $p_{ji}=(\varphi(k_j))^{-1}\varphi(k_jl_i)(\varphi(l_i))^{-1}$, *where* $G=\bigcup_i l_i$ $L=\bigcup_i Kk_j$, $P=(p_{ji})$.

Proof. If A can be contained in a G-monoid S as an ideal, then $S=G\cup A$ by the definition. Now we let $p_{1i}=p_{j1}=e$ for every $i\in I$, $j\in J$, and so $g(e, 1, 1)=\varphi(g)$, $\psi(g)$, 1) and $(e, 1, 1)g=(\varphi(g), 1, \chi(g))$ for any $g\in G$. These for formulas define three mappings of G, φ into H, ψ onto I, and χ onto J. Since $S=\langle G, (e, 1, 1)\rangle$

and $\prod_{i=1}^{n-1}(g_i(e,1,1))g_n=(\prod_{i=1}^{n}\varphi(g_i),\psi(g_1),\chi(g_n))$ where $g_i\in G$ and $n\in N$, $H=\langle\varphi(G)\rangle$ and $I=\psi(G)$, $J=\chi(G)$. Now consider the subsets of G: $K=\{g\in G:\chi(g)=1\}$ and $L=\{g\in G:\psi(g)=1\}$, it is easy to prove that K and L form two subgroups. Furthermore, $(gl)(e,1,1)=(\varphi(gl),\psi(gl),1)$ and $g(l(e,1,1))=g(\varphi(l),1,1))=(\varphi(g)\varphi(l),\psi(g),1)$, from which follows $\varphi(gl)=\varphi(g)\varphi(l)$; as the same reason, $\varphi(kg)=\varphi(k)\varphi(g)$, and so $\varphi(kgl)=\varphi(k)\varphi(g)\varphi(l)$, where $g\in G$, $k\in K$ and $l\in L$. Clearly, for f, $g\in G$ $\psi(f)=\psi(g)$ is equivalent to $g^{-1}f\in L$ and $\chi(f)=\chi(g)$ is equivalent to $gf^{-1}\in K$, these imply $|I|=[G:L]$, $|J|=[G:K]$. Moreover, from the equation

$$(e,1,1)(fg)(e,1,1)=((e,1,1)f)(g(e,1,1))$$

where g, $f\in G$, we have $\varphi(gf)=\varphi(g)p_{\chi(g)\psi(f)}\varphi(f)$, that is,

$$p_{\chi(g)\psi(f)}=(\varphi(g))^{-1}\varphi(gf)(\varphi(f))^{-1}.$$

Hence the condition is essential.

Now we prove the direct part: on the set $S=G\cup A$ we define an operation by declaring $g(h,i,j)=(\varphi(gl_i)(\varphi(l_i))^{-1}h,\psi(gl_i),j)$ and $(h,i,j)g=(h(\varphi(k_j))^{-1}\varphi(k_jg),i,\chi(k_jg))$ for every $g\in G$, $h\in H$, $i\in I$ and $j\in J$, where $\psi(g)=i$ if $g\in l_iL$ and $\chi(g)=j$ if $g\in Kk_j$, and keeping products as they are otherwise. It is easy to check associativity and $S=\langle G,x\rangle$ for any $x\in A$, this shows S forms a G-monoid.

This completes the proof.

In the preceding proof a kind of G-monoids was made, and the theorem also shows that every G-monoid is isomorphic to one constructed in the manner. G-monoid is closely related with the object of this paper, and it is one of important material used for constructing π_1-semigroups.

3. Some Simple Results

In the π_1-semigroups with some special conditions will be determined.

Theorem 3.1. *S is a simple π_1-semigroup if and only if S is isomorphic to $\mathcal{M}[G;I,J;P]$, $\mathrm{Max}\{|I|,|J|\}=3$, or $\mathrm{Max}\{|I|,|J|\}=4$ and $\mathrm{Min}\{|I|,|J|\}\leqq 2$, G is a finite group.*

Proof. Let $S=\mathcal{M}[G;I,J;P]$ be a finite semigroup, then S contains only the subsemigroup in the form of $\mathcal{M}[H;K,L;Q]$ with H a subgroup of G, $K\subseteq I$, $L\subseteq J$ and Q a submatrix of P. If $\mathrm{Max}\{|I|,|J|\}=3$, it is easy to check $\pi(S)=\{2|S|/3\}$; if $\mathrm{Max}\{|I|,|J|\}=4$ and $\mathrm{Min}\{|I|,|J|\}\leqq 2$, then $\pi(S)=\{3|S|/4\}$. Hence the condition is sufficent. For the converse, if S is a π_1-semigroup, we can suppose $\mathrm{Max}\{|I|,|J|\}=|I|\geqq 3$ by the conclusion of [1]. Clearly, for $m=(|I|-1)|S|/|I|$, $(|I|-2)|S|/|I|$, $(|J|-1)|S|/|J|$, or $(|I|-1)(|J|-1)|S|/|I||J|$, S must contain a subsemigroup of order equal to m, and so $\pi(S)=\{(|I|-1)|S|/|I|\}$. Hence $(|I|-2)/|I|\leqq 1/2$, $(|J|-1)(|I|-1)/|I||J|\leqq 1/2$, and $(|J|-1)/|J|\leqq 1/2$ or $=(|I|-1)/|I|$, that is, $|I|=4$ and $J\leqq 2$ or $|I|=3$. This shows the condition is essential.

Theorem 3.2. *S is π_1-monoid if and only if S is one of the following types:*
(1) *the non-simple monoid of order* 3 *or* 4;

(2) *Z_p-monoid of order 2p, p a prime;*
(3) *$G \cup \{x\}$, $xG = Gx = G$, $x^2 = x$, G a finite group admitting no subgroup of index 2;*
(4) *$\langle a, b;\ a = a^{p+1},\ b = b^{p+1},\ ab = ba = b \rangle$, p a prime;*
(5) *G-monoid of order $n < 2|G|$, G a finite group containing only the proper subgroup of order $\leqq (|G| - 1/2n)$ or $= (2|G| - n)$.*

Proof. It is enough to prove the essentiality.

Let e be the identity of S and assume $|S| > 4$, then $\mathrm{Tor}(e)$ forms a subgroup and $S - \mathrm{Tor}(e)$ an ideal by Th. 2.1. We divide the argument into three cases:

Case 1. $|\mathrm{Tor}(e)| = 1/2|S|$. At this time, $\pi(S) = \{1/2|S| + 1\}$ since $\{e\} \cup (S - \mathrm{Tor}(e))$ forms a subsemigroup; moreover, $T \cup (S - \mathrm{Tor}(e))$ also forms a subsemigroup for every subgroup T of $\mathrm{Tor}(e)$, this shows $\mathrm{Tor}(e)$ must be a group of order p, p a prime. Let I be the minimal ideal of S, then $I = S - \mathrm{Tor}(e)$: in fact, if $I \neq S - \mathrm{Tor}(e)$, $I = \{f\}$ since $I \cup \mathrm{Tor}(e)$ forms a proper subsemigroup of order greater than $1/2|S|$ and $\pi(S) = \{1/2|S| + 1\}$. For any $x \in S - \mathrm{Tor}(e)$, if there exists an element $a \in \mathrm{Tor}(e)$ such that $f = xa$, then $x\mathrm{Tor}(e) = x(a\mathrm{Tor}(e)) = (xa)\mathrm{Tor}(e) = f\mathrm{Tor}(e) = \{f\}$; if there is no element $a \in \mathrm{Tor}(e)$ such that $f = xa$, $x\mathrm{Tor}(e) = \{x\}$ since $|x\mathrm{Tor}(e)| \leqq |S - \mathrm{Tor}(e) - I| = |\mathrm{Tor}(e)| - 1$. Therefore $x\mathrm{Tor}(e) \subseteq \{x\} \cup I$, and $\mathrm{Tor}(e)x = \{x\} \cup I$ by the same reason. And there must be a proper subsemigroup Z of $S - \mathrm{Tor}(e)$ such that $I \subset Z \subset S - \mathrm{Tor}(e)$ by the result of [1], clearly, $Z \cup \mathrm{Tor}(e)$ forms a subsemigroup of S. This contradicts the assumption, hence $I = S - \mathrm{Tor}(e)$. If $S - \mathrm{Tor}(e)$ is a group, S is of type (4); otherwise, S is of the type (2).

Case 2. $|\mathrm{Tor}(e)| < |S - \mathrm{Tor}(e)|$. At this time $\pi(S) = \{|S - \mathrm{Tor}(e)|\}$ and so $\mathrm{Tor}(e) = \{e\}$ since $\{e\} \cup (S - \mathrm{Tor}(e))$ forms a subsemigroup. Clearly, $S - \mathrm{Tor}(e)$ contains no subsemigroup of order $\geqq 1/2|S|$, that is, $\pi(S - \mathrm{Tor}(e)) = \emptyset$, hence $S - \mathrm{Tor}(e)$ is an L-semigroup by the result of [1]; moreover, it is evident that $S - \mathrm{Tor}(e)$ contains no subsemigroup of order equal to $1/2|S - \mathrm{Tor}(e)|$, consequently, $S - \mathrm{Tor}(e)$ forms a finite group admitting no subgroup of index 2, and S is of the type (3).

Case 3. $|\mathrm{Tor}(e)| > |S - \mathrm{Tor}(e)|$. Clearly, $S - \mathrm{Tor}(e)$ is the minimal ideal of S, and $\mathrm{Tor}(e)$ is maximal in S, hence S is a $\mathrm{Tor}(e)$-monoid. Evidently, S is of the type (5).

Example 3.1. Here we give a semigroup with its Cayley table, and it is of the type (2): (See [13], or see [14]: NR. 158)

	1	2	3	4	5	6
1	3	2	5	4	1	6
2	4	2	6	4	2	6
3	5	2	1	4	3	6
4	6	2	2	4	4	6
5	1	2	3	4	5	6
6	2	2	4	4	6	6

Theorem 3.3. *Let S be a finite semigroup with right identity, and not a monoid. Then S is a π_1-semigroup if and only if S is one of the following types:*
(1) *the non-simple semigroup with right identity of order 3 or 4, and not a monoid;*

(2) $\mathcal{M}[G; I, J; P]$, $|I|=1$, $|J|=3$ or 4, *G a finite group;*

(3) $R(T, \theta)$, *T a G-monoid of order* $3|G|/2$, *G a finite group admitting no subgroup of index* 3;

(4) $U(T, P, 1, n)$, *T a G-monoid of order* $2|G|$, *G a finite group admitting no subgroup of index* k, $(n, k)=(2, 3)$ *or* $(3, 2)$.

Proof. Clearly, if S is one of the type (1) and (2), $\pi(S)=\{2|S|/3\}$ or $\{3|S|/4\}$: if S is the type (3), it is easy to check that $\pi(S)=\{|T\cup\theta(T)|\}=\{2|S|/3\}$; if S is the type (4), it is easy to check that $\pi(S)=\{|\mathcal{M}[G; I, J; P]|\}=\{3|S|/4\}$.

Now we prove the essentiality and let S be a finite non-simple semigroup with a right identity e of order $\geqq 5$, $eS\neq S$, by Th. 2.1 Tor(e) forms a subgroup of S and $S-\mathrm{Tor}(e)$ a right ideal of S.

Step 1. $|\mathrm{Tor}(e)|<|S-\mathrm{Tor}(e)|$. If $|\mathrm{Tor}(e)|\geqq|S-\mathrm{Tor}(e)|$, we derive a contradiction, dividing the argument into two cases:

Case 1. $|\mathrm{Tor}(e)|>|S-\mathrm{Tor}(e)|$. By the assumption $\mathrm{Tor}(e)\subseteq eS\neq S$ and $\pi(S)=\{|\mathrm{Tor}(e)|\}$, hence $eS=\mathrm{Tor}(e)$, and so $ef=(ef)^2=e$ for every $f\in E(S)$. This shows $E(S)=E(\mathrm{Tor}(e))$, a contradiction.

Case 2. $|\mathrm{Tor}(e)|=|S-\mathrm{Tor}(e)|$. If there exists a $x\in S-\mathrm{Tor}(e)$ such that $ex\in\mathrm{Tor}(e)$, then $ef=e$ for the idempotent f of $\langle x\rangle$. Evidently, $S=\mathrm{Tor}(e)\cup f\mathrm{Tor}(e)$ forms an L-semigroup, a contradiction. Therefore $e(S-\mathrm{Tor}(e))\subseteq S-\mathrm{Tor}(e)$, and so $S-\mathrm{Tor}(e)$ forms an ideal of S. It follows that $\pi(S)=\{1/2|S|+1\}$, and Tor(e) is a group of order p, p a prime. Hence S is a monoid, a contradiction (the discussing is the same as case 1 of Th. 3.2).

Step 2. $|X|\geqq 2$, $X=\{x\in E(S): ex=e\}$.

There exists a $x\in S-\mathrm{Tor}(e)$ such that $ex\in\mathrm{Tor}(e)$. In fact, if $e(S-\mathrm{Tor}(e))\subseteq S-\mathrm{Tor}(e)$, then $S-\mathrm{Tor}(e)$ forms an ideal of S and so $|\mathrm{Tor}(e)|=1$ since $\pi(S)=\{|S|-\mathrm{Tor}(e)|\}$ by step 1. Clearly, $\pi(S)=\{|S|-1\}$ and $\pi(S-\mathrm{Tor}(e))=\emptyset$ or $\{1/2|S|\}$. By the result of [1] and Lemma 4.1 there must exists $f\in E(S-\mathrm{Tor}(e))$ such that $|f(S-\mathrm{Tor}(e))|\geqq 1/2(|S-\mathrm{Tor}(e)|)$. Since $f=fe$, $|e(S-\mathrm{Tor}(e))|\geqq|f(S-\mathrm{Tor}(e))|$. Hence $|eS|=|e(S-\mathrm{Tor}(e))|+1=1/2(|S|+1)$, and so $eS=S$ by $\pi(S)=\{|S|-1\}$. This is a contradiction.

Now we let $x\in S-\mathrm{Tor}(e)$ such that $ex\in\mathrm{Tor}(e)$, then $ef=e$ where f is the idempotent of $\langle x\rangle$. Hence $|X|\geqq 2$.

Step 3. $|X|\leqq 3$.

Let $f\in X-\{e\}$. Since $S-\mathrm{Tor}(e)$ and $S-\mathrm{Tor}(e)-\mathrm{Tor}(f)$ form two proper subsemigroups of S by Th. 2.1, we have $|S-\mathrm{Tor}(e)-\mathrm{Tor}(f)|\leqq 1/2|S|$, that is, $|\mathrm{Tor}(e)\cup\mathrm{Tor}(f)|\geqq 1/2|S|$ by $\pi(S)=\{|S-\mathrm{Tor}(e)|\}$; moreover, by the same reason $|\bigcup_{x\in X-\{e\}}\mathrm{Tor}(x)|\leqq 1/2|S|$ since $\bigcup_{x\in X}\mathrm{Tor}(x)$ and $\bigcup_{x\in X-\{e\}}\mathrm{Tor}(x)$ form two proper subsemigroups of S, hence $|X-\{e\}|\leqq 2$, that is, $|X|\leqq 3$.

Step 4. The conclusion holds.

Case 1. $|X|=2$. Let $X=\{e, f\}$, then it is easy to prove $\mathrm{Tor}(e)\cup\mathrm{Tor}(f)$ forms a subsemigroup of S of order greater than $1/2|S|$ by Th. 2.1. Since $\pi(S)=\{|S-\mathrm{Tor}(e)|\}$ by step 1 and Th. 2.1, $|S-\mathrm{Tor}(e)|=|\mathrm{Tor}(e)\cup\mathrm{Tor}(f)|$ and so $\pi(S)=\{2|\mathrm{Tor}(e)|\}=2|S|/3$. Clearly, $S=eS\cup fS$ and $S-\mathrm{Tor}(e)-\mathrm{Tor}(f)$ forms the minimal ideal of S. When $eS\cap fS\neq\emptyset$, $eS=S-\mathrm{Tor}(f)$ and $fS=S-\mathrm{Tor}(e)$ and $S=U(eS, P, 1, 2)$, hence S must be the type (4); when $eS\cap fS=\emptyset$, $S=R(eS, \theta)$ where θ is the map of eS onto $fS: x\to fx$, and so S is the type (3).

Case 2. $|X|=3$. Let $X=\{e,f,h\}$, then $\pi(S)=\{3|\mathrm{Tor}(e)|\}=\{3|S|/4\}$: in fact, by step 1 $\pi(S)=\{|S-\mathrm{Tor}(e)|\}$ and both $S-\mathrm{Tor}(e)-\mathrm{Tor}(f)$ and $\bigcup_{x\in X}\mathrm{Tor}(x)$ form two proper subsemigroups of S by Th. 2.1 and the assumption, hence $|\mathrm{Tor}(e)\cup\mathrm{Tor}(f)|=1/2|S|$ and the statement is true. Clearly, $S=eS\cup fS\cup hS$ and $S-\bigcup_{x\in X}\mathrm{Tor}(x)$ forms the minimal ideal of S. Furthermore, we can prove further that $eS=S-\mathrm{Tor}(f)-\mathrm{Tor}(h)$ and eS forms a $\mathrm{Tor}(e)$-monoid of order $2|\mathrm{Tor}(e)|$. So far, we have proved that $S=U(eS,\ P,\ 1,\ 3)$, therefore S must be of the type (4).

4. The Main Results

In the section we will determine the π_1-semigroup S with $\pi(S)=\{r\}$, $r\leqq 2|S|/3$.

Lemma 4.1. *Let S be a finite semigroup of order greater than 5. If $\pi(S)=\{r\}$, $r\leqq 2|S|/3$, then there exist two subsets X, Y of $E(S)$ such that*

$$S=\bigcup_{x\in X}xS=\bigcup_{y\in Y}Sy \text{ and } |X|,\ |Y|\leqq 3.$$

Proof. Step 1. $S=\bigcup_{e\in E(S)}eS=\bigcup_{e\in E(S)}Se.$

Let A be a minimal set of generators of S. For each $t\in A$ we let

$$R(t)=t^2S^1\ \bigcup_{x\in A-\{t\}} xS^1.$$

Evidently, $R(t)$ forms a right ideal of S. Since $S-\{t\}$ can not form a subsemigroup of S by the condition, t must be contained in $R(t)\cup tR(t)$. Hence $S=R(t)\cup tR(t)$; moreover, $t^3\in R(t)\cap tR(t)$, and so $S-R(t)$ is a proper subset of $tR(t)$. This implies $|R(t)|>1/2|S|$. Now we derive our argument into two cases:

Case of $S=R(t)$. By the structure of $R(t)$ there exist two elements $x\in S$ and $y\in S^1$ such that $t=xty$. From this equation a new one can be obtained: $t=x^nty^n$, any $n\in N$, and so $t\in\bigcup_{e\in E(S)}eS$.

Case of $S\neq R(t)$. At this time, $J(t)=S$. If $R(t)$ is an L-semigroup, then $R(t)=R(t)t^2R(t)\subseteq S^1tS^1\subseteq J(t)$ by the results of [1], and so $S=R(t)\cup tR(t)=J(t)$; if $R(t)$ is not an L-semigroup, there exists a proper subsemigroup Q of $R(t)$ such that $1/2|R(t)|<|Q|$ by the characterization of L-semigroup and $|Q|\leqslant 1/2|S|$ since $\pi(S)=\{R(t)\}$, and so $|\langle Q,\ S-R(t)\rangle|\geqq|Q|+|S-R(t)|>1/2|R(t)|+(|S|-|R(t)|)=|S|-1/2|R(t)|\geqq 2|S|/3$, hence $S=\langle Q,\ S-R(t)\rangle$; moreover, Q is a subsemigroup of S and $S-R(t)\subset tR(t)\subseteq J(t)$, so $S-Q=\langle Q,\ S-R(t)\rangle-Q\subseteq J(t)$. This shows $|J(t)|\geqq|S-Q|\geqq 1/2|S|$, and now we prove $J(t)=S$:

(1) $|J(t)|\neq 1/2|S|$. Otherwise, $|J(t)|=1/2|S|$ and $J(t)=S-Q$. Clearly, $T\cup J(t)$ forms a subsemigroup of S for every subsemigroup T of Q, hence Q must be a group of order p, p a prime and $\pi(S)=\{1/2|S|+1\}$. Since $Q\subsetneq R(t)$ and $|R(t)|\in\pi(S)$, $R(t)=\{t^2\}\cup Q$ and so $t^2S=Qt=\{t^2\}$ by $J(t)\cap Q=\emptyset$. If $t^2=tq$ for some $q\in Q$, then $tQ=t(qQ)=(tq)Q=t^2Q=\{t^2\}$, and so $tR(t)=\{t^2\}$. This is a contradiction to $S-Q\subseteq tR(t)$. If $t^2\neq tq$ for any element q of Q, then there exist two distinct elements a, b of Q such that $ta=tb$ since $|tQ|\leqq|J(t)-\{t^2\}|=|Q|-1$, hence $tQ=t\{ab^{-1}\}=\{te\}$, e is the identity of Q, and so $|R(t)|\geqq|S-tR(t)|=|S|-2$. This is also a contradiction.

(2) $|J(t)| \neq r$. Otherwise, let $x \in S - J(t)$, then S contains the following proper subsemigroup

$$W = \begin{cases} J(t) \cup E(\langle x \rangle) & \text{if } \langle x \rangle \text{ forms a group,} \\ J(t) \cup (\langle x \rangle - \{x\}) & \text{otherwise,} \end{cases}$$

and the order of W is greater than that of $J(t)$. This is a contradiction to the assumption.

From (1) and (2) we have proved $S = J(t)$. It remains to show $t \in \bigcup_{e \in E(S)} eS$: since $S - \{t\}$ can not form a subsemigroup, there exist two elements w, $z \in S$ such that $t = wz$; moreover, there exist two elements x, $y \in S^1$ such that $z = xty$ by $S = J(t)$, hence $t = wz = w(xty) = (wx)ty$ and so t is contained in $\bigcup_{e \in E(S)} eS$.

By the arbitrary property of t we have proved $S = \langle A \rangle = \bigcup_{e \in E(S)} eS$; similarly, $S = \bigcup_{e \in E(S)} Se$.

Step 2. The conclusion holds.

Otherwise, by step 1 we can let the most short decomposition of S as the following

$$S = Se_1 \cup \ldots \cup Se_n, \; n \geqq 4,$$

where $e_i \in E(S)$, $i = 1, 2, \ldots, n$. Clearly, if $|Se_1 \cup Se_2| > 1/2|S|$, then $|Se_1 \cup Se_2|$ and $|Se_1 \cup Se_2 \cup Se_3|$ belong to the set $\pi(S)$; if $|Se_3 \cup \ldots \cup Se_n| > 1/2|S|$, then both $|Se_2 \cup \ldots \cup Se_n|$ and $|Se_3 \cup \ldots \cup Se_n|$ belong to $\pi(S)$; otherwise, $m = \mathrm{Max}\{|Se_1 \cup Se_3 \cup \ldots \cup Se_n|, |Se_2 \cup \ldots \cup Se_n|\} \geqq 3|S|/4$ belongs to $\pi(S)$. These will derive a contradiction to the assumption $\pi(S) = \{r\}$, $r \leqq 2|S|/3$.

Therefore the conclusion holds.

Theorem 4.1. *Let S be a finite semigroup and $S \neq Se \cup Sf$ for any two elements, $e, f \in E(S)$. Then $\pi(S) = \{r\}$, $r \leqq 2|S|/3$ if and only if $S = \mathcal{M}[G; I, J; P]$, $|I| = 3$, $|J| = 1$ and G a finite group.*

Proof. It is enough to prove the essentiality. By Lemma 4.1 we can assume $S = Se \cup Sf \cup Sh$ is the most short decomposition, $e, f, h \in E(S)$. It is easily proved $|Se| = |Sf| = |Sh| = |S|/3$, and $Sxy = Sy$ for any x, y of $\{e, f, h\}$. Hence S must be simple, and so the conclusion holds based on Th. 3.1.

Theorem 4.2. *Let S be a finite non-simple semigroup without one-side identity, and there exist two elements, e, f, of E(S) such that $S = Se \cup Sf$. Then $\pi(S) = \{r\}$, $r \leqq |S| - 2$ if and only if S is one of the following types:*

(1) *$U(T, P, 2, 2)$, T a G-monoid of order $3|G|$;*

(2) *$R(U(T, P, 2, 1), \theta)$, T a G-monoid of order $2|G|$;*

(3) *$L(U(T, P, 1, 2), \theta)$, T a G-monoid of order $2|G|$;*

(4) *$L(R(T, \theta), \varphi)$, T a G-monoid of order $3|G|/2$, where G is a finite group admitting no subgroup of index 3.*

Proof. If S is one of the type (1)-(4), it is easy to check $\pi(S) = \{2|S|/3\}$. Now we prove the essentiality:

Let $X = \{x \in E(S) : ex = e\}$, $Y = \{y \in E(S) : fy = f\}$, I the minimal ideal of S,

$P=Se-\bigcup_{x\in X}\operatorname{Tor}(x)$ and $Q=Sf-\bigcup_{y\in Y}\operatorname{Tor}(y)$. By the given conditions it is easy to prove that $\operatorname{Tor}(x)\cap Sf=\emptyset$ for any $x\in X$ and $\operatorname{Tor}(y)\cap Se=\emptyset$ for any $y\in Y$, and so each of the following sets

$$\operatorname{Tor}(x),\ \operatorname{Tor}(y),\ P,\ Q,\ Se-P,\ Sf-Q$$

forms a subsemigroup of S by Th. 2.1.

Step 1. $Sxy=Sy$ and $Syx=Sx$ for any $x\in X$, $y\in Y$.

Clearly, $Sx=Se$ and $Sy=Sf$. Now we divide our argument into two cases:

Case of $\pi(S)\cap\{|Se|,\ |Sf|\}=\emptyset$. At this time, $|Se|=|Sf|=1/2|S|$, and so $Sx\cap Sy=\emptyset$. Since both $Sx\cup Sxy$ and $Sx\cup Sxy\cup\{y\}$ are the subsemigroups of S, there must be the equation $S=Sx\cup Sxy$ by the assumption $\pi(S)=\{r\}$, $r\leqq|S|-2$, and so $Sy=Sxy$; similarly, $Syx=Sx$.

Case of $\pi(S)\cap\{|Se|,\ |Sf|\}\neq\emptyset$. Let $|Se|\in\pi(S)$, then $r=|Se|$. Clearly, $S=\langle Sx,y\rangle=Sx\cup Sxy\cup\{y\}$, and so $S=Sx\cup Sxy$ since S contains no subsemigroup of order equal to $|S|-1$, hence $Sxy=Sy$. It remains to prove $Syx=Sx$, and it is enough to show $|Sf|=r$.

Since S is non-simple and $r=|Se|$, x must be out of I by the assumption $\pi(S)=\{r\}$, $r\leqq|S|-2$ and so P at least contains the subset Ix. Clearly, $|P|=|Sx|-|Sx-P|\geqq r-1/2|S|$, hence $|P\cup Sy|=|P|+|Sy-P|=|P|+|Sy-Sx|\geqq(r-1/2|S|)+(|S|-r)=1/2|S|$. If $S\neq\langle P,\ Sy,\ x\rangle$, then x can not belong to $Syx\cup SyP$, and so $Syx\subseteq P$ and $\langle P,Sy\rangle=P\cup Sy$, it follows from the assumption $\pi(S)=\{r\}$, $r\leqq|S|-2$ that $r=1/2|S|+1$; moreover, the subset $T\cup P\cup Sy$ also forms a subsemigroup of S for every subsemigroup T of $Sx-P$ since x does not belong to SyT. Therefore $Sx-P$ is a group of order p, p a prime since it admits only the subsemigroup of order equal to 1 or $|Sx-P|$, and so $|S|=|Sx-P|+|Sy\cup P|=p+1/2|S|$, hence $|S|=2p$. Evidently, $|(Sx-P)y|=|Sy-P|\leqq p-1$ and so $|(Sx-P)y|=1$, that is, $Sxy=\{xy\}$ or P, this is a contradiction to $Sxy=Sy$. Consequently, $S=\langle P,Sy,x\rangle$, this implies $S=P\cup Sy\cup Syx\cup\{x\}$. It follows that $S=P\cup Sy\cup Syx$ and so x is contained in $Syx\cup SyK$, hence $|Sy|\geqq|Sx|$, this shows $|Sy|=|Sx|=r$.

Now we can assume $ef=f$ and $fe=e$.

Step 2. $Pf=Q$, $Qe=P$, $(Se-P)f=Sf-Q$ and $(Sf-Q)e=Se-P$.

Since $Se(Pf)=(SeP)f\subseteq Pf$, it is easy to prove, by step 1, that $Pf\subseteq Q$; similarly, $Qe\subseteq P$, and so $P=P(fe)=(Pf)e\subseteq Qe$. Hence $P=Qe$ and $Q=Pf$. Furthermore, $(Se-P)f=Sf-Q$ and $(Sf-Q)e=Se-P$.

Step 3. $I=P\cup Q$.

By step 2 it is easy to prove that S contains the following subsemigroups

$$P\cup Q,\ P\cup Q\cup\{e\},\ P\cup Q\cup\{e,f\},\ S-P\cup Q,\ (S-P\cup Q)\cup I,$$

hence $S=(S-P\cup Q)\cup I$ by the condition $\pi(S)=\{r\}$, and so $I=P\cup Q$.

Step 4. $|S-I|=r=2|S|/3$ and $S-I$ is an L-semigroup without one-side identity.

Clearly, $|S-I|=|Se\cup I|=|Sf\cup I|=r$, hence $r=|S-I|=|(Se-Sf\cup I)\cup(Sf-Se\cup I)|=2|Se-Sf\cup I|=2(|S|-r)$ and so $r=2|S|/3$, $|I|=|S|/3$. If $S-I$ contains a subsemigroup T of order greater than $1/2|S-I|$, then $I\cup T$ forms a proper subsemigroup of S of order greater than r, a contradiction. So $S-I$ is an L-semigroup. By "S without one-side identity" it is easy to show $S-I$ has no one-side identity: in fact, if the statement is false, $e(S-I)=S-I$ and so $eS=S$, a contradiction.

Step 5. $S=eS\cup hS$, where $h\in E(S-I)-\{e,f\}$.

By step 4 there exists $h\in E(S-I)-\{e, f\}$ such that $S-I=e(S-I)\cup h(S-I)$. Since $I\cap(eS\cup hS)\neq\emptyset$, $S=eS\cup hS$. Here we let $eh=e$ and $he=h$.

Step 6. The conclusion holds.

Tor(e) admits no subgroup of index 3; in fact, if the statement is false, then $S-I$ must have a subsemigroup T such that its order equal to $|S-I|/3$ and so $T\cup I$ forms a subsemigroup of order $5|S|/9$, a contradiction.

If $Se\cap Sf\neq\emptyset$ and $eS\cap hS\neq\emptyset$, then, by the preceding results, we have $I=Se\cap Sf=eS\cap hS$ and so Tor$(e)\cup I$ forms a Tor(e)-monoid of order $3|\text{Tor}(e)|$ since $S=\langle S-I, x\rangle=(S-I)\cup\langle\text{Tor}(e), x\rangle$ for any element x of I, hence $S=U(\text{Tor}(r)\cup I, P, 2, 2)$ is of the type (1); if $Se\cap Sf=eS\cap hS=\emptyset$, then $S=L(Se,\varphi)$ where φ is the map of Se onto $Sf: x\to xf$ and $Se=R(eSe, \theta)$ where θ is the map of eSe onto $hSe: x\to hx$, hence $S=L(R(eSe), \theta), \varphi)$ is of the type (4); if $Se\cap Sf=\varphi$ and $eS\cap hS\neq\varphi$, then $I=eS\cap hS$ and $S=L(Se, \theta)$ where θ is the map of Se onto $Sf: x\to sf$, moreover, it is clear that $\pi(Se)=\{|S|/3\}$ and so $Se=U(eSe, P, 1, 2)$ by Th. 3.3, hence $S=L(U(eSe, P, 1, 2), \theta)$ is of the type (3); if $Se\cap Sf\neq\emptyset$ and $eS\cap hS=\emptyset$, then S is of the type (2) by the same reason.

This completes the proof.

Theorem 4.3. *Let S be a finite semigroup of order greater than 6. Then $\pi(S)=\{r\}$, $r\leq 1/2|S|+1$ if and only if S is a Z_p-monoid of order $2p$ or $S=\langle a, b; a^{p+1}=a, b^{p+1}=b, ab=ba=a\rangle$, p a prime.*

Proof. It is easy to check the direct part. For the converse, S must be a monoid by La4.1, Th. 4.1, Th. 4.2 and Th. 3.3 and so the conclusion holds by Th. 3.2.

Theorem 4.4. *Let S be a finite semigroup. Then $\pi(S)=\{r\}$, $1/2|S|+2\leq r\leq 2|S|/3-1$ if and only if S is a G-monoid of order n, where $3n+12\leq 6|G|\leq 4n-6$, and G is a finite group admitting only the proper subgroup of order $=|G|-1/2n$ or equal to $2|G|-n$.*

Proof. (as the proof of Th. 4.3).

Theorem 4.5. *Let S be a finite non-simple semigroup of order $\neq 6$. Then $\pi(S)=\{2|S|/3\}$ if and only if S contains an L-semigroup L of order $2|S|/3$ as its maxiaml subsemigroup, and L admits no subsemigroup of order $|L|/3$.*

Proof. It is easy to verify that the condition is essential by La 4.1, Th. 3.2, Th. 3.3, Th. 4.1 and Th. 4.2.

Now we prove the direct part:

Assume $|S|\geq 7$ and $e\in E(L)$. If the condition holds, it is easy to prove that $S-L$ forms properly the minimal ideal of S. For any element x of $S-L$, $eSe=\langle\text{Tor}(e), exe\rangle$ since $S=\langle L, exe\rangle$ by the condition, hence eSe is a Tor(e)-monoid. Now consider the proper subsemigroup T of order r, $r\in\pi(S)$: if $T=L$, then $r=2|S|/3$; if $T\neq L$, by the choice of T and the given condition $T\cap L$ forms a subsemigroup of S of order $|S|/3$, and we can prove further that $E(T-L)=E(S-L)$ by the inequality on indices (Th. 1.5.5, [11]) and the structure of G-monoid, hence $T-L=S-L$ by $r\geq 1/2|S|+1$, consequently, $r=|T\cap L|+|S-L|=2|S|/3$. This shows $\pi(S)=\{2|S|/3\}$.

At the final, we conclude with an example for Theorem 4.4:

Example 4.1. Suppose $G=\langle a; a^{78}=a\rangle$ and A is the rectangular band $(I\times J,\cdot)$, $I=J=\{1, 2,\ldots, 7\}$. Let $S=G\cup A$ and on the set S define an operation by the rule that $a^k(i, j)=(\theta(k, i), j)$ and $(i, j)a^k=(i, \theta(k, j))$ for every $i \in I$, $j \in J$, $k=1, 2, \ldots, 77$, and keeping products as they are otherwise, where $\theta(k, n)$ is the minimal nonnegative remanent number of $k+n$ for the module 7. Then S forms a semigroup of order 126 and $\pi(S)=\{|G|\}=\{77\}=\{11|S|/18\}$.

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