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# Approximation by Positive Operators on Infinite Intervals

*S. M. Mazhar**

*Presented by Bl. Sendov*

An operator based on Gamma operator, introduced earlier by A. Lupas and M. Müller, is defined and some basic results concerning this operator are obtained.

## 1. Introduction

Let $g_n(x, u)=\frac{x^{n+1}}{n!}e^{-xu}u^n$, $x>0$, then the operator $G_n(f, x)$ defined

$$G_n(f, x)=\int_0^\infty g_n(x, u)f\left(\frac{n}{u}\right)du$$

is called Gamma operator which was introduced by A. Lupas and M. Müller [1]. We define an operator $F_n(f, x)$ as

$$F_n(f, x)=\int_0^\infty g_n(x, u)\,du\int_0^\infty g_{n-1}(u, t)f(t)\,dt$$

$$=\int_0^\infty \frac{(2n)!}{n!\,(n-1)!}\frac{x^{n+1}t^{n-1}}{(x+t)^{2n+1}}f(t)\,dt,\ n>1,\ x>0$$

$$=\frac{(2n)!}{n!\,(n-1)!}\int_0^\infty \frac{w^{n-1}}{(1+w)^{2n+1}}f(wx)\,dw$$

for any $f$ for which the last integral is convergent.

From the relation

$$\int_0^\infty e^{-ut}t^\alpha dt=\frac{\Gamma(\alpha+1)}{u^{\alpha+1}},$$

we easily find that

* Presently on sabbatical leave from Kuwait University, P. Box 5969, 13060-Safat-KUWAIT

$$F_n(1, x) \equiv 1 \qquad (\text{and so } \|F_n\| = 1).$$
$$F_n(t, x) \equiv x,$$
$$F_n(t^2, x) = \frac{n+1}{n-1} x^2, \ n > 1$$

so that

$$F_n((t-x)^2, x) = \frac{2x^2}{n+1}.$$

We write

$$\Delta_h^2(f, x) = f(x+h) - 2f(x) + f(x-h),$$

$$w_x(f, \delta) = \sup_{\substack{0 \leqq h \leqq \delta \\ x \pm hx \in (0, \infty)}} |\Delta_{hx}^2(f, x)|$$

and denote by $C(0, \infty)$ the class of bounded and continuous functions in $(0, \infty)$. In this note we propose to study some of the basic results concerning the operator $F_n$.

## 2. Degree of approximation

To obtain a result on the degree of approximation we need the following lemma which is a particular case $\varphi(x) = x$ of Theorem 1 in [2].

**Lemma 2.1.** *Let $\{L_n\}$ be a sequence of positive linear operators mapping $C(0, \infty)$ into $C(0, \infty)$ such that $L_n(1, x) \equiv 1$, $L_n(t, x) \equiv x$ and $L_n((t-x)^2, x) \leqq K x^2 \alpha_n^2$, where $\alpha_n \to 0$ and $n \to \infty$, then for $f \in C(0, \infty)$*

$$|L_n(f, x) - f(x)| \leqq K w_x\left(f, \frac{1}{\sqrt{n}}\right).$$

Using the above lemma with $\alpha_n = \dfrac{1}{\sqrt{n}}$ we have

**Theorem 2.2.** *If $f \in C(0, \infty)$, then*

$$|F_n(f, x) - f(x)| \leqq K w_x\left(f, \frac{1}{\sqrt{n}}\right).$$

## 3. Uniform approximation

We now proceed to characterize those functions $f$ which can be approximated uniformly by $F_n(f, x)$. It is easy to see that

$$\int_0^\infty g_{n+1}(x, u)\,du \int_0^\infty g_{n-1}(u, t)\,dt$$

$$= \int_0^\infty \frac{t^{n-1}x^{n+2}}{\Gamma(n)\Gamma(n+2)} \frac{\Gamma(2n+2)}{(x+t)^{2n+2}}\,dt$$

$$= \frac{\Gamma(2n+2)}{\Gamma(n+2)\Gamma(n)} \int_0^1 t^{n+1}(1-t)^{n-1}\,dt = 1$$

and,

$$F'_n(f, x) = \int_0^\infty \frac{\partial}{\partial x} g_n(x, u)\,du \int_0^\infty g_{n-1}(u, t) f(t)\,dt$$

$$= \int_0^\infty \frac{n+1}{x}(g_n(x, u) - g_{n+1}(x, u))\,du \int_0^\infty g_{n-1}(u, t) f(t)\,dt.$$

Hence,

$$|F'_n(f, x)| \leqq \frac{n+1}{x} \int_0^\infty (g_n(x, u) + g_{n+1}(x, u))\,du \int_0^\infty g_{n-1}(u, t) \|f\|\,dt$$

$$\leqq \frac{2(n+1)}{x} \|f\|.$$

Using Theorem 2 in [2] with $\varphi(x) = x$, we obtain the following result on uniform approximation.

**Theorem 3.1.** *For $f \in C(0, \infty)$ the following statements are equivalent:*

(i) $F_n(f) - f = 0(1)$ *uniformly in* $(0, \infty)$,

(ii) $\lim_{h \to 0} f(x + hx) - f(x) = 0$ *uniformly in* $(0, \infty)$,

(iii) $f(e^x)$ *is uniformly continuous in* $(-\infty, \infty)$.

## 4. A global saturation result

In this section we examine the global saturation problem for our operator $F_n(f, x)$. Using Proposition 1 in [2] with $\varphi(x) = x$ and $\alpha_n = \frac{1}{\sqrt{n}}$, we have

**Theorem 4.1.** *$\{F_n\}$ is globally saturated on $(0, \infty)$ with order $\{\frac{1}{n}\}$ and has the saturation class $\{f / f'$ is locally absolutely continuous and $x^2 f''(x) \leqq K\}$.*

Proof. To prove our theorem we have only to verify, in view of Proposition 1 in [2] that

$$F_n(h_{x,\varepsilon}; x) = 0_{x,\varepsilon}\left(\frac{1}{n}\right),$$

where

$$h_{x,\varepsilon}(t)=\begin{cases}(t-x)^2, & |t-x|\geqq\varepsilon,\\ 0 & |t-x|<\varepsilon.\end{cases}$$

Writing

$$G_n(x,\ t)=\frac{(2n)!}{n!\,(n-1)!}\frac{x^{n+1}\,t^{n-1}}{(x+t)^{2n+1}},$$

we have

$$F_n(h_{x,\varepsilon}\,;\ x)=\Big(\int_0^{x-\varepsilon}+\int_{x+\varepsilon}^{\infty}\Big)G_n(x,\ t)(t-x)^2\,dt$$
$$=I_1+I_2,\ \text{say.}$$

Now

$$I_2\leqq\frac{(2n)!}{n!\,(n-1)!}\int_{x+\varepsilon}^{\infty}\frac{x^{n+1}\,t^{n+1}}{(x+t)^{2n+1}}dt$$
$$\leqq K_x 2^{2n}\int_{1+\eta}^{\infty}\frac{u^{n+1}\,du}{(1+u)^{2n+1}},\ \ \eta=\frac{\varepsilon}{x}$$
$$\leqq K_x 2^{2n}\Big(\frac{(1+\eta)}{(2+\eta)^2}\Big)^{n-2}\int_{1+\eta}^{\infty}\frac{u^3}{(1+u)^5}du$$
$$=O\Big(\Big(\frac{4+4\eta}{4+4\eta+\eta^2}\Big)^n\Big)=0_{x,\varepsilon}\Big(\frac{1}{n}\Big).$$

Also

$$I_1\leqq\frac{(2n)!}{n!\,(n-1)!}\int_0^{x-\varepsilon}\frac{x^{n+1}\,t^{n-1}\,x^2}{(x+t)^{2n+1}}dt$$
$$\leqq K_x 2^{2n}\int_0^{1-\eta}\frac{u^{n-1}}{(1+u)^{2n+1}}du$$
$$\leqq K_x 2^{2n}\Big\{\frac{1-\eta}{(2-\eta)^2}\Big\}^{n-1}=O\Big\{\Big(\frac{4-4\eta}{4-4\eta+\eta^2}\Big)^{n-1}\Big\}$$
$$=0_{x,\varepsilon}\Big(\frac{1}{n}\Big).$$

This proves Theorem 4.1.

## 5. Lipschitz class

Finally we prove the following theorem:

**Theorem 5.1.** *Let* $0<\alpha<1$ *and* $f\in C(0,\ \infty)$. *Then the following statements are equivalent*:

(a) $F_n(f)-f=O(n^{-\alpha})$ *uniformly in* $(0, \infty)$,
(b) $w_x(f, \delta)=O(\delta^{2\alpha})$,
(c) $f(e^x)\in Lip_2 2\alpha$,

*where* $Lip_2\beta=\{g\in C(0, \infty)/\Delta_h^2(g, x)\leqq K_g h^\beta, h>0\}$.

Proof. To prove that (a) $\longleftrightarrow$ (b), in view of the Corollary in ([2], page 167) it is enough to prove that

$$|x^2F_n''(f, x)|\leqq Kn\|f\|, f\in C(0, \infty) \tag{5.1}$$

and

$$|x^2F_n''(f, x)|\leqq K(\|\varphi^2 f''\|+\|f\|), \tag{5.2}$$

where $\varphi(x)=x$, $f\in C(0, \infty)$ and $f'$ is locally absolutely continuous.

Now it is easy to see that

$$x^2F_n''(f, x)=\int_0^\infty \frac{(2n)!}{n!(n-1)!}H_n(x, t)\frac{x^{n+1}t^{n-1}}{(x+t)^{2n+1}}f(t)\,dt,$$

where

$$\begin{aligned}H_n(x, t)&=n(n+1)-2(n+1)(2n+1)\frac{x}{x+t}\\&\quad+(2n+1)(2n+2)\frac{x^2}{(x+t)^2}\\&=O(n)+n^2\left(1-\frac{4x}{x+t}+\frac{4x^2}{(x+t)^2}\right)\\&=O(n)+n^2\left(1-\frac{2x}{(x+t)}\right)^2.\end{aligned}$$

Since

$$\int_0^\infty \frac{x^a t^b}{(x+t)^{a+b+1}}dt=\frac{(a-1)!\,b!}{(a+b)!},\quad a, b,\in N,$$

we get

$$\begin{aligned}|x^2F_n''(f, x)|&\leqq Kn\|f\|+\left|\int_0^\infty \frac{(2n)!}{n!(n-1)!}\frac{x^{n+1}t^{n-1}}{(x+t)^{2n+1}}n^2\left(1-\frac{2x}{x+t}\right)^2 f(t)\,dt\right.\\&=\|f\|\left\{O(n)+\frac{(2n)!}{n!(n-1)!}n^2\left(\frac{n!(n-1)!}{(2n)!}-\frac{4(n+1)!(n-1)!}{(2n+1)!}\right.\right.\\&\qquad\left.\left.+\frac{4(n+2)!(n-1)!}{(2n+2)!}\right)\right\}=\|f\|O(n).\end{aligned}$$

Also

$$F_n(f, x)=\int_0^\infty \frac{x^{n+1}}{n!}e^{-xu}.u^n\,du\int_0^\infty \frac{u^n}{(n-1)!}e^{-ut}t^{n-1}f(t)dt$$

$$=\frac{1}{n!\,(n-1)!}\int_0^\infty y^n e^{-y}\,dy\int_0^\infty e^{-z} z^{n-1} f\left(\frac{xz}{y}\right)dz,$$

so $$F_n''(f, x)=\frac{1}{n!\,(n-1)!}\int_0^\infty y^n e^{-y}\,dy\int_0^\infty e^{-z} z^{n-1} f''\left(\frac{xz}{y}\right)\left(\frac{z}{y}\right)^2 dz.$$

Thus

$$x^2 F_n''(f, x)=\frac{1}{n!\,(n-1)!}\int_0^\infty y^n e^{-y}\,dy\int_0^\infty e^{-z} z^{n-1}\left(\frac{xz}{y}\right)^2 f''\left(\frac{xz}{y}\right)dz$$
$$\leqq \|\varphi^2 f''\| \leqq K(\|\varphi^2 f''\|+\|f\|).$$

This verifies (5.1) and (5.2). The proof of the equivalence of (b) and (c) is omitted as it is already contained in [2].

## References

[1] A. Lupas, M. Müller. Approximationseigenschaften der Gammaoperatoren. *Math. Zeit*, **98**, 1967, 208-226.
[2] V. Totik. Uniform approximation by positive operators on infinite interval. *Anal. Math.*, **10**, 1984, 163-182.

*Dept. of Mathematics*
*Ohio State University*
*Columbus, OH 43210-1174*
*U.S.A.*

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