Mathematica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

Mathematica Balkanica

New Series Vol. 5, 1991, Fasc. 2

Rigorous Sensitivity Analysis for Real Symmetric Matrices with Uncertain Data

Christian Jansson

Presented by Bl. Sendov

At present, interval methods for the solution of linear algebraic systems with interval input data assume that all input data very independently between their given lower and upper bounds. Solving with those methods linear symmetric systems a severe overestimation may occur. In this paper we introduce a new method calculating very sharp bounds for linear symmetric systems with uncertain input data.

1. Introduction

We consider the numerical solution of a system of linear algebraic equations

(1.1) $$Ax=b,$$

where A is a symmetric real $n\times n$-matrix and b is a real column vector with n components. Such systems arise in many practical applications in physics and engineering. Moreover, in most applications the coefficients of A, b are uncertain due to measurements.
Interval arithmetic (cf. [1], [2], [14], [15], [18]) provides a useful tool for solving linear systems the coefficients of which are uncertain. In interval arithmetic the real input data are replaced by real compact intervals

(1.2) $$[a]:=[\underline{a},\ \bar{a}]\ :=\{a\in \mathrm{R}\,|\,\underline{a}\leqq a\leqq \bar{a}\},$$

where $\underline{a}\leqq \bar{a}$. It is assumed that the reader is familiar with the basic results of this theory. We will use the following notations. R, R^n, $\mathrm{R}^{n\times n}$ are the sets of real numbers, real column vectors with n components and real $n\times n$ matrices. II R denotes the set of all real compact intervals. II R^n is the set of all real interval vectors

(1.3) $$[x]:=[\underline{x},\ \bar{x}]\ :=\{x\in \mathrm{R}^n\,|\,\underline{x}\leqq x\leqq \bar{x}\},\quad \underline{x}\leqq \bar{x}$$

with n real interval components $[x_i]=[\underline{x}_i, \bar{x}_i]$, $i=1,\dots, n$. II $\mathrm{R}^{n\times n}$ is the set of all real interval matrices

(1.4) $$[A]:=[\underline{A},\ \bar{A}]\ :=\{A\in \mathrm{R}^{n\times n}\,|\,\underline{A}\leqq A\leqq \bar{A}\},\quad \underline{A}\leqq \bar{A}$$

with n^2 real interval coefficients $[a_{ij}]=[\underline{a}_{ij}, \bar{a}_{ij}]$, $i, j=1,\dots, n$ ($\leqq$ is to be understand componentwise). We will use $*\in\{+, -, \cdot, /\}$ as well for the real arithmetic operations as for the interval operations in $\mathbb{I}\mathbb{R}$, $\mathbb{I}\mathbb{R}^n$, $\mathbb{I}\mathbb{R}^{n\times n}$; from the context it will be always clear whether it is an operation on reals or on intervals. For a set $X \subseteq \mathbb{R}^n$ the i-th component is marked by

$$(1.5) \qquad X_i := \{x_i \in \mathbb{R} \mid x = (x_i) \in X\}.$$

For any interval $[a] = [\underline{a}, \bar{a}]$ the diameter of $[a]$ is

$$(1.6) \qquad d([a]) := \bar{a} - \underline{a}$$

and the midpoint of $[a]$ is

$$(1.7) \qquad m([a]) := \underline{a} + 0.5 \cdot d([a]).$$

For interval vectors and interval matrices diameter and midpoint are defined componentwise.
For an interval vector $[x] \in \mathbb{I}\mathbb{R}^n$

$$(1.8) \qquad \operatorname{int}([x]) := [x \in \mathbb{R}^n \mid \underline{x} < x < \bar{x}\}$$

denotes the interior of $[x]$.

A linear interval system is a family of linear systems

$$(1.9) \qquad Ax = b, \quad A \in [A] \in \mathbb{I}\mathbb{R}^{n\times n} \text{ and } b \in [b] \in \mathbb{I}\mathbb{R}^n.$$

We denote a solution of this family by $x = x(A, b) := A^{-1}b$ to indicate the dependency on the input data A, b. The corresponding solution set is defined by

$$(1.10) \qquad \Sigma := \{x(A, b) \in \mathbb{R}^n \mid A \in [A],\ b \in [b]\}$$

Several methods (cf. [4], [5], [11], [16], [17], [18]) for computing outer bounds $[x] = ([\underline{x}_i, \bar{x}_i])$ of the solution set are known. That means these methods compute outer bounds for each component of Σ, i.e.

$$(1.11) \qquad \underline{x}_i \leqq \inf \Sigma_i \leqq \sup \Sigma_i \leqq \bar{x}_i \text{ for } i = 1, \ldots, n.$$

Depending on the method the outer bounds may overestimate some or all components Σ_i of the solution set. Recently, S. M. Rump [20] has developed a method for calculating additionally inner bounds $[y] = ([\underline{y}_i, \bar{y}_i])$ with

$$(1.12) \qquad \inf \Sigma_i \leqq \underline{y}_i \leqq \bar{y}_i \leqq \sup \Sigma_i, \quad i = 1, \ldots, n.$$

These inner bounds allow to determine the degree of sharpness of the calculated outer bounds. Moreover, a guaranteed sensitivity analysis for a linear system with interval input data is given.
For the above methods it is always assumed that the input data vary independently. In the case of symmetric matrices with uncertain data, dependencies occur and it is of interest to calculate inner and outer bounds of the corresponding solution set

$$(1.13) \qquad \Sigma^{\text{sym}} := \{x(A, b) \mid A \in [A],\ b \in [b],\ A \text{ symmetric}\}.$$

Obviously $\Sigma^{\text{sym}} \subseteq \Sigma$ but Σ^{sym} may be small compared to the latter. At present, methods for calculating outer or inner bounds of Σ^{sym} are not known (compare [15], Chapter 3).

In Section 2 we present a method calculating very sharp inner and outer bounds for the solution set Σ^{sym}. In Section 3 an algorithm with some remarks about the convergence and some details of implementation on digital computers is described. Section 4 contains some numerical experiments.

2. Basic Theorem

In the following we always assume that $[A]\in \mathrm{II\,R}^{n\times n}$ with $[a_{\mu\nu}]=[a_{\nu\mu}]$ for $\nu,\mu=1,\dots,n$, $[b]\in \mathrm{II\,R}^n$, R is a real $n\times n$-matrix with row vectors r^i for $i=1,\dots,n$ and $\tilde{x}\in \mathrm{R}^n$.
In applications usually R is a calculated approximative inverse of the midpoint of $[A]$ and $\tilde{x}$ is an approximative solution of the midpoint system $m([A])x=m([b])$. If $A\in[A]$, $b\in[b]$, $[w]\in \mathrm{II\,R}^n$ and

$$(2.1)\qquad R(b-A\tilde{x})+(I-RA)\cdot[w]\subseteq \operatorname{int}[w]$$

then from a Theorem of S. M. Rump ([18], page 59) it follows that R, A are nonsingular and

$$(2.2)\qquad x(A,\ b):=A^{-1}b\in\tilde{x}+R(b-A\tilde{x})+(I-RA)\cdot[w].$$

Hence, (1.13) yields

$$(2.3)\qquad \Sigma^{sym}\subseteq \cup\{\tilde{x}+R(b-A\tilde{x})+(I-RA)\cdot[w]\mid A\in[A],\ b\in[b],\ A \text{ symmetric}\}$$

if (2.1) is fulfilled for all $A\in[A]$, $b\in[b]$, A symmetric. To obtain very sharp bounds for the solution set Σ^{sym} we have to examine carefully the right hand side of (2.3). If the diameter of [A] is of moderate size then the quantities

$$I-RA,\ [w] \text{ and } R(b-A\tilde{x})$$

are of small order of the magnitude for all $A\in[A]$, $b\in[b]$. Because the set

$$(2.4)\qquad \cup\{(I-RA)[w]\mid A\in[A],\ A \text{ symmetric}\}$$

is the product of two small quantities this set is in general small compared to

$$(2.5)\qquad Q:=\{R\cdot(b-A\tilde{x})\mid A\in[A],\ b\in[b],\ A \text{ symmetric}\}.$$

The first set is obviously contained in $(I-R[A])\cdot[w]$. In the following we construct sharp bounds for every component of Q.
The i-th component of $R(b-A\tilde{x})$ satisfies

$$r^i(b-A\tilde{x})=\sum_{\mu=1}^{n} r_{i\mu}b_\mu-\sum_{\mu=1}^{n} r_{i\mu}\Big(\sum_{\nu=1}^{n} a_{\mu\nu}\tilde{x}_\nu\Big)$$

$$=\sum_{\mu=1}^{n} r_{i\mu}(b_\mu-a_{\mu\mu}\tilde{x}_\mu)-\sum_{\substack{\mu,\nu=1\\ \mu\neq\nu}}^{n} r_{i\mu}a_{\mu\nu}\tilde{x}_\nu.$$

The symmetry $a_{\mu\nu}=a_{\nu\mu}$ yields

$$(2.6)\qquad r^i(b-A\tilde{x})=\sum_{\mu=1}^{n} r_{i\mu}(b_\mu-a_{\mu\mu}\tilde{x}_\mu)-\sum_{\substack{\mu,\nu=1\\ \mu<\nu}}^{n}(r_{i\mu}\tilde{x}_\nu+r_{i\nu}\tilde{x}_\mu)a_{\mu\nu}.$$

By defining

$$(2.7)\qquad [z_i] := \sum_{\mu=1}^{n} r_{i\mu}([b_\mu]-[a_{\mu\mu}]\tilde{x}_\mu) - \sum_{\substack{\mu,\nu=1\\ \mu<\nu}}^{n} (r_{i\mu}\tilde{x}_\nu + r_{i\nu}\tilde{x}_\mu)[a_{\mu\nu}]$$

for $i=1,\ldots,n$ from a Theorem of R. E. Moore ([14], page 23) it follows that

$$(2.8)\qquad Q_i = [z_i] \text{ for } i=1,\ldots,n,$$

where $Q_i := \{r^i(b-A\tilde{x}) \mid A\in[A],\ b\in[b],\ A \text{ symmetric}\}$ is the i-th component of Q. This is because in (2.7) each interval variable occurs only once and to the first power.

Now, by defining

$$(2.9)\qquad [x_i]=[\underline{x}_i, \bar{x}_i] := \tilde{x}_i + [z_i] + (e_i^t - r^i[A])\cdot[w] \text{ for } i=1,\ldots,n$$

(e_i denotes the i-th canonical unit vector) from (2.3), (2.8) and from elementary properties of the interval operations it follows that

$$(2.10)\qquad \underline{x}_i \leqq \inf(\Sigma_i^{sym}) \leqq \sup(\Sigma_i^{sym}) \leqq \bar{x}_i \text{ for } i=1,\ldots,n.$$

To be perfectly clear, we mention that because of (2.8) exact bounds for Q are calculated. Therefore the outer bounds (2.10) are very sharp for moderate problems where $d((I-R[A]\cdot[w])$ is small compared to $d(Q)$.

Now, we assume the existence of an interval vector $[w]\in II\,\mathbb{R}^n$ such that

$$(2.11)\qquad [z]+(I-R[A])[w] \subseteq \operatorname{int}[w].$$

Then (2.1) is satisfied for all symmetric $A\in[A]$, $b\in[b]$ and by (2.9) outer bounds $[x]$ of the solution set Σ^{sym} are given. In the following, we show how to calculate the inner bounds.

Obviously, for all $A\in[A]$, $b\in[b]$ then equation

$$(2.12)\qquad A^{-1}b = \tilde{x} + R(b-A\tilde{x}) + (I-RA)\cdot(A^{-1}b-\tilde{x})$$

holds. Looking at the i-th component of (2.12) it follows that

$$(2.13)\qquad \Sigma_i^{sym} = \tilde{x}_i + V_i$$

with

$$V_i := (r^i(b-A\tilde{x}) + (e_i^t - r^iA)(A^{-1}b-\tilde{x}) \mid A\in[A],\ b\in[b],\ A \text{ symmetric}\}$$

for $i=1,\ldots,n$. By defining

$$(2.14)\qquad [\Delta] := (I-R[A])\cdot([x]-\tilde{x})$$

using $\Sigma^{sym} \subseteq [x]$ yields

$$\{(e_i^t - r^iA)\cdot(A^{-1}b-\tilde{x}) \mid A\in[A],\ b\in[b],\ A \text{ symmetric}\} \subseteq [\Delta_i]$$

for $i=1,\ldots,n$ and with (2.5), (2.8), (2.13) it follows that

$$(2.15a)\qquad \inf(V_i) \leqq \inf([z_i]) + \sup([\Delta_i]) = \underline{z}_i + \bar{\Delta}_i,$$

$$(2.15b)\qquad \sup(V_i) \geqq \sup([z_i]) + \inf([\Delta_i]) = \bar{z}_i + \underline{\Delta}_i.$$

Therefore

$$(2.16)\qquad \inf(\Sigma_i^{sym}) \leqq \underline{y}_i \leqq \bar{y}_i \leqq \sup(\Sigma_i^{sym}),$$

where $[y]=([\underline{y}_i, \bar{y}_i])$ is defined for $i=1,\ldots, n$ by

(2.17a) $$\underline{y}_i := \tilde{x}_i + \underline{z}_i + \bar{\Delta}_i,$$

(2.17b) $$\bar{y}_i := \tilde{x}_i + \bar{z}_i + \underline{\Delta}_i.$$

With same arguments as before the inner bounds $[y]$ are very sharp for moderate problems. Even for system where the diameter of $[z]$ is smaller than the diameter of Δ for some components, very sharp inner bounds may be computed for the remaining components of the solution set Σ^{sym}.

We summarize the obtained results in the following theorem:

Theorem: *Let* $[A]\in II\,\mathsf{R}^{n\times n}$ *with* $[a_{\mu\nu}]=[a_{\nu\mu}]$ *for* ν, $\mu=1,\ldots, n$ *and* $[b]\in II\,\mathsf{R}^n$, $R\in\mathsf{R}^{n\times n}$, $\tilde{x}\in\mathsf{R}^n$. *Let*

$$[z_i] := \sum_{\mu=1}^{n} r_{i\mu}([b_\mu]-[a_{\mu\mu}]\tilde{x}_\mu) - \sum_{\substack{\mu,\nu=1\\ \mu<\nu}}^{n} (r_{i\mu}\tilde{x}_\nu + r_{i\nu}\tilde{x}_\mu)[a_{\mu\nu}]$$

for $i=1,\ldots, n$ *and let* $[w]\in II\,\mathsf{R}^n$ *with*

$$[z]+(I-R[A])\cdot[w] \subseteq \operatorname{int}([w]).$$

Then

a) $[z_i]=\{r^i(b-A\tilde{x}) \mid A\in[A],\ b\in[b],\ A$ *symmetric) for* $i=1,\ldots, n$

b) *By defining*

$$[x_i]=\tilde{x}_i+[z_i]+(e_i^t - r^i[A])\cdot[w] \text{ for } i=1,\ldots, n$$

the inequalities

$$\underline{x}_i \leqq \inf(\Sigma_i^{\text{sym}}) \leqq \sup(\Sigma_i^{\text{sym}}) \leqq \bar{x}_i \text{ for } i=1,\ldots, n$$

are valid.

c) *By defining*

$$[\Delta_i] := (e_i^t - r^i[A])\cdot([x]-\tilde{x}) \text{ for } i=1,\ldots, n$$
$$\underline{y}_i := \tilde{x}_i + \underline{z}_i - \bar{\Delta}_i, \text{ for } i=1,\ldots, n$$
$$\bar{y}_i := \tilde{x}_i + \bar{z}_i + \underline{\Delta}_i, \text{ for } i=1,\ldots, n$$

the inequalities

$$\inf(\Sigma_i^{\text{sym}}) \leqq \underline{y}_i \leqq \bar{y}_i \leqq \sup(\Sigma_i^{\text{sym}})$$

are valid for all i *with* $\underline{y}_i \leqq \bar{y}_i$.

3. The Algorithm

The parts a) and b) of the theorem of section 2 show how the outer bounds $[x]$ are calculated and that the solution set Σ^{sym} is very well described by these bounds if the diameter of $(I-R[A])[w]$ is small compared to the diameter of $[z]$. The main problem is then to find an appropriate interval vector $[w]\in II\,\mathsf{R}^n$ satisfying condition (2.11). To get $[w]$ we apply an iteration scheme which was first introduced by S. M. Rump (cf. [18], page 62) and we modify it for the symmetric case:

(1) Calculate with some standard algorithm an approximation R of $m([A]$ and an approximation $\tilde{x}$ of the solution of the midpoint system $m([A])x = m([b])$.
(2) Calculate $[z]$ by (2.7); $[v] := [z]$; $k := 0$;

repeat $\quad [w] := [v] \cdot [1-\varepsilon,\ 1+\varepsilon] + [-\mu,\ \mu]$;

$\qquad k := k+1$;

$\qquad [v] := [z] + (I - R \cdot [A]) \cdot [w]$;

until $\quad [v] \subseteq \mathrm{int}([w])$ or $k > k_{\max}$.

(3) if $[v] \subseteq \mathrm{int}([w])$ then

a) By $[x_i] := \tilde{x}_i + [v_i]$, $i = 1, \ldots, n$ outer bounds of the solution set Σ^{sym} are given.

b) Calculate $\underline{y}_i$, $\bar{y}_i$, $i = 1, \ldots, n$ by (2.17a, b).

If $\underline{y}_i \leqq \bar{y}_i$ then

$$\inf(\Sigma_i^{\mathrm{sym}}) \leqq \underline{y}_i \leqq \bar{y}_i \leqq \sup(\Sigma_i^{\mathrm{sym}}).$$

A proper value of ε is 0.1 and μ is usually a vector the coefficients of which are equal to $m \cdot \mu_0$, where m is a small integer and μ_0 is the smallest positive floating-point number. The maximal number of iterations $k_{\max}$ can be specified by the user. In general only one or two iterations in step (2) are necessary such that $k_{\max} \approx 10$ is sufficient.

Now, the main question is the convergence of this method, i. e. under which assumptions an interval vector $[w] \in \mathrm{II\,R}^n$ is computed such that $[v] \subseteq \mathrm{int}[w]$. Recently, S. M. Rump ([21]) has proved the following Theorem:

Theorem: *Let* $[C] \in \mathrm{II\,R}^{n \times n}$, $[z]$, $[v]^0 \in \mathrm{II\,R}^n$. *Then the following two conditions are equivalent*:

(i) *Using for* $0 < \varepsilon \in \mathrm{R}$, $0 < \mu \in \mathrm{R}^n$ *the iteration*

$$[w] := [v]^k \cdot [1-\varepsilon,\ 1+\varepsilon] + [-\mu,\ \mu], \tag{3.1}$$

$$[v]^{k+1} := [z] + [C] \cdot [v]^k \tag{3.2}$$

there exists a $k \in \mathrm{N}$ *with* $[v]^{k+1} \subseteq \mathrm{int}([w])$.

(ii) $\rho(|[C]|) < 1$.

Remark: For $[C] \in \mathrm{II\,R}^n$ the absolute value $|[C]|$ is defined as the real $n \times n$-matrix the coefficients of which are equal to $\max\{|c_{ij}| \mid c_{ij} \in [c_{ij}]\}$ and ρ denotes the spectral radius of $|[C]|$.

By step (2) if follows that the algorithm converges in a finite number of steps if the spectral radius $\rho(|(I - R[A])|) < 1$.

For a practical implementation of the algorithm some remarks are given. The operations in step (1) are the ordinary floating-point operations. According to formulae (2.9), (2.10), we have to calculate on a computer floating-point bounds $[\tilde{x}_i] = [\underline{\tilde{x}}_i,\ \bar{\tilde{x}}_i]$ containing $[x_i] = ([\underline{x}_i,\ \bar{x}_i])$ for $i = 1, \ldots, n$. Obviously, using the ordinary floating-point interval operations in (2.9) will yield such validated bounds.

The computation of the inner bounds must be implemented carefully. According to (2.15a, b), (2.16), (2.17a, b) we have to calculate on a computer upper bounds of $\inf([z_i]) = \underline{z}_i$ and upper bounds of $\sup([\Delta_i])$ for $i = 1, \ldots, n$. The latter are calculated by using ordinary floating-point interval operations. Using systematically the monotone roundings available on a computer, it is possible to get upper bounds of $\underline{z}_i$. Analogously lower bounds of $\bar{z}_i$ and Δ_i are obtained.

Better results can be achieved by using U. W. Kulisch's arithmetic [12] for vector and matrix operations if the diameter of the interval input data are small.

4. Computational Results

In the following we list some numerical results for the algorithm described in the previous section. The algorithm described in section 3 is implemented by the programming language CALCULUS [19]. CALCULUS is an interactive programming environment supporting Kulisch's arithmetic, LINPACK, EISPACK and the algorithms of the ACRITH subroutine library [9], [10]. We use CALCULUS on a IBM 4361 (base 16, machine unit eps$=16^{-13}=0.22\ldots 10^{-16}$).

In our first example we consider the real symmetric matrix

$$A=\begin{pmatrix} -758.0284 & 8.971284 & -507.7297 & -260.2576 \\ 8.971284 & -507.7118 & 7.705539 & 508.9875 \\ -507.7297 & 7.705539 & -5.192805 & -510.2374 \\ -260.2576 & 508.9875 & -510.2374 & -259.0101 \end{pmatrix}$$

of dimension 4. This matrix is well-conditioned with an l_1-condition number $5.33\cdot 10^3$. The right hand side is given by $[b]=A\cdot x$ with exact solution

$$x=(1,\ -1,\ 1,\ -1)^t.$$

A symmetric interval matrix $[A]$ is defined by A in the following way:

$$[\underline{a}_{ij},\ \bar{a}_{ij}]:=[a_{ij}\cdot(1-r),\ a_{ij}\cdot(1+r)]$$

for

$$(i,\ j)\in\{(1,3),\ (3,1),\ (2,3),\ (3,2),\ (3,4),\ (4,3)\}$$

and

$$[\underline{a}_{ij},\ \bar{a}_{ij}]:=[a_{ij},\ a_{ij}]$$

for all other $(i,\ j)$. We display some results for this symmetric interval system $[A]x=[b]$ where $r=10^{-7}$. In Table 4.1 the inner and outer bounds $[x]$, $[y]$ of the solution set Σ^{sym} calculated by the algorithm of section 3 are displayed. These results are compared with the outer bounds $[u]$ of the solution set Σ calculated by the routine DILIN of ACRITH and the inner bounds $[v]$ calculated by the algorithm of S. M. Rump [20]. These bounds are displayed in Table 4.2.

Table 4.1
Outer and inner bounds $[x]$, $[y]$ of Σ^{sym} for $r=10^{-7}$

$[x]$	
$[\ 9.99999998436 \cdot 10^{-1}$,	$1.000000001562 \cdot 10^{0}]$
$[-1.00000020157 \cdot 10^{0}$,	$-9.999997984292 \cdot 10^{-1}]$
$[\ 9.99999898940 \cdot 10^{-1}$,	$1.000000101059 \cdot 10^{0}]$
$[-1.00000020259 \cdot 10^{0}$,	$-9.999997974089 \cdot 10^{-1}]$

$[y]$	
$[\ 9.999999984787 \cdot 10^{-1}$,	$1.000000001521 \cdot 10^{0}]$
$[-1.000000201528 \cdot 10^{0}$,	$-9.999997984710 \cdot 10^{-1}]$
$[\ 9.999998989821 \cdot 10^{-1}$,	$1.000000101018 \cdot 10^{0}]$
$[-1.000000202549 \cdot 10^{0}$,	$-9.999997974507 \cdot 10^{1}]$

Table 4.2
Outer and inner bounds $[u]$, $[v]$ of Σ for $r=10^{-7}$

$[u]$	
$[\ 9.997798545488 \cdot 10^{-1}$,	$1.00020514545 \cdot 10^{0}]$
$[-1.000204944371 \cdot 10^{0}$,	$-9.99795055628 \cdot 10^{-1}]$
$[\ 9.997950545768 \cdot 10^{-1}$,	$1.00020494542 \cdot 10^{0}]$
$[-1.000204945406 \cdot 10^{0}$,	$-9.99795054593 \cdot 10^{-1}]$

$[v]$	
$[\ 9.997949470319 \cdot 10^{-1}$,	$1.000205052967 \cdot 10^{0}]$
$[-1.000204851979 \cdot 10^{0}$,	$-9.997951480205 \cdot 10^{-1}]$
$[\ 9.997951469698 \cdot 10^{-1}$,	$1.000204853030 \cdot 10^{0}]$
$[-1.000204853013 \cdot 10^{0}$,	$-9.997951469868 \cdot 10^{-1}]$

Comparing Tables 4.1, 4.2 shows that a drastical overestimation occurs if the input data of a linear interval system with symmetric system matrix are handled independently. Moreover, comparing in each case the inner and the outer bounds, it follows that the solution sets Σ^{sym} resp. Σ are very well described by the computed outer bound $[x]$ resp. $[u]$. To be perfectly clear we display in Table 4.3 the ratio of diameters $d([x_i])/d([u_i])$, $d([x_i])/d([y_i])$ and $d([u_i])/d([v_i])$ for $i=1,\dots,4$.

Table 4.3
Ratio of diameters for $r=10^{-7}$

$d([x_i])/d([u_i])$	$d([y_i])/d([x_i])$	$d([v_i])/d([u_i])$
$7.618\cdot 10^{-6}$	0.9732	0.9995
$9.835\cdot 10^{-4}$	0.9997	0.9995
$4.931\cdot 10^{-4}$	0.9995	0.9995
$9.885\cdot 10^{-4}$	0.9997	0.9995

The ratios in the first column show the phenomenon of overestimation whereas the ratios in the second and third column of Table 4.3 demonstrate the sharpness of the calculated bounds for the solution sets Σ^{sym} and Σ. It is rather surprising, that for a linear system with small dimension and small condition number for example the diameter of the first component of the solution set Σ^{sym} is equal to $7.6\cdot 10^{-1}.d(\Sigma_1)$.

If x is the exact solution of $Ax=b$, $\tilde{A}$ is a perturbation of A and $\tilde{x}$ is the exact solution $\tilde{A}\tilde{x}=b$ then

$$\frac{\|x-\tilde{x}\|}{\|\tilde{x}\|}\approx \operatorname{cond}(A).\frac{\|A-\tilde{A}\|}{\|\tilde{A}\|}.$$

This is a well-known fact by the theory of Wilkinson. Because of the relative perturbation $r=10^{-7}$ and a condition number of about 5.10^3 it would be expected that only 3 or 4 figure is correct. But from the results of Table 4.1 it follows that depending on the components of the solution set between 6 and 8 figures are correct.

In Tables 4.4, 4.5 the ratio of diameters with $r=10^{-10}$, resp. $r=10^{-13}$, are given. As expected the phenomenon of overestimation is decreasing if r is decreasing. Obviously if $r=0$ then $\Sigma^{sym}=\Sigma$. Comparing the diameters of inner and outer bounds in both cases very sharp bounds of the corresponding solution sets are calculated.

Table 4.4
Ratio of diameters for $r=10^{-10}$

$d([x_i])/d([u_i])$	$d([y_i])/d([x_i])$	$d([v_i])/d([u_i])$
$9.734\cdot 10^{-6}$	0.9999	0.9999
$9.859\cdot 10^{-4}$	0.9999	0.9999
$4.954\cdot 10^{-4}$	0.9999	0.9999
$9.909\cdot 10^{-4}$	0.9999	0.9999

Table 4.5
Ratio of diameters for $r=10^{-13}$

$d([x_i])/d([u_i])$	$d([y_i])/d([x_i])$	$d([v_i])/d([u_i])$
$2.252\cdot10^{-3}$	0.9999	0.9999
$3.299\cdot10^{-3}$	0.9998	0.9999
$2.777\cdot10^{-3}$	0.9999	0.9999
$3.163\cdot10^{-3}$	0.9998	0.9999

The degree of overestimation depends strongly on the solution resp. on the right hand side of the linear system $Ax=b$ and the corresponding interval input data. In Table 4.6 the ratio of diameters is listed in the example case where

$$x=(1,\ 1,\ 1,\ 1)$$

is the exact solution, $[b]=A\cdot x$ and

$$[\underline{a}_{ij},\ \bar{a}_{ij}]=[a_{ij}(1-r),\ a_{ij}(1+r)]$$

for all $i,\ j=1,\dots,\ 4$ with $r=10^{-10}$.

Table 4.6
Ratio of diameters with $r=10^{-10}$

$d([x_i])/d([u_i])$	$d([y_i])/d([x_i])$	$d([v_i])/d([u_i])$
0.6018	0.9999	0.9999
0.6023	0.9999	0.9999
0.6018	0.9999	0.9999
0.6024	0.9999	0.9999

In this case inner and outer bounds of the corresponding solution sets are practically equal whereas according to the first column of Table 4.6 there occurs only an overestimation of about 80%.

In the second example we discuss the well-known (symmetric) Hilbert matrix

$$(H^{(n)}ij):=(1\,\mathrm{cm}\,(1,\dots,2n-1))/(i+j+1)$$

of dimension $n=7$. The least common multiple of all denominators is denoted by 1 cm. Contrary to the first example $H^{(7)}$ is an ill-conditioned matrix with l_1-condition number $7.45.10^8$. The right hand side is given by $[b]=H^{(7)}\cdot x$, where

$$(x=1,\ -0.5,\ 0.375,\ -0.312,\ 0.273,\ -0.246,\ 0.226)^t$$

and the corresponding symmetric interval matrix [A] is defined by

$$[a_{i,\,i+1}]:=[a_{i+1,\,i}]:=[H^{(7)}_{i,\,i+1}\cdot(1-r),\ H^{(7)}_{i,\,i+1}\,(1-r)],\ i=1,\dots,\ n-1$$

and $[a_{ij}]=H^{(7)}_{ij}$ for all other coefficients.

In the following we display in Table 4.7 for some r the minimum value of the ratio of diameters.

Table 4.7
Minimum value of the ratio of diameters

r	$\min_{i=1}^{4} \frac{d([x_i])}{d([u_i])}$	$\min_{i=1}^{4} \frac{d([y_i])}{d([x_i])}$	$\min_{i=1}^{4} \frac{d([v_i])}{d([u_i])}$
10^{-9}	$5.496 \cdot 10^{-3}$	0.8643	0.8929
10^{-10}	$5.439 \cdot 10^{-3}$	0.9862	0.9889
10^{-11}	$5.455 \cdot 10^{-3}$	0.9986	0.9988
10^{-12}	$5.712 \cdot 10^{-3}$	0.9999	0.9999
10^{-13}	$7.700 \cdot 10^{-3}$	0.9999	0.9999
10^{-14}	$2.715 \cdot 10^{-2}$	0.9999	0.9999

Likewise, this example demonstrates a severe overestimation if bounds are calculated for Σ and not for the solution set Σ^{sym}.

References

1. G. Alefeld, J. Herberger, Einführung in die Intervallrechnung. B. I. Wissenschaftsverlag, 1974.
2. H. Bauch, K.-U. Jahn, D. Oelschlägel, H. Süsse, V. Wiebigke. Intervallmathematik, Theorie und Anwendungen. Mathematische-Naturwissenschaftliche Bibliothek, Band 72, B. B. Teubner, Leipzig, 1987.
3. J. H. Bleher, S. M. Rump, U. W. Kulisch, M. Metzger, Ch. Ullrich, W. Walter. FORTRAN-SC: A Study of Computation with Access to ACRITH. *COMPUTING*, **39**, 1987, 93-110, Springer.
4. E. Hansen, R. Smith. Interval Arithmetic in Matrix Computations, Part II. *SIAM J. Numer. Anal.*, **4**, 1967, 1-9.
5. E. Hansen. On the Solution of Linear Algebraic Equations with Interval Coefficients. *Linear Algebra Appl.*, **2**, 1969, 153-165.
6. D. Husung. ABACUS – Programmierwerkzeug mit hochgenauer Arithmetik für Algorithmen mit verifizierten Ergebnissen, Diplomarbeit, Karlsruhe, 1988.
7. D. Husung. Precompiler for Scientific Computation (TPX), Informatik III, TU Hamburg-Harburg, 1989.
8. D. Husung, S. M. Rump. ABACUS, Proceedings SCAN'89, Tagung "Wissenschaftliches Rechnen und Programmiersprachen", Basel, 1989.
9. IMB High-Accuracy Arithmetic Subroutine Library (ACRITH), General Information Manual, GC 33-6163-02, 3rd Edition, 1986.
10. IBM High-Accuracy Arithmetic Subroutine Library (ACRITH), Program Description and User's Guide, SC 33-6164-02, 3rd Edition, 1986.
11. R. Krawczyk. Newton-Algorithmen zur Bestimmung von Nullstellen mit Fehlerschranken. *COMPUTING*, **4**, 1969, 187-201.
12. U. W. Kulisch. Computer Arithmetic in Theory and Practice. ACADEMIC PRESS, New York, 1981.
13. U. W. Kulisch, (ed.) PASCAL-SC: A PASCAL Extension for Scientific Computation, Information Manual and Floppy Disks, Version IBM PC. Stuttgart, B. G. Teubner; Chichester: John Wiley & Sons, 1987.
14. R. E. Moore. Methods and Applications of Interval Analysis. SIAM Philadelphia, 1979.
15. A. Neumaier. Interval Methods for Systems of Equations. Cambridge University Press, to be published.
16. J. Rohn. Solving Interval Linear Systems. *Freiburger Intervallberichte*, **84/7**, 1984, 1-30.

17. S. M. Rump. Kleine Fehlerschranken bei Matrixproblemen, Dr.-Dissertation, Institut für Angewandte Mathematik, Universität Karlsruhe, 1980.
18. S. M. Rump. Solving Algebraic Problems with High Accuracy, Habilitationsschrift. – In: A New Approach to Scientific Computation, Hrsg. U. W. Kulisch und W. L. Miranker, ACADEMIC PRESS, 1983, 51-120.
19. S. M. Rump, CALCULUS, in "Wissenschaftliches Rechnen mit Ergebnisverifikation", ed. U. Kulisch, Vieweg und Akademie Verlag Berlin, 1989.
20. S. M. Rump. Rigorous Sensitivity Analysis for Systems of Linear and Nonlinear Equations. To appear in *MATH. OF COMP.*, 1990.
21. S. M. Rump. Estimation of the Sensitivity of Linear and Nonlinear Algebraic Problems. To appear in *Linear Algebra and its Applications*, 1990.

Technishe Universität Hamburg
Harburg
GERMANY

Received 25.07.1990