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A Bernstein Type Operator on the Simplex

T. N. T. Goodman^{+}, *A. Sharma*$^{++}$

Presented by Bl. Sendov

1. Introduction

Let $T=T_d$ denote the standard simplex in R^d, $d\geqq 1$: $T=\{\mathbf{x}\in\mathrm{R}^d : \mathbf{x}\geqq 0, |\mathbf{x}|\leqq 1\}$, when $|\mathbf{x}|=\sum_{j=1}^{d} x_j$. Then for any multi-index $\mathbf{i}=(i_1,\dots,i_{d+1})$, we denote by $p_i := p_i^d$ the Bernstein basis polynominal on T:

$$p_{\mathbf{i}}(\mathbf{x})=\frac{|\mathbf{i}|\,!}{\mathbf{i}!}(\mathbf{x},\ 1-|\mathbf{x}|)^{\mathbf{i}},\ \ \mathbf{x}\in T.$$

Above we have used the usual notation

$$|\mathbf{i}|=i_1+\dots+i_{d+1},\ \ \mathbf{i}!=i_1!\dots i_{d+1}!,$$
$$(\mathbf{x},\ 1-|\mathbf{x}|)^{\mathbf{i}}=x_1^{i_1}\dots x_d^{i_d}(1-|\mathbf{x}|)^{i_{d+1}}.$$

In [7] J. L. Durrmeyer defines for $n\geqq 0$, the operator M_n from the space of inegrable functions on T to the space Π_n of polynomials of total degree at most n on T by

$$(1.1)\quad (M_nf)(\mathbf{x}) := (M_{n,d}f)(\mathbf{x}) = \sum_{|\mathbf{i}|=n} p_{\mathbf{i}}(\mathbf{x})\int_T \frac{(n+d)!}{n!}p_{\mathbf{i}}(\mathbf{t})f(\mathbf{t})\,\mathrm{d}\mathbf{t},\ \ \mathbf{x}\in T.$$

This operator has been studied by M. M. Derriennic [4], [5], Z. Ditzian and K. Ivanov [6] and others.

Here we shall define a related but essentially different operator U_n from the space $C(T)$ of continuous function on T to Π_n. To define this, it will be convenient to consider certain simplex splines (or multivariate B-splines) (see [10]). For any multi-index $\mathbf{i}=(i_1,\dots,i_{d+1})$, we denote by $B_{\mathbf{i}}(x)$ the simplex spline on R^d with knots at the points given by the vectors $\mathbf{e}_1=(1,0,\dots,0)$, $\mathbf{e}_2=(0,1,0,\dots,0),\dots$, $\mathbf{e}_d=(0,0,\dots,0,1)$ and the origin with multiplicities $i_1,\dots,i_{d+1}$ respectively.

For simplicity and to preserve symmetry with respect to the faces of the simplex, we shall use the convention that

$$(1.2)\quad \int_{T_d} f(t_1,\dots,t_{d+1})\,\mathrm{d}t_1\dots\mathrm{d}t_{d+1}=\int_{T_d} f(t_1,\dots,t_d,\ 1-t_1-t_2\dots-t_d)\,\mathrm{d}t_1\dots\mathrm{d}t_d.$$

Then $B_i(\mathbf{x})$ is the distribution defined by

$$(1.3)\quad \int_{\mathsf{R}^d} B_{\mathbf{i}}(\mathbf{x})f(\mathbf{x})\,d\mathbf{x}=(|\mathbf{i}|-1)!\int_{T_{|\mathbf{i}|-1}} f\Big(\sum_{v=1}^{d} e_v \sum_{j=1}^{i_v} t_{(i_1+\dots+i_{v-1})+j}\Big)\,dt_1\dots dt_{|\mathbf{i}|}.$$

We then define for $n\geqq 1$,

$$(1.4)\quad (U_n f)(\mathbf{x}) := (U_{n,d} f)(\mathbf{x}) = \sum_{|\mathbf{i}|=n} p_{\mathbf{i}}(\mathbf{x}) \int_{\mathsf{R}^d} B_{\mathbf{i}}(\mathbf{t})f(\mathbf{t})\,d\mathbf{t} \quad \mathbf{x}\in T.$$

To see the connection between (1.1) and (1.3), we recall ([10], Corollary 2), that

$$(1.5)\quad p_{\mathbf{i}}(\mathbf{x}) = \frac{|\mathbf{i}|!}{(|\mathbf{i}|+d)!} B_{\mathbf{i}+\mathbf{1}},$$

where $\mathbf{1}$ denotes the multi-index $(1,\dots,1)$.

We also note that if $i_j=0$, then the support of $B_{\mathbf{i}}(\mathbf{x})$ lies in the set $\{\mathbf{x}: x_j=0\}$ if $1\leqq j\leqq d$, and in the set $\{\mathbf{x}: x_1+\dots+x_d=1\}$ if $j=d+1$. In such cases, we can regard $B_{\mathbf{i}}$ as a distribution on R^{d-1}. Applying (1.5), we can then express $U_n f$ directly in terms of Bernstein basis polynomials. As examples we have

$$(1.6)\quad (U_{n,1}f)(x) := f(0)p_{0,n}(x) + \sum_{i=1}^{n-1} p_{i,n-i}(x)\int_0^1 (n-1)p_{i-1,n-i-1}(t)f(t)\,dt + f(1)p_{n,0}(x),\quad 0\leqq x\leqq 1.$$

$$\begin{aligned}(1.7)\quad (U_{n,2}f)(x,y) :=\ & f(0,0)p_{0,0,n}(x,y)+f(1,0)p_{n,0,0}(x,y) + f(0,1)p_{0,n,0}(x,y)\\ &+ \sum_{i=1}^{n-1} p_{0,i,n-i}(x,y)\int_0^1 (n-1)p_{i-1,n-i-1}(t)f(0,t)\,dt\\ &+ \sum_{i=1}^{n-1} p_{i,0,n-i}(x,y)\int_0^1 (n-1)p_{i-1,n-i-1}(t)f(t,0)\,dt\\ &+ \sum_{i=1}^{n-1} p_{i,n-i,0}(x,y)\int_0^1 (n-1)p_{i-1,n-i-1}(t)f(t,1-t)\,dt\\ &+ \sum_{\substack{i+j+k=n\\ i,j,k\geqq 1}} p_{i,j,k}(x,y)\int_{T_2} (n-1)(n-2)p_{i-1,j-1,k-1}(u,v)f(u,v)\,du\,dv.\end{aligned}$$

The operator U_n shares, in some sense, the advantages of both the operators M_n and the Bernstein operator B_n. Like M_n, U_n has the commutative property, i.e., $U_n U_m = U_m U_n$. Moreover, U_n has a basis of eigenfunctions in Π_m, $m\leqq n$ which is independent of n and can be represented explicitly. These properties, together with other basic properties are shown in Section 2. Like the Bernstein operator, U_n reproduces linear functions and $U_n f$ interpolates f at the vertices of T. That U_n shares all the shape-preserving properties of B_n is shown in Section 3. It is well-known that the rate of convergence of the Bernstein operator is slow, and one of its main advantages, therefore, lies in its shape preserving properties. Thus it is interesting to observe that these properties are shared by U_n. Finally in Section 4, we show some convergence properties of U_n. Without going into a detailed analysis of convergence, we show that U_n shares with B_n the elegant form of asymptotic estimate due to Voronoskaya, and if any derivative $D^\alpha f$ of f is continuous, then $D^\alpha U_n f$ converges uniformly on T to $D^\alpha f$.

The operator $U_{n,1}$ is a special case of a spline operator considered in [8]. It is easily seen that $(U_{n,1}f)' = M_{n-1,n}(f)'$ and hence some properties of $U_{n,1}$, like commutativity, can be easily deduced from the corresponding properties of $M_{n,1}$. However, for $d \geqq 2$ there is no such simple relationship between $U_{n,d}$ and $M_{n,d}$.

2. Basic properties

Clearly we have

(2.1) $$U_n \text{ is linear and positive,}$$

(2.2) $$(U_n f)(\mathbf{x}) = f(\mathbf{x}) \text{ if } \mathbf{x} \text{ is any vertex of } T,$$

(2.3) $$\|U_n f\|_\infty \leqq \|f\|_\infty, \; f \in C(T).$$

Theorem 1. *If for some $r \leqq d-1$, $f \in C(T_d)$ is essentially a function only of r variables, i.e.,*

$$f(x_1, \dots, x_d) = g(x_1, \dots, x_r),$$

then

$$U_{n,d} f(x_1, \dots, x_d) = U_{n,r} g(x_1, \dots, x_r), \quad \mathbf{x} \in T_d.$$

Proof. It is easily seen from (1.3) that

(2.4) $$\int_{\mathbf{R}^d} B_{i_1, \dots, i_{d+1}}(\mathbf{x}) f(\mathbf{x})\, d\mathbf{x} = \int_{\mathbf{R}^r} B_{i_1, \dots, i_r, (i_{r+1} + \dots + i_{d+1})}(\mathbf{x})\, g(\mathbf{x})\, d\mathbf{x}.$$

We also note that for fixed $\mathbf{j} = (j_1, \dots, j_r)$,

$$\sum_{\substack{|\mathbf{i}|=n \\ (i_1, \dots, i_r) = \mathbf{j}}} p_{i_1, \dots, i_{d+1}}(\mathbf{x})$$

$$= \frac{n!\, x_1^{j_1} \dots x_r^{j_r}}{\mathbf{j}!\,(n-|\mathbf{j}|)!} \sum_{i_{r+1} + \dots + i_{d+1} = n - |\mathbf{j}|} \frac{(n-|\mathbf{j}|)!}{i_{r+1}! \dots i_{d+1}!} x_{r+1}^{i_{r+1}} \dots x_d^{i_d} (1-|\mathbf{x}|)^{i_{d+1}}$$

$$= p_{\mathbf{j}, n-|\mathbf{j}|}(x_1, \dots, x_r).$$

So from (1.4) and (2.4) we obtain

$$(U_{n,d} f)(\mathbf{x}) = \sum_{|\mathbf{i}|=n} p_{\mathbf{i}}(\mathbf{x}) \int B_{i_1, \dots, i_r, n-i_1-\dots-i_r}(\mathbf{t}) g(\mathbf{t})\, d\mathbf{t}$$

$$= \sum_{|\mathbf{j}| \leqq n} \int B_{j_1, \dots, j_r, n-|\mathbf{j}|}(\mathbf{t}) g(\mathbf{t})\, d\mathbf{t} \sum_{\substack{|\mathbf{i}|=n \\ (i_1, \dots, i_r) = \mathbf{j}}} p_{\mathbf{i}}(\mathbf{x})$$

$$= \sum_{|\mathbf{j}| \leqq n} p_{\mathbf{j}, n-|\mathbf{j}|}(x_1, \dots, x_r) \int B_{\mathbf{j}, n-|\mathbf{j}|}(\mathbf{t}) g(\mathbf{t})\, d\mathbf{t}$$

$$= U_{n,r} g(x_1, \dots, x_r). \qquad \blacksquare$$

By symmetry it follows from Theorem 1 that if for $r \leqq d-1, f \in C(T_d)$ satisfies

$$f(x_1, \ldots, x_d) = g(x_1, \ldots, x_{r-1}, \; 1 - x_1 - \ldots - x_d), \; \mathbf{x} \in T_d,$$

then

$$U_{n,d} f(x_1, \ldots, x_n) = U_{n,r} g(x_1, \ldots, x_{r-1}, \; 1 - x_1 - \ldots - x_d), \; \mathbf{x} \in T_d.$$

The following result shows, in particular, that U_n reproduces linear functions and preserves the degree of the polynomials in Π_n with respect to each variable (coordinatewise). Henceforward we adopt the convention that negative factorials are omitted except when $(-1)!/(-1)! = 1$.

Theorem 2. *If* $f(\mathbf{x}) = \mathbf{x}^{\boldsymbol\alpha}$, $\mathbf{x} \in T_d$, $\boldsymbol\alpha \in Z^d_+$, *then*

$$(2.5) \qquad (U_{n,d} f)(\mathbf{x}) = \frac{(n-1)!\,\boldsymbol\alpha!}{(n-1+|\boldsymbol\alpha|)!} \sum_{\mathbf{i}=0}^{\boldsymbol\alpha} \frac{n!}{\mathbf{i}!\,(n-|\mathbf{i}|)!} \binom{\boldsymbol\alpha-1}{\mathbf{i}-1} \mathbf{x}^{\mathbf{i}},$$

where

$$\binom{\boldsymbol\alpha-1}{\mathbf{i}-1} := \frac{(\boldsymbol\alpha-1)!}{(\mathbf{i}-1)!\,(\boldsymbol\alpha-\mathbf{i})!} \quad and \quad \sum_{\mathbf{i}=0}^{\boldsymbol\alpha} := \sum_{i_1=0}^{\alpha_1} \sum_{i_2=0}^{\alpha_2} \ldots \sum_{i_d=0}^{\alpha_d}.$$

Proof. If $f(\mathbf{x}) = \mathbf{x}^{\boldsymbol\alpha}$, then by (1.3), we have

$$(2.6) \qquad \int_{R^d} f(\mathbf{x}) B_{\mathbf{i}}(\mathbf{x})\, d\mathbf{x} = (|\mathbf{i}|-1)! \int_{T_{|\mathbf{i}|-1}} \mathbf{v}^{\boldsymbol\alpha}\, dt_1 \ldots dt_{|\mathbf{i}|},$$

where

$$\mathbf{v} := (v_1, \ldots, v_d) = \Big(\sum_{j=1}^{i_1} t_j, \; \sum_{j=1}^{i_2} t_{i_1+j}, \ldots, \sum_{j=1}^{i_d} t_{(i_1+\cdots+i_{d-1})+j} \Big).$$

In the integral on the right in (2.5), we replace the variables $t_1, t_{i_1+1}, \ldots t_{(i_1+\cdots+i_{d-1})+1}$ by $v_1, \ldots, v_d$ and integrate with respect to the other variables. Then

$$\int_{T_{|\mathbf{i}|-1}} \mathbf{v}^{\boldsymbol\alpha}\, dt_1 \ldots dt_{|\mathbf{i}|} = \int_{T_d} \mathbf{v}^{\boldsymbol\alpha} F(\mathbf{v})\, dv_1 \ldots dv_d,$$

where $F(\mathbf{v})$ is the product of the volumes of the following simplices:

$$v_1 T_{i_1-1}, \; v_2 T_{i_2-1}, \ldots, v_d T_{i_d-1}, \; (1-|\mathbf{v}|) T_{i_{d+1}-1}.$$

Thus we have

$$\int_{T_{|\mathbf{i}|-1}} \mathbf{v}^{\boldsymbol\alpha}\, dt_1 \ldots dt_{|\mathbf{i}|} = \int_{T_d} \mathbf{v}^{\boldsymbol\alpha} \cdot \frac{(v_1, \ldots, v_{d+1})^{\mathbf{i}-1}}{(\mathbf{i}-1)!} dv_1 \ldots dv_{d+1}$$

$$= \frac{(\boldsymbol\alpha+\mathbf{i}-1)!}{(\mathbf{i}-1)!\,(|\mathbf{i}|-1+|\boldsymbol\alpha|)!}.$$

By (1.4), we therefore obtain

$$(2.7) \qquad (U_{n,d} f)(\mathbf{x}) = \frac{(n-1)!}{(n-1+|\boldsymbol\alpha|)!} \sum_{|\mathbf{i}|=n} p_{\mathbf{i}}(\mathbf{x}) \frac{(\boldsymbol\alpha+\mathbf{i}-1)!}{(\mathbf{i}-1)!}.$$

Putting $z=1-|\mathbf{x}|$ and differentiating the binomial expansion keeping z fixed yields

$$(2.8)\qquad \sum_{|\mathbf{i}|=n} p_{\mathbf{i}}(\mathbf{x})\frac{(\boldsymbol{\alpha}+\mathbf{i}-\mathbf{1})!}{(\mathbf{i}-\mathbf{1})!}=x_1 x_2 \dots x_d D^{\boldsymbol{\alpha}}(\mathbf{x}^{\boldsymbol{\alpha}-\mathbf{1}}(|\mathbf{x}|+z)^n).$$

Now using Leibnitz's formula, we get

$$(2.9)\qquad D^{\boldsymbol{\alpha}}(\mathbf{x}^{\boldsymbol{\alpha}-\mathbf{1}}(|\mathbf{x}|+z)^n)=\sum_{\mathbf{i}=\mathbf{0}}^{\boldsymbol{\alpha}}\binom{\boldsymbol{\alpha}}{\mathbf{i}}\frac{(\boldsymbol{\alpha}-\mathbf{1})!\,n!}{(\mathbf{i}-\mathbf{1})!\,(n-|\mathbf{i}|)!}x^{\mathbf{i}-\mathbf{1}}(|\mathbf{x}|+z)^{n-|\boldsymbol{\alpha}|},$$

so that (2.5) follows from (2.7), (2.8) and (2.9) after rearrangement.

We can now prove that $U_{n,d}$, $U_{m,d}$ have the commutative property. We shall prove

Theorem 3. *For n, $m\geqq 1$, $(U_{n,d}U_{m,d}f)(\mathbf{x})=(U_{m,d}U_{n,d}f)(\mathbf{x})$ for all $f\in C(T_d)$.*

Proof. It follows from (2.3) and the fact that polynomials are dense in $C(T)$, that it suffices to prove the result for $f(\mathbf{x})=\mathbf{x}^{\boldsymbol{\alpha}}$, $\boldsymbol{\alpha}\in Z^d_+$. In this case, from (2.5) we get

$$(U_{m,d}U_{n,d}f)(\mathbf{x})=\frac{(n-1)!\,\boldsymbol{\alpha}!}{(n-1+|\boldsymbol{\alpha}|)!}\sum_{\mathbf{i}=\mathbf{0}}^{\boldsymbol{\alpha}}\frac{n!}{\mathbf{i}!\,(n-|\mathbf{i}|)!}\binom{\boldsymbol{\alpha}-\mathbf{1}}{\mathbf{i}-\mathbf{1}}(U_{m,d}\mathbf{t}^{\mathbf{i}})(\mathbf{x}),$$

where

$$(U_{m,d}\mathbf{t}^{\mathbf{i}})(\mathbf{x})=\frac{(m-1)!\,\mathbf{i}!}{(m-1+|\mathbf{i}|)!}\sum_{\mathbf{j}=\mathbf{0}}^{\mathbf{i}}\frac{m!}{\mathbf{i}!\,(m-|\mathbf{j}|)!}\binom{\mathbf{i}-\mathbf{1}}{\mathbf{j}-\mathbf{1}}\mathbf{x}^{\mathbf{j}}.$$

Interchanging the order of summation, we have

$$(U_{m,d}U_{n,d}f)(\mathbf{x})=\frac{c_m c_n c_{\boldsymbol{\alpha}}}{(n-1+|\boldsymbol{\alpha}|)!}\sum_{\mathbf{j}=\mathbf{0}}^{\boldsymbol{\alpha}}\frac{\mathbf{x}^{\mathbf{j}}}{c_{\mathbf{j}}(m-|\mathbf{j}|)!}\sum_{\mathbf{i}=\mathbf{j}}^{\boldsymbol{\alpha}}\frac{1}{(n-|\mathbf{i}|)!\,(\boldsymbol{\alpha}-\mathbf{i})!\,(\mathbf{i}-\mathbf{j})!\,(m-1+|\mathbf{i}|)!},$$

where we have set $c_m=m!\,(m-1)!$ and $c_{\boldsymbol{\alpha}}=\boldsymbol{\alpha}!\,(\boldsymbol{\alpha}-\mathbf{1})!$. The second summation above can be rewritten as

$$\frac{1}{(m+n-1)!\,(\boldsymbol{\alpha}-\mathbf{j})!}\sum_{\mathbf{i}=\mathbf{0}}^{\boldsymbol{\alpha}-\mathbf{j}}\binom{m+n-1}{n-|\mathbf{i}+\mathbf{j}|}\binom{\boldsymbol{\alpha}-\mathbf{j}}{\mathbf{i}}.$$

It is easy to see by combinatorial considerations or by multiplying suitable binomial expressions that

$$\sum_{\mathbf{i}=\mathbf{0}}^{\boldsymbol{\alpha}-\mathbf{j}}\binom{m+n-1}{n-|\mathbf{i}+\mathbf{j}|}\binom{\boldsymbol{\alpha}-\mathbf{j}}{\mathbf{i}}=\binom{m+n-1+|\boldsymbol{\alpha}|-|\mathbf{j}|}{n-|\mathbf{j}|}.$$

Thus we obtain

$$(U_{m,d}U_{n,d}f)(\mathbf{x})=\frac{c_m c_n c_{\boldsymbol{\alpha}}}{(m-1+|\boldsymbol{\alpha}|)!\,(n-1+|\boldsymbol{\alpha}|)!\,(m+n-1)!}$$
$$\times\sum_{j=0}^{a}\frac{\mathbf{x}^{j}(m+n-1+|\mathbf{a}|-|\mathbf{j}|)!}{c_j(m-|\mathbf{j}|)!\,(n-|\mathbf{j}|)!\,(\alpha-\mathbf{j})!}.$$

The symmetry of the above expression in m and n completes the proof. ∎

Henceforward in this section we shall for simplicity restrict our attention to $d=1$ or 2. In general it can be easily seen how the results extend to $d \geqq 3$.

Theorem 4. *The operator $U_{n,1}$ has eigenvalues*

$$\lambda_{n,m} = \frac{(n-1)!\,n!}{(n-1+m)!\,(n-m)!}, \quad m=0, 1, \ldots, n, \tag{2.10}$$

and for $m \geqq 2$, corresponding eigenfunctions F_{m-2}, where

$$F_m(x) = \frac{d^m}{dx^m}(x^{m+1}(1-x)^{m+1}). \tag{2.11}$$

Proof. We have already seen that $U_{n,1}$ reproduces linear functions and so $\lambda_{n,0} = \lambda_{n,1} = 1$ and the corresponding eigenfunctions are 1 and x. Now recall that $(U_{n,1}f)' = M_{n-1,1}(f')$. It is shown in [4] that $M_{n,1}$ has eigenvalues $\lambda_{n+1,m+1}$, $m=0, 1, \ldots, n$ with corresponding eigenfunctions $P_m(x)$, the Legendre polynomials of degree m given by (upto a constant factor)

$$P_m(x) = \frac{d^m}{dx^m}(x^m(1-x)^m).$$

Defining F_m by (2.11), we see that $F'_m = P_{m+1}$ and so

$$(U_{n,1}F_{m-2})' = M_{n-1,1}P_{m-1} = \lambda_{n,m}P_{m-1} = \lambda_{n,m}F'_{m-2}.$$

Since $U_{n,1}F_{m-2}(0) = F_{m-2}(0) = 0$, we have

$$U_{n,1}F_{m-2} = \lambda_{n,m}F_{m-2}.$$ ∎

Theorem 5. *The operator $U_{n,2}$ also has eigenvalues $\lambda_{n,m}$, $m=0, 1, \ldots, n$ and for $m \geqq 2$, the corresponding eigenspace has as a basis the $m+1$ functions*

$$F_{m-2}(x),\ F_{m-2}(y),\ F_{m-2}(1-x-y) \text{ and } F_{r,m-3-r}(x, y),\ 0 \leqq r \leqq m-3,$$

where F_{m-2} is given by (2.11) and

$$F_{r,s}(x, y) = \frac{\partial^{r+s}}{\partial x^r \partial y^s}(x^{r+1}y^{s+1}(1-x-y)^{r+s+1}), \quad r+s=m-3. \tag{2.12}$$

Proof. As in Theorem 4, $\lambda_{n_0} = \lambda_{n_1} = 1$ are eigenvalue with eigenspace the linear functions. By Theorem 1 and 4, we know that for $m \geqq 2$, $F_{m-2}(x)$, $F_{m-2}(y)$ and $F_{m-2}(1-x-y)$ are eigenfunctions of $U_{n,2}$ with eigenvalue $\lambda_{n,m}$.

Now for $l \geqq 3$, set

$$\Pi_l^0 := \{p \in \Pi_l : p(x, y) = xy(1-x-y)q(x, y),\ q \in \Pi_{l-3}\}.$$

For fixed $n \geqq 3$, we shall prove by induction on m that for $m=3, \ldots, n$, $U_{n,2}$ has, corresponding to the eigenvalue $\lambda_{n,m}$ an eigenspace of dimension $m-2$ in Π_m^0 but not in Π_{m-1}^0. For $m=3$, we see that

$$\Pi_3^0 = \{p \in \Pi_3 : p(x, y) = cxy(1-x-y),\ c \text{ constant}\}.$$

By Theorem 2, $U_{n,2}p$ preserves the degree of p, and since $p \in \Pi_3^0$ vanishes on the sides of T, it follows that $U_{n,2}p = c_1 p$, where from (2.5), we see that

$c_1=\lambda_{n,3}=\dfrac{(n-1)(n-2)}{(n+1)(n+2)}$. Thus the result is true for $m=3$. So take $4\leqq m\leqq n$ and suppose the result is true for $m-1$. If p is a polynomial in Π_m^0 but not in Π_{m-1}^0, then by (2.5)

$$U_{n,2}p=\lambda_{n,m}p+q, \tag{2.13}$$

for some q in Π_{m-1}^0. We note that $\lambda_{n,m}$ is strictly decreasing in m for $m>1$. By the inductive hypothesis, we know that $U_{n,2}$ restricted to Π_{m-1}^0 has eigenvalues $\lambda_{n,j}$, $0\leqq j\leqq m-1$ and thus does not have $\lambda_{n,m}$ as an eigenvalue. So there is a unique polynomial r in Π_{m-1}^0 with

$$(U_{n,2}-\lambda_{nm})r=q$$

which from (2.13) gives

$$U_{n,2}(p-r)=\lambda_{n,m}(p-r).$$

Hence $p-r$ is an eigenfunction with eigenvalue $\lambda_{n,m}$ and the inductive hypothesis is established.

Now suppose that $p(x, y)$ is an eigenfunction in Π_l^0 corresponding to the eigenvalue $\lambda_{n,l}$ and $\hat{p}(x, y)$ is an eigenfunction in Π_m^0 corresponding to $\lambda_{n,m}$, $l\neq m$. Then $p(x, y)=xy(1-x-y)q(x, y)$, $\hat{p}(x, y)=xy(1-x-y)\hat{q}(x, y)$, where $q(x,y)\in\Pi_{l-3}$ and $\hat{q}(x, y)\in\Pi_{m-3}$. Also

$$\lambda_{n,l}\int_T xy(1-x-y)q(x, y)\hat{q}(x, y)$$
$$=\int_T U_{n,2}p(x, y)\hat{q}(x, y)\,\mathrm{d}x\mathrm{d}y$$
$$=\int_T \hat{q}(x, y)\sum_{\substack{i+j+k=n\\ i,j,k\geqq 1}} p_{i,j,k}(x, y)\int_T (n-1)(n-2)p_{i-1,j-1,k-1}(u, v)p(u, v)\,\mathrm{d}u\mathrm{d}v\mathrm{d}x\mathrm{d}y$$

on using (1.7). Since

$$p_{i,j,k}(x, y)p_{i-1,j-1,k-1}(u, v)uv(1-u-v)=xy(1-x-y)p_{i-1,j-1,k-1}(x, y)p_{i,j,k}(u, v)$$

it follows from the above that

$$\lambda_{n,l}\int_T xy(1-x-y)q(x, y)\hat{q}(x, y)\,\mathrm{d}x\mathrm{d}y=\int_T U_{n,2}\hat{p}(u, v)q(u, v)\,\mathrm{d}u\mathrm{d}v$$
$$=\lambda_{n,m}\int_T xy(1-x-y)q(x, y)\hat{q}(x, y)\,\mathrm{d}x\mathrm{d}y.$$

Since $\lambda_{n,l}\neq\lambda_{n,m}$, we must have

$$\int_T xy(1-x-y)q(x, y)\hat{q}(x, y)\,\mathrm{d}x\mathrm{d}y=0.$$

From the theory of orthogonal polynomials ([1], Chapter 6) the polynomials given by (2.12), for $0\leqq r+s\leqq m-3$, are orthogonal with respect to the weight function $xy(1-x-y)$ and span the space Π_m^0. It follows that the eigenspace in Π_m^0 corresponding to eigenvalue $\lambda_{n,m}$ must lie in the span of $F_{r,m-3-r}$, $0\leqq r\leqq m-3$. Since this eigenspace has dimension $m-2$, the result follows.

Corollary. *For* $0 \leq m \leq n$, *the eigenfunctions of* $U_{n,2}$ *corresponding to eigenvalue* $\lambda_{n,m}$, *all satisfy the differential equation*

$$x(1-x)f_{xx}+y(1-y)f_{yy}-2xyf_{xy}+m(m-1)f=0. \tag{2.14}$$

Proof. It is known [11] that $f=F_m(x)$ given by (2.11) satisfies the differential equation

$$x(1-x)f_{xx}+(m+2)(m+1)f=0$$

and it follows immediately that $F_{m-2}(x)$, $F_{m-2}(y)$, $F_{m-2}(1-x-y)$ satisfy (2.14).

It is known ([1], p. 103, formula (18)) that $g(x, y):=x^{-1}y^{-1}F_{r,s}(x, y)$ satisfies the following differential equations:

$$\begin{cases} x(1-x)g_{xx}-xyg_{xy}+(2+(s-2)x)g_x-(2+r)yg_y+(2+r)(r+s+1)g=0 \\ y(1-y)g_{yy}-xyg_{xy}+(2+(r-2)y)g_y-(2+s)xg_x+(s+2)(r+s+1)g=0. \end{cases} \tag{2.15}$$

Putting $f(x, y)=F_{r,s}(x, y)=xyg(x, y)$, we see easily that

$$\begin{aligned} &x(1-x)f_{xx}+y(1-y)f_{yy}-2xyf_{xy} \\ &=xy[x(1-x)g_{xx}+y(1-y)g_{yy}-2xyg_{xy}+(2-4x)g_x+(2-4y)g_y-2g] \\ &=-xy(r+s+2)(r+s+3)g(x, y) \end{aligned}$$

on adding the two equations in (2.15). Thus for $r+s=m-3$, we see that $f(x, y)=F_{r,s}(x, y)$ satisfies (2.14). ∎

For $n \geqq 0$, we set

$$\begin{aligned} U_n^{(1)}f(\mathbf{x}) &:= \sum_{i+j+k=n} p_{i,j,k}(\mathbf{x}) \int B_{i,j+1,k+1}(\mathbf{t})f(\mathbf{t})\,d\mathbf{t}, \\ U_n^{(2)}f(\mathbf{x}) &:= \sum_{i+j+k=n} p_{i,j,k}(\mathbf{x}) \int B_{i+1,j,k+1}(\mathbf{t})f(\mathbf{t})\,d\mathbf{t}, \\ U_n^{(3)}f(\mathbf{x}) &:= \sum_{i+j+k=n} p_{i,j,k}(\mathbf{x}) \int B_{i+1,j+1,k}(\mathbf{t})f(\mathbf{t})\,d\mathbf{t}, \end{aligned} \tag{2.16}$$

as operators from $C(T_2)$ to Π_n. Thus it is easy to see on using (1.5) that, for example

$$\begin{aligned} U_n^{(2)}f(\mathbf{x}) = &\sum_{\substack{i+j+k=n \\ j\geq 1}} p_{i,j,k}(\mathbf{x}) \int (n+1)np_{i,j-1,k}(\mathbf{t})f(\mathbf{t})\,d\mathbf{t} \\ &+\sum_{i=0}^{n} p_{i,0,n-i}(\mathbf{x}) \int_0^1 (n+1)p_{i,n-i}(t)f(t, 0)\,dt. \end{aligned} \tag{2.17}$$

We shall prove

Theorem 6. *For* $f\in C(T_2)$ *with* $\frac{\partial f}{\partial x}$ *continuous, we have*

$$\frac{\partial}{\partial x}(U_{n,2}f)(x, y)=\left(U_{n-1}^{(2)}\left(\frac{\partial f}{\partial x}\right)\right)(x, y), \quad (x, y)\in T_2. \tag{2.18}$$

Proof. Ignoring Bernstein basis functions with any negative subscripts, we have

$$\frac{\partial}{\partial x}(U_nf)(x, y) = \sum_{i+j+k=n} n(p_{i-1,j,k}(x, y) - p_{i,j,k-1}(x, y)) \int_T B_{i,j,k}(u, v)f(u, v)\,dudv$$
$$= \sum_{i+j+k=n-1} p_{i,j,k}(x, y) \int_T n(B_{i+1,j,k}(u, v) - B_{i,j,k+1}(u, v))f(u, v)\,dudv$$
$$= \sum_{i+j+k=n-1} p_{i,j,k}(x, y) \int_T \left(-\frac{\partial}{\partial u} B_{i+1,j,k+1}(u, v)\right)f(u, v)\,dudv$$
$$= \sum_{i+j+k=n-1} p_{i,j,k}(x, y) \int_T B_{i+1,j,k+1}(u, v)\left(\frac{\partial f}{\partial u}\right)dudv,$$

by the definition of the derivative of a distribution. This gives (2.18).

Similarly, we can see that

$$\frac{\partial}{\partial y}(U_{n,2}f)(x, y) = U^{(1)}_{n-1}\left(\frac{\partial f}{\partial y}\right)(x, y), \tag{2.19}$$

$$\left(\frac{\partial}{\partial y} - \frac{\partial}{\partial x}\right)(U_{n,2}f)(x, y) = U^{(3)}_{n-1}\left(\frac{\partial f}{\partial y} - \frac{\partial f}{\partial x}\right)(x, y). \tag{2.20}$$

3. Convexity

For $f \in C(T)$, we denote by B_nf the Bernstein polynomial

$$(B_nf)(\mathbf{x}) = \sum_{|i|=n} p_i(\mathbf{x}) f\left(\frac{i_1}{n}, \ldots, \frac{i_d}{n}\right). \tag{3.1}$$

As a special case of Theorem 2 of [8], we know that the operator $U_{n,1}$ is variation-diminishing and so for any $f \in C[0, 1]$ and linear function l, $S(U_{n,1}f - l) \leqq S(f - l)$, where $S(g)$ denotes the number of strict sign changes of a function g on [0, 1]. In particular, we see that if f is convex, then so is $U_{n,1}f$.

Henceforward in this section, we assume that $d = 2$. In [2], G. Z. Chang and P. J. Davis have given an example of a convex function f in $C(T)$ for which the quadratic Bernstein polynomial B_2f is not convex. For the same function it can be easily checked on using (1.7) that U_2f is also not convex and so U_n does not, in general, preserve convexity. However, like B_n, U_n does preserve a stronger form of convexity which we now describe.

We say that a function f in $C(T)$ is strongly convex, with respect to T if we have

$$\begin{cases} f(x, y) + f(x+h, y) \leqq f(x, y+h) + f(x+h, y-h), \\ f(x, y) + f(x, y+h) \leqq f(x+h, y) + f(x-h, y+h), \\ f(x, y) + f(x+h, y-h) \leqq f(x+h, y) + f(x, y-h), \end{cases} \tag{3.2}$$

whenever these points lie in T. That these conditions imply convexity can be easily seen as follows. From Theorem 5 [2] it follows that if f is strongly convex with respect to T, then B_nf is convex. Since B_nf converges to f as $n \to \infty$, it follows that f is convex.

If $f \in C^2(T)$, then f is strongly convex with respect to T if and only if

(3.3) $$f_{xx} \geqq f_{xy} \geqq 0 \text{ and } f_{yy} \geqq f_{xy} \geqq 0,$$

and it can be seen immediately that these conditions imply convexity.

We can now prove

Theorem 7. *If f is strongly convex with respect to T, then so is $U_n f$.*

Proof. First suppose $f \in C^2(T)$. Then as in Theorem 6, we can show that

(3.4) $$\frac{\partial^2}{\partial x \partial y}(U_n f)(x, y)$$

$$= \sum_{i+j+k=n-2} p_{i,j,k}(x, y) \int B_{i+1,j+1,k+2}(u, v) \frac{\partial^2 f}{\partial u \partial v}(u, v) du dv \geqq 0,$$

by (3.3).

Similarly, we can show that

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)(U_n f) \geqq 0, \quad \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y} - \frac{\partial}{\partial x}\right)(U_n f) \geqq 0,$$

and the result follows from (3.3).

For a general function f in $C(T)$, which is strongly convex with respect to T, we note that G. Z. Chang and P. J. Davis [2] have shown that $B_m f$ is also strongly convex. Thus $U_n(B_m f)$ is strongly convex with respect to T. Since $B_m f$ converges to f as $m \to \infty$, it follows that $U_n(B_m f) \to U_n f$ as $m \to \infty$. It follows from (3.2) that $U_n f$ is strongly convex. ■

Theorem 8. *If $f \in C(T)$ is convex, thus*

$$U_n f(x, y) \geqq B_n f(x, y).$$

Proof. Since both B_n and U_n reproduce linear functions, we have

$$x = \sum_{i+j+k=n} \frac{i}{n} p_{i,j,k}(x, y) = \sum_{i+j+k=n} p_{i,j,k}(x, y) \int B_{i,j,k}(u, v) u du dv$$

and so

$$\frac{i}{n} = \int B_{i,j,k}(u, v) u du dv, \quad \frac{j}{n} = \int B_{i,j,k}(u, v) v du dv, \quad i+j+k=n.$$

So by convexity of f, we have

$$f\left(\frac{i}{n}, \frac{j}{n}\right) \leqq \int B_{i,j,k}(u, v) f(u, v) du dv.$$

Thus

$$B_n f(x, y) = \sum_{i+j+k=n} f\left(\frac{i}{n}, \frac{j}{n}\right) p_{i,j,k}(x, y)$$

$$\leqq \sum_{i+j+k=n} p_{i,j,k}(x, y) \int B_{i,j,k}(u, v) f(u, v) du dv = U_n f(x, y).$$

■

To finish this section we shall show that if f in $C(T)$ is convex, then the sequence $(U_n f)$ is decreasing. This will require two lemmas. First we show that the average value of a convex function on a simplex is bounded by the average value of the function on the boundary of the simplex. To be precise we have

Lemma 1. *If $f \in C(T_d)$ is convex, then*

$$d! \int_{T_d} f(t_1, \ldots, t_d)\, dt_1 \ldots dt_d \tag{3.5}$$

$$\leq \frac{1}{d+1} \sum_{i=1}^{d+1} (d-1)! \int_{T_{d-1}} f(t_1 \ldots t_{i-1},\ 0,\ t_{i+1} \ldots t_d)\, d\hat{t}_1 \ldots d\hat{t}_i \ldots dt_{d+1},$$

where we use the symbol $d\hat{t}_i$ to denote that dt_i is omitted in the integration. We also recall the convention defined in (1.2).

Proof. If $\mathbf{t}=(t_1, \ldots, t_d)$ is any point in the interior of T, and if t_i and t_j are replaced by a and b respectively, we shall denote the new point by $\mathbf{t}(i,j;\ a,b)$. Similarly if t_i is replaced by a we shall denote the point by $\mathbf{t}(i;\ a)$. With this convention, it is easy to see that

$$\mathbf{t}=\frac{2}{d(d+1)} - \left\{ \sum_{i=1}^{d} \left[\frac{1-|\mathbf{t}|}{t_i+1-|\mathbf{t}|}\mathbf{t}(i;\ 0)+\frac{t_i}{t_i+1-|\mathbf{t}|}\mathbf{t}(i;\ t_i+1-|\mathbf{t}|)\right]\right.$$

$$\left. + \sum_{1 \leq i<j \leq d} \left[\frac{t_i}{t_i+t_j}\mathbf{t}(i,j;\ t_i+t_j,\ 0)+\frac{t_j}{t_i+t_j}\mathbf{t}(i,j;\ 0,\ t_i+t_j)\right] \right\}.$$

Since f is convex, we have

$$f(\mathbf{t}) \leq \frac{2}{d(d+1)} \left\{ \sum_{i=1}^{d} \left[\frac{1-|\mathbf{t}|}{t_i+1-|\mathbf{t}|}f(\mathbf{t}(i;0))+\frac{t_i}{t_i+1-|\mathbf{t}|}f(\mathbf{t}(i;\ t_i+1-|\mathbf{t}|))\right]\right.$$

$$\left. + \sum_{1 \leq i<j \leq d} \left[\frac{t_i}{t_i+t_j}f(\mathbf{t}(i,j;\ t_i+t_j,\ 0))+\frac{t_j}{t_i+t_j}f(\mathbf{t}(i,j;\ 0,\ t_i+t_j))\right] \right\}.$$

Integrating both sides over T_d, we obtain

$$\frac{d(d+1)}{2}\int_{T_d} f(\mathbf{t})\, d\mathbf{t} \leq \sum_{i=1}^{d} \int_{T_{d-1}} \frac{1}{2}(t_i+1-|\mathbf{t}|)f(\mathbf{t}(i;\ 0))\, dt_1 \ldots d\hat{t}_i \ldots dt_d$$

$$+ \sum_{i=1}^{d} \int_{T_{d-1}} \frac{1}{2}(t_i+1-|\mathbf{t}|)f(\mathbf{t}(i;\ t_i+1-|\mathbf{t}|))\, dt_1 \ldots d\hat{t}_i \ldots dt_d$$

$$+ \sum_{1 \leq i<j \leq d} \int_{T_{d-1}} \frac{1}{2}t_i f(\mathbf{t}(j;\ 0))\, dt_1 \ldots d\hat{t}_j \ldots dt_d$$

$$+ \sum_{1 \leq i<j \leq d} \int_{T_{d-1}} \frac{1}{2}t_j f(\mathbf{t}(i;\ 0))\, dt_1 \ldots d\hat{t}_i \ldots dt_d,$$

where in the last two summations we have relabelled t_i+t_j as t_i and t_j respectively. In the second summation we now make a change of variable and use convection (1.2). This yields

$$\frac{d(d+1)}{2}\int_{T_d} f(\mathbf{t})\,d\mathbf{t} \leqq \sum_{i=1}^{d}\int_{T_{d-1}}\frac{1}{2}(t_i+1-|\mathbf{t}|)f(\mathbf{t}(i;\ 0))\,dt_1\dots d\hat{t}_i\dots dt_d$$
$$+\sum_{i=1}^{d}\int_{T_{d-1}}\frac{1}{2}t_i f(t_1,\dots,t_d)dt_1\dots dt_d$$
$$+\sum_{i=1}^{d}\sum_{\substack{j=1\\ j\neq i}}^{d}\int_{T_{d-1}}\frac{1}{2}t_j f(\mathbf{t}(i;\ 0))dt_1\dots d\hat{t}_i\dots dt_d.$$

Combining the first and last summations and observing that

$$t_i+1-|\mathbf{t}|+\sum_{\substack{j=1\\ j\neq i}}^{d} t_j=1,$$

we get

$$d(d+1)\int_{T_d} f(\mathbf{t})d\mathbf{t} \leqq \sum_{i=1}^{d}\int_{T_{d-1}} f(\mathbf{t}(i;\ 0))dt_1\dots d\hat{t}_i\dots dt_d+\int_{T_{d-1}} f(t_1,\dots,t_d)dt_1\dots dt_d.$$

On using the convention (1.2), we see that this gives the formula (3.5). ■

Lemma 2. *If $f\in C(T_2)$ is convex and $i\geqq 0$, $j\geqq 0$, $i+j\leqq n+1$, and if*

(3.7) $$g(t_1,\dots,t_n):=f(t_1+\dots+t_i,\ t_{i+1}+\dots+t_{i+j}),$$

then $g(t_1,\dots,t_n)$ is convex on T_n, where $\sum_{1}^{n+1} t_j=1$.

Proof. First suppose $i+j\leqq n$. Then for $\mathbf{a}$, $\mathbf{b}\in T_n$, $0\leqq\lambda\leqq 1$,

$$g(\lambda\mathbf{a}+(1-\lambda)\mathbf{b})=f\Big(\lambda\sum_{v=1}^{i}a_v+(1-\lambda)\sum_{v=1}^{i}b_v,\ \lambda\sum_{v=i+1}^{i+j}a_v+(1-\lambda)\sum_{v=i+1}^{i+j}b_v\Big)$$
$$=f\Big(\lambda\Big(\sum_{v=1}^{i}a_v,\ \sum_{v=i+1}^{i+j}a_v\Big)+(1-\lambda)\Big(\sum_{v=1}^{i}b_v,\ \sum_{v=i+1}^{i+j}b_v\Big)\Big)$$
$$\leqq\lambda f\Big(\Big(\sum_{v=1}^{i}a_v,\ \sum_{v=i+1}^{i+j}a_v\Big)+(1-\lambda)f\Big(\sum_{v=1}^{i}b_v,\ \sum_{v=i+1}^{i+j}b_v\Big)\Big)$$
$$=\lambda g(\mathbf{a})+(1-\lambda)g(\mathbf{b}).$$

If $i+j=n+1$, then $g(t_1,\dots,t_n)=f(t_1,\dots,t_i,\ 1-t_1-\dots-t_i)$ and the result follows similarly. ■

We can now prove

Theorem 9. *If $f\in C(T_2)$ is convex, then for all $n\geqq 1$,*

(3.8) $$U_{n+1}f\leqq U_n f.$$

Proof. Since for $i+j+k=n$, we have

$$p_{i,j,k}(x, y)=\frac{n!}{i!j!k!}x^i y^j(1-x-y)^k[x+y+(1-x-y)]$$

$$=\frac{i+1}{n+1}p_{i+1,j,k}(x, y)+\frac{j+1}{n+1}p_{i,j+1,k}(x, y)+\frac{k+1}{n+1}p_{i,j,k+1}(x, y)$$

it follows that

$$U_nf(x, y)=\sum_{i+j+k=n}\left\{\frac{i+1}{n+1}p_{i+1,j,k}(x, y)+\frac{j+1}{n+1}p_{i,j+1,k}(x, y)\right.$$
$$\left.+\frac{k+1}{n+1}p_{i,j,k+1}(x, y)\right\}\int B_{i,j,k}(u, v)f(u, v)\,\mathrm{d}u\mathrm{d}v$$

$$=\sum_{i+j+k=n+1}p_{i,j,k}(x, y)\int\left\{\frac{i}{n+1}B_{i-1,j,k}(u, v)+\frac{j}{n+1}B_{i,j-1,k}(u, v)\right.$$
$$\left.+\frac{k}{n+1}B_{i,j,k-1}(u, v)\right\}f(u, v)\,\mathrm{d}u\mathrm{d}v.$$

Comparing it with the definition of $U_{n+1}f(x, y)$, we see that to prove the result, it is sufficient to prove that for $i+j+k=n+1$, we must have

$$\text{(3.9)}\quad \int B_{i,j,k}(u, v)f(u, v)\,\mathrm{d}u\mathrm{d}v\leqq\int\left\{\frac{i}{n+1}B_{i-1,j,k}(u, v)+\frac{j}{n+1}B_{i,j-1,k}(u, v)\right.$$
$$\left.+\frac{k}{n+1}B_{i,j,k-1}(u, v)\right\}f(u, v)\,\mathrm{d}u\mathrm{d}v.$$

But from (1.3), we know that

$$\int_{\mathbf{R}^n} B_{i,j,k}(u, v)f(u, v)\,\mathrm{d}u\mathrm{d}v=n!\int_{T_n} f(t_1+\ldots+t_i,\ t_{i+1}+\ldots+t_{i+j})\,\mathrm{d}t_1\,\mathrm{d}t_2\ldots\mathrm{d}t_{n+1}.$$

By Lemma 2, $f(t_1+\ldots+t_i,\ t_{i+1}+\ldots+t_{i+j})$ is convex T_n and so applying Lemma 1, and thus relabelling the variables, suitably, we obtain

$$\int_{\mathbf{R}^n} B_{i,j,k}(u, v)f(u, v)\,\mathrm{d}u\mathrm{d}v\leqq\frac{(n-1)!}{n+1}\left\{i\int_{T_{n-1}} f(t_1+\ldots t_{i-1},\ t_i+\ldots+t_{i+j-1})\,\mathrm{d}t_1\ldots\mathrm{d}t_n\right.$$
$$+j\int_{T_{n-1}} f(t_1+\ldots+t_i,\ t_{i+1}+\ldots+t_{i+j-1})\,\mathrm{d}t_1\ldots\mathrm{d}t_n$$
$$\left.+k\int_{T_{n-1}} f(t_1+\ldots+t_i,\ t_{i+1}+\ldots+t_{i+j})\,\mathrm{d}t_1\ldots\mathrm{d}t_n\right\}.$$

The result now follows on applying formula (1.3) again to each of the integrals on the right. ■

4. Convergence

We shall now give a Voronovskaja type result for the operator U_n. To do so we shall prove some lemmas.

Lemma 3. $U_{n,2}(\cdot - a)^4(a) = O(n^{-2})$.

Proof. From (2.5) and (2.10), we see that if $f(x) = x^k$, then

$$U_{n,1}f(x) = \lambda_{n,k}\left(x^k + \frac{k(k-1)}{n-k+1}x^{k-1}\right) + O(n^{-2}).$$

Expanding $(x-a)^4$ and applying $U_{n,1}$ gives the result after some elementary calculations. ∎

Lemma 4. *If* $f_{k,l} = x^k y^l$, *then*

$$\lim_{n\to\infty} n(U_{n,2}f_{20}(x, y) - x^2) = 2x(1-x), \tag{4.1}$$

$$\lim_{n\to\infty} n(U_{n,2}f_{11}(x, y) - xy) = -2xy. \tag{4.2}$$

Proof. Simple calculation from (2.5) shows that

$$U_{n,2}f_{20}(x, y) = \frac{n-1}{n+1}x^2 + \frac{2x}{n+1}$$

$$U_{n,2}f_{11}(x, y) = \frac{n-1}{n+1}xy.$$

The lemma follows immediately from the above. ∎

It follows from (4.1) on using a well-known result of Shisha and Mond ([3], p. 28, Theorem 2.3) that if $\omega(f; \delta)$ is the modulus of continuity of f, then $\|U_{n,1}f - f\| \leqq 2\omega(f; \frac{1}{\sqrt{n}})$.

We now prove

Theorem 10. *If* $f \in C(T_d)$ *is twice differentiable at* $\mathbf{a} = (a_1, \dots, a_d)$ *in* T_d, *then*

$$\lim_{n\to\infty} n(U_{n,d}f - f)(\mathbf{a}) = \sum_{i=1}^{d} a_i(1-a_i)\frac{\partial^2 f}{\partial x_i^2}(\mathbf{a}) - 2\sum_{1\leqq i<j\leqq d} a_i a_j \frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{a}). \tag{4.3}$$

Proof. For $\mathbf{x} \in T_d$, we have

$$f(\mathbf{x}) = f(\mathbf{a}) + \sum_{i=1}^{d}(x_i - a_i)\frac{\partial f}{\partial x_i}(\mathbf{a}) + \frac{1}{2}\sum_{i=1}^{d}(x_i - a_i)^2\frac{\partial^2 d}{\partial x_i^2}(\mathbf{a})$$

$$+ \sum_{1\leqq i<j\leqq d}(x_i - a_i)(x_j - a_j)\frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{a}) + R(\mathbf{x}),$$

where $R(\mathbf{x}) = \eta(\mathbf{x}) \sum_{i=1}^{d}(x_i - a_i)^2$, η being a bounded function in T_d with $\eta(\mathbf{x}) \to 0$ as $\mathbf{x} \to \mathbf{a}$.

Recalling Theorem 4 and the fact that U_n reproduces linear functions, we have

$$U_{n,d}f(\mathbf{x})=f(\mathbf{a})+\frac{1}{2}\sum_{i=1}^{d}(U_{n,1}(t^2)(a_i)-a_i^2)\frac{\partial^2 f}{\partial x_i^2}(\mathbf{a})$$

$$+\sum_{1\leq i<j\leq d}(U_{n,2}(st)(a_ia_j)-a_ia_j)\frac{\partial^2 f}{\partial x_i\partial x_y}(\mathbf{a})+U_{n,d}R(\mathbf{x}).$$

From Lemma 2, all that remains to prove is that

$$\lim_{n\to\infty} nU_{n,d}R(\mathbf{x})=0.$$

Take $\varepsilon>0$ and choose $\delta>0$ such that $|\eta(\mathbf{x})|<\varepsilon$ whenever $\sum_{i=1}^{d}(x_i-a_i)^2<\delta$. Then if $|\eta(\mathbf{x})|<k$ for $\mathbf{x}\in T_d$, we have

$$|R(\mathbf{x})|\leqq\varepsilon\sum_{i=1}^{d}(x_i-a_i)^2+\frac{k}{\delta}\Big(\sum_{i=1}^{d}(x_i-a)^2\Big)^2.$$

Thus we have

$$nU_{n,d}R(\mathbf{x})\leqq\varepsilon n\sum_{i=1}^{d}\{U_{n,1}(t^2)(a_i)-a_i^2\}+\frac{ndk}{\delta}\sum_{i=1}^{d}U_{n,1}(t-a_i)^4.$$

The result now follows from Lemmas 3 and 4. ∎

If f has continuous partial derivatives of order $\boldsymbol{\alpha}=(\alpha_1,\dots,\alpha_d)$, then we can give estimates of the difference $(D^{\alpha}U_nf-D^{\alpha}f)(x)$ in terms of the modulus of continuity of $D^{\alpha}f$. More precisely, we have

Theorem 11. *If $D^{\alpha}f\in C(T)$, then we have*

$$\|D^{\alpha}U_nf\|\leqq\|D^{\alpha}f\| \tag{4.4}$$

and

$$\|D^{\alpha}U_nf-D^{\alpha}f\|\leqq C_1\omega\Big(D^{\alpha}f;\frac{1}{\sqrt{n}}\Big)+\frac{C_2}{n}\|D^{\alpha}f\|, \tag{4.5}$$

where for g in $C(T)$, $\|g\|:=\sup_{x\in T}|g(\mathbf{x})|$ and C_1, C_2 are constants independent of n.

Proof. We shall prove the result when $d=2$. If $\boldsymbol{\alpha}=(\alpha_1,\alpha_2)$, then from (1.4), on applying the method of proof of Theorem 6 repeatedly, we see that

$$D^{\alpha}U_{n,2}f(\mathbf{x})=a_n\sum_{i+j+k=n-|\alpha|}p_{i,j,k}(\mathbf{x})\int_T B_{i+\alpha_1,j+\alpha_2,k+\alpha_1+\alpha_2}(\mathbf{t})(D^{\alpha}f)(\mathbf{t})\,d\mathbf{t}, \tag{4.6}$$

where we have set

$$a_n:=\frac{(n-1)!\,n!}{(n-1+|\alpha|)!\,(n-|\alpha|)!}.$$

Since $a_n<1$ and

$$\int_T B_{i+\alpha_1,\, j+\alpha_2,\, k+\alpha_1+\alpha_2}(\mathbf{t})\,d\mathbf{t}=1,$$

from (4.6), we get (4.4).

In order to prove (4.5), we observe that

$$(4.7) \qquad (D^{\alpha}U_n f - D^{\alpha}f)(\mathbf{x}) = (1-a_n)(D^{\alpha}f)(\mathbf{x}) + a_n I_n,$$

where

$$I_n := \sum_{i+j+k=n-|\alpha|} p_{i,j,k}(\mathbf{x}) \int_T B_{i+\alpha_1,\, j+\alpha_2,\, k+\alpha_1+\alpha_2}(\mathbf{t})(D^{\alpha}f(t) - D^{\alpha}f(x))\,d\mathbf{t}.$$

Then for any $\delta>0$ we have

$$|I_n| \leqq \omega(D^{\alpha}f;\ \delta) \sum_{i+j+k=n-|\alpha|} p_{i,j,k}(\mathbf{x}) \int_T \Big(1+\frac{|\mathbf{t}-\mathbf{x}|}{\delta}\Big) B_{i+\alpha_1,\, j+\alpha_2,\, k+\alpha_1+\alpha_2}(\mathbf{t})\,d\mathbf{t}$$

$$(4.8)$$

$$\leqq \omega(D^{\alpha}f;\ \delta)\Big[1+\frac{1}{\delta} \sum_{i+j+k=n-|\alpha|} p_{i,j,k}(\mathbf{x}) \int_T |\mathbf{t}-\mathbf{x}|\, B_{i+\alpha_1,\, j+\alpha_2,\, k+\alpha_1+\alpha_2}(\mathbf{t})\,d\mathbf{t}\Big].$$

In order to estimate the sum above in the brackets, we first observe that by Schwarz inequality

$$(4.9) \quad \int_T B_{i+\alpha_1,\, j+\alpha_2,\, k+\alpha_1+\alpha_2}(\mathbf{t})\,|\mathbf{t}-\mathbf{x}|\,d\mathbf{t} \leqq \Big[\int_T B_{i+\alpha_1,\, j+\alpha_2,\, k+\alpha_1+\alpha_2}(\mathbf{t})\,|\mathbf{t}-\mathbf{x}|^2 d\mathbf{t}\Big]^{1/2}.$$

Now using Schwarz inequality on the sum, we have

$$(4.10) \qquad \sum_{i+j+k=n-|\alpha|} p_{i,j,k}(x)\Big[\int_T B_{i+\alpha_1,\, j+\alpha_2,\, k+\alpha_1+\alpha_2}(\mathbf{t})\,|\mathbf{t}-\mathbf{x}|^2\,d\mathbf{t}\Big]^{1/2}$$

$$\leqq \Big[\sum_{i+j+k=n-|\alpha|} p_{i,j,k}(x) \int_T B_{i+\alpha_1,\, j+\alpha_2,\, k+\alpha_1+\alpha_2}(\mathbf{t})\,|\mathbf{t}-\mathbf{x}|^2\,d\mathbf{t}\Big]^{1/2}.$$

In order to estimate the above, we shall require the integral

$$(4.11) \qquad A_p(i,\ j,\ k) := \int_T B_{i+\alpha_1,\, j+\alpha_2,\, k+\alpha_1+\alpha_2}(\mathbf{t})\, t_1^p\,d\mathbf{t}.$$

An easy calculation on using (1.5) shows that

$$(4.12) \qquad A_p(i,\ j,\ k) := \frac{(n+|\alpha|-1)!\,(p+i+\alpha_1-1)!}{(i+\alpha_1-1)!\,(n+p+|\alpha|-1)!},\quad p=0,\ 1, \ldots$$

Hence we have

$$\sum_{i+j+k=n-|\alpha|} p_{i,j,k}(\mathbf{x}) \int_T B_{i+\alpha_1,\, j+\alpha_2,\, k+\alpha_1+\alpha_2}(\mathbf{t})(t_1-x_1)^2\,d\mathbf{t}$$

$$\sum_{i+j+k=n-|\alpha|} p_{i,j,k}(\mathbf{x})[A_2(i,\ j,\ k) - 2x_1 A_1(i,\ j,\ k) + x_1^2 A_0(i,\ j,\ k)]$$

$$(4.13) \qquad = \frac{2x_1(1-x_1)}{n} + Q\Big(\frac{1}{n^2}\Big),$$

on using (4.12) and simple properties of Bernstein polynomials on the simplex, viz.

$$\sum_{i+j+k=n-|\alpha|} p_{i,j,k}(\mathbf{x})\left(\frac{i}{n-|\alpha|}\right)^{\nu}=\begin{cases}1, & \nu=0\\ x, & \nu=1\\ \dfrac{x(1-x)}{n-|\alpha|}+x^2, & \nu=2.\end{cases}$$

Thus combining (4.8), (4.9), (4.10) and (4.13), we have

$$|I_n|\leqq\omega(D^{\alpha}f;\ \delta)\left[1+\frac{2}{\delta}\left\{\frac{x_1(1-x_1)+x_2(1-x_2)}{n}+O\left(\frac{1}{n^2}\right)\right\}^{1/2}\right].$$

Choosing $\delta=n^{-1/2}$, we get $|I_n|\leqq C_1\omega(D^{\alpha}f;\ \frac{1}{\sqrt{n}})$. Since $1-a_n=\frac{C_2}{n}+O(\frac{1}{n^2})$, (4.5) follows from (4.7) and the estimate of $|I_n|$. ∎

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+ *University of Texas*
College Station
Texas
U.S.A
++ *University of Alberta*
Edmonton, Alberta
T6J 3A6
CANADA

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