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Characterization of Tikhonov Well-Posedness for Preorders*

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In this article several topological characterization of the Tikhonov well-posedness for minimum problems for preorders are given. It is investigated what happens with the well-posedness in the quotient space which is obtained by identifying the elements that are not "distinguished" with respect to the preorder. Possible generalizations, not requiring uniqueness of the solution, for both the well-posed unconstrained and constrained minimum problem are proposed. Characterizations for them are also proved.

1. Introduction and preliminaries

The main goal of this paper is to provide some topological characterizations of Tikhonov well-posedness for minimum problems for preorders. This property is a direct extension of the analogous one for functions to be minimized: we refer to [4, 6, 12 and 14] for functions and to [10, 11] for preorders.

Before proceeding further in the description of our results let us see what is the meaning of well-posedness both for functions and preorders.

Let (X, τ) be a topological space and $f: X \to \mathrm{R}$ be a real-valued function on it. The problem of finding a minimum of f on X (which we denote by (X, f)) is said to be Tikhonov well-posed [14] if the function f has unique minumum x_0 in X and every minimizing sequence $(x_n) \subset X$ (that is $f(x_n) -> inf(X, f) := inf\{f(x): x \in X\}$) converges to x_0. If (X, f) is Tikhonov well-posed then every minimizing net (not only every minimizing sequence) converges to its unique solution (see [3, 9]). As usual, $arg\,min(X, f)$ will denote the set of all minima of the problem (X, f).

Further let us remind that a relation $\preceq$ on some set X is said to be a preorder if it is reflexive and transitive; if $\preceq$ is also total, the preorder is said to be total. Notice that, given $f: X \to \mathrm{R}$, this f induces a total preorder on X by means of the following:

$$(*) \qquad x \preceq y \text{ iff } f(x) \leq f(y).$$

Conversely, under some restrictions a total preorder can be represented by a real-valued function (i.e. so that $(*)$ is satisfied): see [5].

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Given a preorder $\preceq$ on X and a non-empty subset A of X, we shall denote by $(A, \preceq)$ the problem of finding a minimum for $\preceq$ on A, i.e. to find $x_0 \in A$ such that $x_0 \preceq x$ for every $x \in A$. The notation $min(A, \preceq)$ will designate the set of the minima of $(A, \preceq)$. Following the usual terminology $(A, \preceq)$ will be called unconstrained provided $A = X$, otherwise $(A, \preceq)$ is constrained. In the first three sections we consider only unconstrained minimum problems for preorders.

The following two definitions are given in [10].

Definition 1.1. A net (x_λ) is said to be a minimizing net for $(X, \preceq)$ iff for every $x \in X$ which is not minimum for $\preceq$ on X, we have $x_\lambda \prec x$ eventually.

Here $x \prec y$, $x, y \in X$, means $x \preceq y$, but not $y \preceq x$.

Definition 1.2. The problem $(X, \prec)$ is said to be Tikhonov well-posed (briefly w.p.) iff:

1) there is a unique minimum x_0 for $\preceq$ on X;

2) every minimizing net of $(X, \preceq)$ converges to x_0.

A constrained minumum problem $(A, \preceq)$, $A \subset X$ is called w.p. if $(A, \preceq | A)$ is w.p., where $\preceq | A$ means the restriction of $\preceq$ on the set A.

It is worth being mentioned that w.p. of a $f: X \to \mathrm{R}$ and of the preorder $\preceq$ on X represented by f are related: w.p. of $\preceq$ implies w.p. for f; moreover given f, a function $\tilde{f}: X \to \mathrm{R}$ can be defined so that $\tilde{f}$ still represents $\preceq$ and w.p. for $\preceq$ and for $\tilde{f}$ are equivalent (see [10]).

To get characterizations of w.p., we shall involve the topology on X directly coming from the preorder, denoted here by τ_{pr}. By means of this topology it will be immediate to characterize minimizing nets, and to identify w.p. through a comparison of the two topologies, τ and τ_{pr}. Moreover, we shall express w.p. as upper semicontinuity of the "level set" correspondence.

Furthermore, the topology τ_{pr} will allow us also to provide some conditions under which sequential w.p. (that is the condition 2) in Definition 1.2 fulfilled for sequences instead of nets; see Section 2) implies w.p., and to clarify to some extent the relationship between w.p. for $\preceq$ and w.p. of a real-valued function representing it.

Sections 3 and 4 will be devoted to related topics: in Section 3 we shall compare w.p. of a total preorder $\preceq$ and w.p. of the induced (quotient) total ordering. In Section 4 generalizations of w.p. and of the strong well-posedness for (constrained) minimum problems for preorders (introduced in [11]) are given. They are inspired by the well-known Furi-Vignoli generalization [6] of Tikhonov well-posedness for functions and do not require the uniqueness of the minimum. We get characterizations of generalized w.p. similar to those obtained for w.p. (in particular using τ_{pr}). Furthermore we shall get also a characterization of generalized strong w.p. by means of Kuratowski measure of non-compactness, quite similar to that one proved in [6].

2. Topological characterizations of Tikhonov well-posedness

This Section is devoted to the use of the natural topology on X generated by a preorder $\preceq$ on X for study of w.p. of $(X, \preceq)$.

Definition 2.1. Given a set X and a preorder $\preceq$ on it, the preorder topology τ_{pr} on X is the topology which has as a subbase the family of sets $\{(\leftarrow, x[: x \in X\}$, where $(\leftarrow, x[= \{y \in X : y < x\}$.

Notice that, when $\preceq$ is total, the family above becomes a base, provided we add the whole space X to it.

It should be clear that the role of τ_{pr} is to express the idea of minimizing net in topological terms: so, the following result is foreseen, and its proof is straightforward.

Lemma 2.2. *Let $\preceq$ be a preorder on X, and assume that there is a minimum x_0 for $\preceq$ on X. Then a net (x_λ) is minimizing for the problem $(X, \preceq)$ iff $x_\lambda \xrightarrow{\tau_{pr}} x_0$.*

Proof: Let x_0 be a minimum for the preorder $\preceq$ on X and (x_λ) is a minimizing net for the problem $(X, \preceq)$. Take a τ_{pr}-neighbourhood U of x_0 from the base. Then $U = \bigcap_{i\in F} (\leftarrow, y_i[$, F-finite, non-empty. Since $x_0 \in (\leftarrow, y_i[$ for every $i\in F$, y_i are not minima for $(X, \preceq)$. Now using the fact that (x_λ) is minimizing we have $x_\lambda \in (\leftarrow, y_i[$ eventually for every $i\in F$. Hence (x_λ) is eventually in the intersection.

Conversely, let (x_λ) be τ_{pr}-convergent to x_0. Given y not minimum, notice that it must be $x_0 \preceq y$: so $(\leftarrow, y[$ is a neighbourhood of x_0 for τ_{pr}. Hence (x_λ) eventually enters it, which means that $x_\lambda \preceq y$ eventually. Consequently (x_λ) is minimizing for $(X, \preceq)$. The proof is completed.

Let us recall that a correspondence T acting from a topological space Y into the subsets of a topological space Z is said to be upper semicontinuous (briefly usc) at $y_0 \in Y$ if for every open in Z set U such that $T(y_0) \subseteq U$ there is an open set $V \ni y_0$ with $T(y) \subseteq U$ whenever $y \in V$. For more details about correspondences the reader is referred to [7].

Now we are able to prove the first characterization of w.p. This is achieved through the level set correspondence $Lev(x) = \{y\in X;\ y \preceq x\}$, $x\in X$ (see [9] for an analogous result for minimum problems for functions).

From now on X is endowed with a topology τ and a preorder $\preceq$.

Theorem 2.3. *Let $\preceq$ be total. Then the minimum problem $(X, \preceq)$ is w.p. iff there is $x_0 \in X$ such that the level set correspondence $Lev\colon (X, \tau_{pr}) \to (X, \tau)$ is usc and single-valued at x_0.*

In this case it will turn out that $min(X, \preceq) = \{x_0\}$.

Proof. Let $(X, \preceq)$ be w.p. with unique solution x_0. Then obviously Lev is single-valued at x_0. Suppose Lev is not τ_{pr}-τ usc at x_0. This means that there is a τ-neighbourhood U of x_0 such that for every τ_{pr}-neighbourhood V of x_0 we can find $x_V \in V$ such that $Lev(x_V)\setminus U \neq \emptyset$. Let us take z_V from this non-empty difference and direct the local base of τ_{pr}-neighbourhoods of x_0 by inclusion. In this way we obtain two nets (x_V) and (z_V). Since $x_V \xrightarrow{\tau_{pr}} x_0$, (x_V) is a minimizing net for the minimum problem $(X, \preceq)$. From $z_V \preceq x_V$ we get that (z_V) is minimizing for $(X, \preceq)$ too. This is a contradiction with the well-posedness of $(X, \preceq)$, since $z_V \neq U$ for every V.

Conversely, let Lev be single-valued and τ_{pr}=t usc at some x_0. Then, using totalness of $\preceq$, it is routine to check that x_0 is a solution of the minimum problem $(X, \preceq)$. Further, single-valuedness of Lev at x_0 shows that x_0 is the unique minimum of $(X, \preceq)$. At the end let (x_λ) be a minimizing net for $(X, \preceq)$. Then

$x_\lambda \xrightarrow{\tau_{pr}} x_0$. Since $x_\lambda \in Lev(x_\lambda)$, from the τ_{pr}-τ upper semicontinuity of *Lev* we obtain $x_\lambda \xrightarrow{\tau} x_0$. The proof is completed.

We note a fact that will be present also in the following characterization of w.p. (Theorem 2.4). In the "only if" part of the above theorem totalness of $\preceq$ is not needed. For the "if" part totalness enters only in proving that x_0 is a minimum point: the uniqueness of the minimum and convergence of minimizing nets is assured without totalness. So the following variant of Theorem 2.3 is true not assuming totalness of the preorder.

Theorem 2.3′. *Assume that* $min(X, \preceq)$ *is non-empty. Then* $(X, \preceq)$ *is w.p. iff there is* $x_0 \in X$ *such that* $Lev: (X, \tau_{pr}) \to (X, \tau)$ *is usc and single-valued at* x_0.

A further characterization of w.p. is the following

Theorem 2.4. *Let the topology* τ *be* T_1 *and* $\preceq$ *is total. Then* $(X, \preceq)$ *is w.p. iff there is* $x_0 \in X$ *such that each* τ*-neighbourhood of* x_0 *is a* τ_{pr}*-neighbourhood of* x_0.

As in Theorem 2.3 it turns out that $(x_0) = \min(X, \preceq)$.

Proof: Suppose $(X, \preceq)$ is w.p. with unique solution x_0. Take a τ-neighbourhood U of x_0. Assume that for every τ_{pr}-neighbourhood V from the local base of x_0 we have $x_V \in V \setminus U$. Now we can proceed as in the proof of the "only if" part of the Theorem 2.3 to get a contradiction.

Conversely, assume there is x_0 from X such that each τ-neighbourhood of x_0 is its τ_{pr}-neighbourhood. We prove that x_0 is a minimum for $(X, \preceq)$. Suppose there is $y \in X$ such that $y < x_0$. Take a τ-neighbourhood U of x_0 such that $y \notin U$. Since U is also a τ_{pr}-neighbourhood of x_0 there is some $x \in X$ such that $x_0 \in (\leftarrow, z[\subseteq U$. But we have $y < x_0 < z$, so $y \in (\leftarrow, z[$. The last contradicts $y \notin U$. To prove that x_0 is the unique minimum for the problem $(X, \preceq)$ we have to assume the existence of another minimum y and to proceed as above replacing $y < x_0$ by $y \preceq x_0$.

Now, let (x_λ) be a minimizing net for $(X, \preceq)$. Then (Lemma 2.2) $x_\lambda \xrightarrow{\tau_{pr}} x_0$, hence $x_\lambda \xrightarrow{\tau} x_0$, thanks to the assumptions.

The theorem above has also a variant without assuming that $\preceq$ is total. For, let $N = \{x \in X$: every τ-neighbourhood of x is a τ_{pr}-neighbourhood of $x\}$. Then the following is true.

Theorem 2.4′. *Let the topology* τ *be* T_1. *Then* $(X, \preceq)$ *is w.p. iff both* $\min(X, \preceq)$ *and* N *are non-empty.*

The proof of this theorem is a simple consequence of the proof of the "only if part" of Theorem 2.4 and the following

Proposition 2.5. *Suppose* τ *is* T_1 *and* $N \neq \emptyset$. *Then* $min(X, \preceq) \subset N$.

Proof. Let $y_0 \in min(X, \preceq)$. If $y_0 \notin N$, there is $x_0 \in N$ such that $x_0 \neq y_0$. Hence we can find a τ-neighbourhood U of x_0 such that $y_0 \notin U$. Since U is also τ_{pr}-neighbourhood of x_0 there are $y_1, \ldots, y_k \in X$ such that $x_0 \in \bigcap_{i=1}^{k} (\leftarrow, y_i[\subset U$.

Therefore $y_0 \preceq x_0 < y_0$ for every $i=1, 2 \ldots k$. From this we get $y_0 < y_i$, hence $y_0 \in \bigcap_{i=1}^{k} (\leftarrow, y_i[$, so $y_0 \in U$, which is a contradiction.

The conclusion of Proposition 2.5 is not true if we omit the assumption $N \neq 0$ (see the example in Remark 3.3 below).

Another interesting use that can be made of τ_{pr} is the following theorem. In it, seq-w. p. (in short from sequential w. p.), means that $(X, \preceq)$ satisfies Definition 1.2 with sequences instead of nets in part 2).

Theorem 2.6. *Assume that there is a unique minimum x_0 for $\preceq$ on X and that x_0 has a countable τ_{pr}-local base. Then seq-w. p. for $(X, \preceq)$ implies w. p.*

Proof: Let $(X, \preceq)$ be seq-w. p. with unique solution x_0. Let (x_λ) be a minimizing net for $(X, \preceq)$. Suppose (x_λ) does not converge to x_0 with respect to the topology τ. Then there is a subnet (x_{λ_μ}) of (x_λ) and a τ-neighbourhood U of x_0 such that $x_{\lambda_\mu} \notin U$ for every μ. Take now a countable τ_{pr}-local base for x_0 $\{V_i\}_{i=1}^{\infty}$ such that $V_{i+1} \subseteq V_i$ for every i. Since (x_{λ_μ}) is minimizing for $(X, \preceq)$, for every i there is $x_{\lambda_{\mu_i}} \in V_i$: so, $(x_{\lambda_{\mu_i}})$ is a sequence τ_{pr}-converging to x_0, hence τ-converging to x_0 because of seq-w. p. This is a contradiction, since this sequence is always outside U.

Remark 2.7. If $\preceq$ can be represented by $f: X \to \mathsf{R}$ and x_0 is a minimum for $(X, \preceq)$, obviously we are assured there is a countable τ_{pr}-local base for x_0 (and consequently seq-w. p. and w. p. for $(X, \preceq)$ coincide). It is known (see [5]) that the existence of f representing $\preceq$ is equivalent to the existence of a countable subset E of X such that for every $x, y \in X$ with $x<y$, there is $z \in E$ such that $x<z<y$. However, this condition is not necessary for x_0 to have a countable τ_{pr} — local base, as is seen by the following example.

Example 2.8. Let $X=\{(x, y) \in \mathsf{R}^2 : 0 \leqq x \leqq y\}$. Consider the total order $\preceq$ on X defined as follows: $(x_1, y_1) \preceq (x_2, y_2)$ iff $x_1 < x_2$ or ($x_1 = x_2$ and $y_1 \leqq y_2$). It is immediate to see that $(0, 0)$ is the unique minimum for $(X, \preceq)$ and that this problem is seq-w. p. Moreover, the sets $\{(x, y) \in X : (x, y) < (1/n, 0)\}$, $n=1, 2, \ldots$, provide a countable τ_{pr}-local base for $(0, 0)$: so, we have w. p. for $(X, \preceq)$ too. Notice, however, that it is not possible to represent $\preceq$ by a real-valued function, not only on X, but even on any interval $[(0, 0), (x, y)]$ (of course with $(0, 0) \neq (x, y)$).

Another interesting feature that is worth being mentioned in this context is the following

Remark 2.9. Given $f: X \to \mathsf{R}$ and $\preceq$ induced by f on X, two topologies related with w. p. can be introduced on X. One of them is, of course, τ_{pr}, and the other one τ_f has as a base the sets from the form $(x \in X : f(x) < a\}$, $a \in \mathsf{R}$. It is immediate to see that τ_f is finer than τ_{pr}, but they are not obliged to coincide. To see one example, consider the following two functions which represent the same preorder

$$\bar{f}(x) = \begin{cases} x & \text{if } 0 \leqq x \leqq 1 \\ 1/x & \text{if } x > 1 \end{cases} \qquad \text{and} \quad f(x) = \begin{cases} \bar{f}(x) & \text{if } x > 0 \\ -1 & \text{if } x = 0. \end{cases}$$

Then $\{0\} \notin \tau_{pr} = \tau_{\bar{f}}$, while $\{0\} \in \tau_f$. But if f represents the preorder $\preceq$ the following function

$$\tilde{f}(x)=\begin{cases} inf(f(x): x\in X\backslash \arg\min(X,f)) & \text{if } x\in\arg\min(X,f)\\ f(x) & \text{otherwise}\end{cases}$$

represents the same preorder (see Theorem 3.5 from [10]) and has the property that $\tau_{\tilde{f}}$ and τ_{pr} coincide. As a matter of fact, the coincidence of $\tau_{\tilde{f}}$ and τ_{pr} is another way to see the equivalence between w.p. of $\tilde{f}$ and $\preceq$, as proved in Corollary 3.7 from [10].

Moreover, it is easily recognized that results like Lemma 2.2 or theorem 2.3 and 2.4 can be proved, to express w.p. for a given f by means of τ_f. Therefore the topologies τ_f and τ_{pr} provide another way to understand the relationship between w.p. for f and for the induced preorder $\preceq$.

3. About well-posedness for quotient ordering

Let be given a topological space (X, τ) and a total preorder $\preceq$ on it. The relation $\sim$, defined by $x\sim y$ iff $x\preceq y$ and $y\preceq x$, is an equivalence relation and the corresponding quotient space $X/\sim$ of the equivalence classes $[x]$, $x\in X$, will become a totally ordered set with the preorder $\preceq_q$ defined in the obvious way $[x]\preceq_q [y]$ iff $x\preceq y$. We want to investigate the relationship between the usual quotient topology $(\tau_{pr})_q$ on $X/\sim$ generated by τ_{pr} and the topology on $X/\sim$ induced by the total order $\preceq_q$, denoted by σ_{pr}.

Let us recall that when a topological space (Z, τ) and an equivalence relation ρ on it are given the usual quotient topology τ_q generated by τ on the set of equivalence classes Z/ρ consists of all sets $U\subseteq Z/\rho$ such that $q^{-1}(U)$ is open in Z. Here $q: Z\to Z/\rho$ is the natural quotient mapping assigning to every $z\in Z$ the corresponding equivalence class $[z]$.

In our setting we prove:

Proposition 3.1. *The quotient mapping $q:(X, \tau_{pr})\to (X/\sim, (\tau_{pr})_q)$ is open and a base for the topology $(\tau_{pr})_q$ on $X/\sim$ is the family $\{q((\leftarrow, y[): y\in X\}\cup\{X/\sim\}$.*

Proof: Let $W\neq X$ be open in (X, τ_{pr}). Then $W=\bigcup_{\alpha\in A}(\leftarrow, y_\alpha[$ for some set A. We have $q(W)=\bigcup_{\alpha\in A} q(\leftarrow, y_\alpha[)$. Since $q^{-1}q((\leftarrow, y_\alpha[)=(\leftarrow, y_\alpha[$, it follows that $q(W)$ is open in $(X/\sim, (\tau_{pr})_q)$. To prove the second statement, notice that for $U\in(\tau_{pr})_q$, $q^{-1}(U)\in\tau_{pr}$, hence $q^{-1}(U)=\bigcup_{\alpha\in A}(\leftarrow, y_\alpha[$ for some A, so $U=q(q^{-1}(U))$ $=\bigcup_{\alpha\in A} q((\leftarrow, y_\alpha[)$.

From this proposition we see that $(\tau_{pr})_q=\sigma_{pr}$, because, by $q((\leftarrow, y[)=(\leftarrow, [y][$ it follows that they have a common base of open sets. Eventually we can prove:

Theorem 3.2. *Let $(X, \preceq)$ be w.p. with respect to the topology τ. Then $(X/\sim, \preceq_q)$ is w.p. with respect to the quotient topology τ_q.*

Proof: Let x_0 be the unique minimum for $(X, \preceq)$. Of course, $[x_0]$ is the unique minimum for $(X/\sim, \preceq_q)$. Let $([x_\lambda])$ be minimizing net for $(X/\sim, \preceq_q)$. Take $y\in X$ such that $x_0<y$. Then, $[x_0]<_q[y]$ and consequently $[x_\lambda]<_q[y]$ for large λ, hence $x_\lambda<y$ for large λ. Therefore (x_λ) is a minimizing net for $(X, \preceq)$. The last

entails $x_\lambda \xrightarrow{\tau} x_0$ due to the well-posedness. Hence, since as is well-known the quotient mapping is $\tau-\tau_q$ continuous, we obtain $[x_\lambda] \xrightarrow{\tau_0} [x_0]$. The proof is completed.

Remark 3.3. The converse of the above theorem is not true, even if we have uniqueness of the minimum for $(X, \preceq)$. Take $X \subset R^2$ defined as follows: $(x, y) \in X$ iff $((x, y)=(0, 0)$ or $(x>0$ and $y \in [-1, 1]$)). Define $(x_1, y_1) \preceq (x_2, y_2)$ iff $x_1 \leqq x_2$ and consider on X the inherited Euclidean topology τ. Clearly, $(X/\sim, \preceq_q)$ is w.p., while $(X, \preceq)$ is not.

4. Generalized Tikhonov well-posedness

This last Section is devoted to the generalizations of two notions: one is the idea of generalized Tikhonov well-posedness for preorders, clearly inspired by the analogous idea for functions [6]; the other one is to look at constrained problems, and to do the same with the strong well-posedness for preorders (introduced and studied in [11]).

Let X be a topological space with a topology τ and a preorder $\preceq$. Following an idea from [6] we give the following

Definition 4.1. The problem $(X, \preceq)$ is said to be generalized Tikhonov well-posed (briefly g. w. p.) if the set of minima for $(X, \preceq)$ is non-empty and every minimizing net for $(X, \preceq)$ has a subnet converging to a minimum for $(X, \preceq)$.

For the corresponding notion for functions the reader is referred to [3] and to [1] for other possible ideas of generalization. Observe that if $(X, \preceq)$ is g. w. p. then $min(X, \preceq)$ is (non-empty) and compact.

We remark that there are preorders without any minimizing net. For example take $X=[(0, 0), (0, 1)] \cup [(1, 0), (1, 1)] \subset R^2$ with (x_1, y_1) (x_2, y_2) iff $x_1=x_2$ and $y_1 \leqq y_2$, $(x_i, y_i) \in X$, $i=1, 2$. It is easy to see that there is no minimizing net for the problem $(X, \preceq)$.

The following is a characterization of g. w. p. similar to Theorem 2.3. For functions an analogous result was observed by R. Lucchetti and the second author (private communication).

Theorem 4.2. *Let $\preceq$ be total. Then $(X, \preceq)$ is g. w. p. iff there is a non-empty τ-compact subset K of X such that the following two conditions are fulfilled:*

(i) *$Lev(x)=K$ for every $x \in K$;*

(ii) *the level set correspondence $Lev:(X, \tau_{pr}, \to (X, \tau)$ is usc at any $x \in K$.*

Proof: Let $(X, \preceq)$ be g. w. p. Then $K=\min(X, \preceq)$ is non-empty and compact. Obviously i) is fulfilled for K. Also to ii), assume the contrary. Then, the Lev correspondence is not $\tau_{pr}-\tau$ usc at some $x_0 \in K$. This means the existence of a non-empty τ-open set U of X such that $Lev(x_0)=K \subseteq U$ and for every τ_{pr}- neighbourhood V of x_0 there is a point $y_V \in V$ with $Lev(y_V) \setminus U \neq \emptyset$. Proceeding as in the proof of Theorem 2.3 we obtain a minimizing net (z_V) for $(X, \preceq)$such that $z_V \notin U$ for every V. Hence no subnet of (z_V) converge to a point of $K=\min(X, \preceq)$. This is a contradiction.

Conversely, let a non-empty τ-compact subset K of X exist satisfying i) and ii). We prove that $K = \min(X, \preceq)$. First, $K \subseteq \min(X, \preceq)$. Indeed, let $x \in K$: if $y \in K$ we have $x \preceq y$ since $x \in Lev(y)$; if $y \notin K$, $y \notin Lev(x)$ yielding $x < y$. That $K \supseteq \min(X, \preceq)$ is a consequence of the fact that for every $x \in X$ $Lev(x) \supseteq \min(X, \preceq)$.

Let, further, (x_λ) be minimizing net for $(X, \preceq)$ which means that $x_\lambda \to x$ in the τ_{pr}-topology for every $x \in K$. We prove that there is a point $x_0 \in K$ such that each τ-neighbourhood of x_0 contains a cofinal part of the net (x_λ). Suppose the contrary: for every $x \in K$ there is a τ-neighbourhood U_x of x and λ_x such that $x_\lambda \notin U_x$ for every $\lambda \geqq \lambda_x$. Since K is compact there are $x_1, \ldots, x_k$ from K such that $K \subseteq \bigcup_{i=1}^{\infty} U_{x_i}$. Take $U := \bigcup_{i=1}^{\infty} U_{x_i}$ and let $\lambda \geqq \max\{\lambda_{x_i}: i = 1, \ldots, k\}$. Then $x_\lambda \notin U$. This is a contradiction with the $\tau_{pr} - \tau$ upper semicontinuity of Lev at any point of K. Now, having some x_0 from K, each τ-neighbourhood of which contains a cofinal part of the net (x_λ), it is a standard procedure to organize a subnet of (x_λ) converging to x_0. The proof is completed.

At the end of this section we shall give a generalization of the strong well-posedness for constrained minimum problems for preorder in the setting of metric spaces, that was introduced in [11]. For the generalization of the corresponding notion for functions we refer to [13]. Another approach for functions is given in [2].

Let X be a metric space with a metric d, $\preceq$ be a preorder on X and $\emptyset \neq A \subseteq X$. A sequence (x_n) is called in [11] a minimizing sequence in generalized sense (briefly g. m. s.) for the problem $(A, \preceq)$ if

1) $d(x_n, A) \to 0$;
2) for every $x \in X$ such that there is $y \in A$ with $y < x$, we have eventually $x_n < x$.

Here $d(x, A) = \inf\{d(x, y) : y \in A\}$ is the distance function generated by the set A. The problem $(A, \preceq)$ is strong well-posed ([11]) if it has unique solution towards every g. m. s. converges. It is seen that strong well-posedness of $(A, \preceq)$ implies seq-w. p. of $(A, \preceq)$. Here we give the following (natural) relaxation of this notion.

Definition 4.3. The problem $(A, \preceq)$ is said to be generalized strongly well-posed (g. s. w. p.) if the set of minima for $(A, \preceq)$ is non-empty and every g. m. s. has a subsequence converging to a minimum for $(A, \preceq)$.

Observe that if $(A, \preceq)$ is g. s. w. p. then $min(A, \preceq)$ is a non-empty compact subset of X.

The preorder $\preceq$ is said to be lower semicontinuous (briefly l. s. c.) if all sets from the type $\{y \in X: y \preceq x\}$, $x \in X$, are closed. Let us assume that (X, d) is complete, A is closed and $\preceq$ is l. s. c. Consider M, the set of all g. m. s. (x_n) for $(A, \preceq)$ such that

1) $d(x_{n+1}, A) \leqq d(x_n, A)$ for every n;
2) x_{n+1} x_n for every n;
3) for every n there is $y \in A$ such that $y < x_n$.

Given $(x_n) \in M$, we define

$$L_n := \{x \in X : x \preceq x_n,\ d(x, A) \leqq d(x_n, A) + 1/n\}.$$

Note that, under our assumptions, $x_n \in L_n$, $L_{n+1} \subset L_n$, L_n are closed and $\bigcap_{n=1}^{\infty} L_n = \min(A, \preceq)$. Moreover, if $z_n \in L_n$ for every n, then (z_n) is g. m. s. for (A, f).

Before proving Theorem 4.4 we must introduce some more terminology. Given $E \subseteq X$, by $\alpha(E)$ we denote the Kuratowski measure of non-compactness of E which is the infimum of all numbers $\varepsilon > 0$ such that E can be covered by a finite number of subsets of diameters less or equal to ε (for more details see [8]).

Theorem 4.4. *If there is $(x_n) \in M$ such that for the corresponding L_n we have $\alpha(L_n) \to 0$, then $(A, \preceq)$ is g. s. w. p. Conversely, if $(A, \preceq)$ is g. s. w. p. then for every sequence in M we have $\alpha(L_n) \to 0$ for the corresponding L_n.*

Proof: Let there be a sequence (x_n) from M such that for the corresponding L_n $\alpha(L_n) \to 0$. Then $\bigcap_{n=1}^{\infty} L_n = \min(A, \preceq)$ is non-empty and compact (see [8]). Take (y_n) a g. m. s. for $(A, \preceq)$, and let $x_0 \in \min(A, \preceq)$.

Taking x_1, we have $x_0 < x_1$ and consequently $y_{n_1} < x_1$ for some n_1. It is no loss of generality to assume (having in mind that $d(y_n, A) \to 0$) that $d(y_{n_1}, A) \leqq d(x_1, A) + 1$. Further, for x_2 in the same way we find $n_2 > n_1$ such that $y_{n_2} < x_2$ and $d(y_{n_2}, A) \leqq d(x_2, A) + 1/2$. Proceeding in this way we obtain a subsequence (y_{n_k}) of (y_n) such that $y_{n_k} < x_k$ and $d(y_{n_k}, A) \leqq d(x_k, A) + 1/k$ for every k. Therefore $y_{n_k} \in L_k$ for all k. From the last, a standard procedure gives a further subsequence of (y_n) converging to a point from $\prod_{k=1}^{\infty} L_k$.

Conversely, let $(A, \preceq)$ be g. s. w. p. Take any $(x_n) \in M$ and consider the corresponding L_n. Suppose $\alpha(L_n) > \delta > 0$ for some δ and every n. Since $min(A, \preceq)$ is compact, there are finitely many elements $z_1, \dots, z_k$ from $min(A, \preceq)$ such that $\bigcup_{i=1}^{k} B(z_i, \delta) \supseteq min(A, \preceq)$ (here $B(z_i, \delta)$ are the open balls centered at z_i with radius δ). Consider the open set $U := \bigcup_{i=1}^{k} B(z_i, \delta)$. Obviously each L_n contains a point y_n not belonging to U, because otherwise $\alpha(L_n) \leqq \delta$ for some n. Therefore (y_n) is a g. m. s. which cannot have any subsequence converging to a point of $min(A, \preceq)$. The last is a contradiction.

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Page	Written	Must be
146-11 from below	$f(x_n)->$	$f(x_n) \to$
148-15 from above	$x_0 \preceq y$	$x_0 \prec y$
148-16 from above	$x_\lambda \preceq y$	$x_\lambda \prec y$
148-4 from below	$\tau_{pr} = t$	$\tau_{pr} - \tau$
149-14 from above	$(x_0) = \min(X, \preceq)$	$x_0 = \min(X, \preceq)$
149-21 from above	$y < x_0$	$y \prec x_0$
149-22 from above	$x \in X$	$z \in X$
149-23 from above	$y < x_0 < z$	$y \prec x_0 \prec z$
149-25 from above	$y < x_0$	$y \prec x_0$
150-1 from above	$y_0 \preceq x_0 < y_0$	$y_0 \preceq x_0 \prec y_i$
150-1 from above	$y_0 < y_i$	$y_0 \prec x_i$
150-22 from above	$x < y$	$x \prec y$
150-22 from above	$x < z < y$	$x \prec z \prec y$
150-16 from below	$(x,y) < (1/n,0)$	$(x,y) \prec (1/n,0)$
151-1 from above	$f(x) = inf(f(x) : x$	$\tilde{f}(x) = inf\{f(x') : x'$
151-3 from below	$[x)]$	$([x])$
151-2 from below	$x_0 < y$	$x_0 \prec y$
151-2 from below	$[x_0] <_q [y]$	$[x_0] \prec_q [y]$
151-2 from below	$[x_\lambda] <_q [y]$	$[x_\lambda] \prec_q [y]$
151-1 from below	$x_\lambda < y$	$x_\lambda \prec y$
152-2 from above	$[x_\lambda] \xrightarrow{\tau_0} [x_0]$	$[x_\lambda] \xrightarrow{\tau_q} [y]$
152-18 from below	$(x_1, y_1) \quad (x_2, y_2)$	$(x_1, y_1) \preceq (x_1, y_2)$
152-9 from below	$(X, \tau_{pr},$	(X, τ_{pr})
152-7 from below	Also	As
153-3 from above	$x < y$	$x \prec y$
153-10 from above	$\bigcup_{i=1}^{\infty} U_{x_i}$ (twice)	$\bigcup_{i=1}^{k} U_{x_i}$
153-21 from below	$y < x$	$y \prec x$
153-21 from below	$x_n < x$	$x_n \prec y$
153-6 from below	$x_{n+1} \quad x_n$	$x_{n+1} \preceq x_n$
153-5 from below	$y < x_n$	$y \prec x_n$
154-11 from above	$x_0 < x_1$	$x_0 \prec x_1$
154-11 from above	$y_{n_1} < x_1$	$y_{n_1} \prec x_1$
154-14 from above	$y_{n_2} < x_2$	$y_{n_2} \prec x_2$
154-15 from above	$y_{n_k} < x_k$	$y_{n_k} \prec x_k$
154-17 from above	$\prod_{k=1}^{\infty} L_k$	$\bigcap_{k=1}^{\infty} L_k$
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Due to mailing problems the authors were not in a position to make the corrections in the proofs of their paper. The editors apologize for this annoyance and publish the printing errors herein.