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On Some Properties of Universal Groups

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1. Introduction

P. Neumann [15] and P. Hall [9] proved, independently, that the following is true for the SQ-universal groups:

Theorem (Neumann, Hall): *If H is a subgroup of finite index in a group G, then*

G is SQ-universal if and only if H is SQ-universal.

(A group G is SQ-universal if every countable group can be embedded in some factor group of G.) The above result has several interesting consequences. For example, it gives a positive answer to the question whether the triangle groups

$$(k,\ m,\ n)=\langle a,\ b \mid a^k=b^m=(ab)^n=1\rangle$$

are SQ-universal if $1/k+1/m+1/n<1$. It follows that the modular group

$$M=\langle a,\ b \mid a^2=b^3=1\rangle$$

is SQ-universal, as well as every finitely generated Fuchsian group which does not have an abelian subgroup of finite index (see [15]).

We shall prove here that the analogue of the above theorem, as well as some more general statements, are also true for universal groups.

Let C be a class of groups; a group G is universal for the class C if every group from C can be embedded (as a subgroup) into G. With F, R and G we shall denote the classes of all finitely presented (*fp*) groups, recursively presented (*rp*), and finitely generated (*fg*) groups, respectively. In [10] G. Higman proved that there exist universal *fp* groups for the class F. From Higman's remarkable theorem that an *fg* group H is a subgroup of an *fp* group G if and only if H is *rp*, it immediately follows that every group U universal for the class F of groups is also universal for the class R. In what follows, for an *fg* group G which is universal for F, we will just say that G is universal.

Let P be an algebraic (i.e. preserved under isomorphisms) property of *fg* groups. For the property P we shall say that it is strong hereditary (*SH*) if

(i) P is hereditary,

(ii) both P and res P are proper properties on F.

Here, "hereditary" means that whenever group G has some property P, every *fg* subgroup H of G has P also. (In what follows we do not need more special case when every subgroup H of G has P.) Property "residually P" is denoted with res P, where

$$\text{res}\, P(G) \leftrightarrow \bigcap_{N} \{N \lhd G \mid P(G/N)\} = \{1\},$$

or equivalently, G is res P if for every nontrivial element $g \in G$ there exists a normal subgroup $N_g \lhd G$ such that $g \notin N_g$ and $P(G/N_g)$.

For a property P we say that it is proper on some class C of groups if $P \neq \{1\}$ and $P \neq$ C.

From the above definition, it follows that $P \subseteq \text{res}\, P$ for every property P of *fg* groups. For some P, $P = \text{res}\, P$ (e.g. for P "being Abelian"), but in general $P \neq \text{res}\, P$ (e.g. for P "being finite"). Note that P can be proper on C, but res P can contain the whole class C. To illustrate that we shall use the notion of subdirect decomposability.

A group G is a subdirect product (*sdp*) of groups $G_i (i \in I)$ if G is a subgroup of the direct product $\prod_i G_i$ such that for the i-th projection

$$\pi_i : \prod_i G_i \to G \ (i \in I) \text{ the following is true: } \pi_i(G) = G_i.$$

A group G is subdirect decomposable (*sdd*) if it contains a family $\{N_i \mid i \in I\}$ of non-trival normal subgroup N_i such that $\bigcap_i N_i = \{1\}$. If such a family doesn't exist for a group G, then G is subdirect indecomposable (*sdi*).

For the following statements, see e.g. [8].

Theorem 1: *A group G enjoys* res P *if and only if G is a subdirect product of groups possesing the property P.*

Theorem 2 (Birkhoph): *Every group is a subdirect product of subdirect indecomposable groups.*

More generally, the above theorem is true for arbitrary algebras.

As an example of a property P whose res P is not proper, we can take now the property "to be subdirect indecomposable". A group G is res P if and only if G is a subdirect product of groups with P. By theorem of Birkhoph, it means that every group is res P.

In what follows we shall consider hereditary properties P such that superproperties res P do not contain the whole class F of finitely presented groups, i.e. strong hereditary properties. The class of strong hereditary properties contains some very important and well explored algebraic properties of groups, such as: being solvable, nilpotent, torsion-free, one-relator, *rp* simple, ets. (for more examples and for a Lemma giving a sufficient condition for the property to be strong hereditary, see [7]). Specially, Ch. Miller proved in [13] that the property "having solvable word problem" is, using our terminology, strong hereditary.

2. Statements and proofs of results

The notion of a universal group turned out to be very important in some aspects of the theory of *fp* groups. For example, it enabled a new formulation of the notion of Markov property (see [3]). Also, some of the properties of universal groups give new and effective criteriums for checking algorithmical recognizability of properties of *fp* groups (see [4], [5], and [6]). Similarly, the following Lemma about universal groups has several interesting consequences, and it represents an important step in proving the results given bellow.

Lemma: *If P is an SH property, then there exists and fp group U_P universal in* F, *such that every nontrivial factor group of U_P enjoys NOT P.*

Proof: Let U be a universal *fp* group which has a non-trivial factor group enjoying the property P. (For example, let $U = U_1 \times G_1$, where U_1 is an arbitrary universal group and G_1 is an *fp* group enjoying the proper property P.) Let $N_P = \bigcap_N \{N \lhd U \mid P(U/N)\}$. Since U has a normal subgroup whose corresponding factor group has P, N_P is not empty. Further, $N_P \ncong \{1\}$, since res P is proper in F and none of universal groups can enjoy res P. That means that there exists $v \in N_P$, $v \neq 1$. Let us use now the well-known construction of Rabin (see [16]), which effectively maps every pair (Π, w), where II is a presentation and w a word from that presentation, onto another presentation Π_w such that:

$$\text{if } \vdash_{\Pi} w = 1 \quad \text{then } G_{\Pi_\infty} \cong \{1\}, \text{ and}$$

$$\text{if not } \vdash_{\Pi} w = 1 \quad \text{then } G_{\Pi} < G_{\Pi_\infty}.$$

It is easily seen that construction is such that $C_{\Pi_\infty}/N \cong \{1\}$ if $w \in N$. We shall apply this construction to a presentation Π_U of the above group U and a word from Π_U representing the above element v, which we also denote by v for simplicity. Let $\Pi(v)$ be the presentation obtained by the Rabin's construction, and let $U(v)$ be a group determined by this presentation. Since $v \neq 1$, it means that $U < U(v)$, and hence $U(v)$ is a universal group also. We shall prove now that $U(v)$ has no nontrivial factor group with property P.

Assume, on the contrary, that $H \lhd U(v)$ such that $P(U(v)/H)$. Since H is proper, according to the construction, $v \notin H$. Let $K = U \cap H$. As $HU/H < U(v)/H$, and P is hereditary, it means that $P(HU/H)$, and hence $P(U/K)$. It follows that $N_P < K$, i.e. $v \in K$, what gives $v \in H$, contrary to the above that $v \notin H$. So, $U(v)/H$ cannot enjoy P. ■

This Lemma was used in [7] to prove that for every *SH* property P, there exists a finitely axiomatizable theory $T(P)$ having no models with property P. Here, we shall use it to discuss some features of universal groups themselves, as well as to prove algorithmical undecidability of some properties of *fp* groups.

Theorem 1. *Let P be an arbitrary strong hereditary property of groups. If N is an fg normal subgroup of an fp group U such that $P(U/N)$, then U is universal if and only if N is universal.*

Proof. If subgroup N is universal, then U is evidently universal. Suppose now that U is universal *fp* group and that N is an *fg* normal subgroup of U such that U/N has strong hereditary property P. Using the above Lemma, there exists universal *fp* group V_P such that none of its factor groups has P. Since U is universal, there exists subgroup U_P in U such that $U_P \cong V_P$. If $U_P \cap N = M$, then U_P/M is isomorphic to a subroup of the group U/N. As P is a hereditary property, then $P(U_P/M)$. But, U_P has no nontrivial factor group with P, meaning that U_P/M can only be the trivial group, i.e. $U_P = M$. Hence, $U_P < N$, so that N is a universal group. ∎

It is easy now to prove the analog of Neumann-Hall theorem for universal groups.

Corrolary 1. *If H is a subgroup of finite index in an fp group U, then H is universal if and only if U is universal group.*

Proof. If U is *fp* universal group and if $|U:H| < \infty$, then H is also an *fp* group (see i.e. [12]). If U_k is universal group which is subgroup in U and isomorphic to the group whose all factor groups are infinite, then $U_k \cap H$ would have finite index in U_k; so $U_k < H$.

3. Applications

The above results may be useful in the following two considerations.

A. Construction of universal *fp* groups

The problem of explicit construction of a universal *fp* group, whose existence was proved by Higman in 1961, was posed in 1969 [11] by Greendlinger. Since every countable group can be embedded into a group with two generators, and furthermore, every *fp* group with m relations can be embedded into two-generator group with m relations, only number of relations is relevant. In 1973, M. K. Valiev [18] constructed an *fp* universal group with 42 relations. Soon, in 1974, W. W. Boone and D. J. Collins [2] constructed such a group with 26 relations. M. K. Valiev again [19] in 1977 improved the number of relations to the 21, etc. Now, the problem is to find an example of an universal *fp* group with as small number of defining relations as possible. Of course, that number must be greater than or equal to 2, since there are no one-relator universal *fp* groups (every one-relator group has solvable word problem).

Theorem I and its corrolaries give new possibilities of construction of universal *fp* groups. It follows that for an *fp* universal group U, the following of its subgroups are universal as well:

- the commutant K, as well as every *fg* subgroup which contains K,
- every subgroup of finite index,
- every normal *fg* subgroup N (if such a subgroup exists) such that U/N has any of the following properties: being solvable, nilpotent, torsion free, *rp* simple, one relator, with solvable word problem, free product of groups with P, direct product of groups with P (where $P \subseteq R$, and R is "solvable word problem"), etc.

So, starting with known examples of universal groups, one could find finite presentations of the corresponding subgroups. There are already some elaborated methods and computer programs for finding such presentations, and it could be useful to develop some other ones also. Considerations in this direction are in progress.

B. Algorithmic unsolvability of some problems of *fp* groups. Algebraic property P of *fp* groups is algorithmic unrecognizable if there is no algorithm which for arbitrary finite presentation Π answers whether group G_Π defined by Π has property P or not.

Corrolary 2: *Property C of groups, to have "context-free word problem" is unrecognizable.*

Proof: Let G_Π be an *fp* group determined by the finite presentation $\Pi=\langle a_1,\dots,a_n;\ R_1,\dots,R_m\rangle$ and let

$$S(G_\Pi)=\{w\in\Pi \mid \underset{\Pi}{\vdash} w=1\},$$

i.e. $S(G_\Pi)$ is the set of all the words w in alphabet $\{a_1,\dots,a_n,a_1^{-1},\dots,a_n^{-1}\}$ for which $w=1$ is consequence of relations $R_1=1,\dots,R_m=1$. If set $S(G_\Pi)$ is the contex-free language (in the sence of formal language theory), and if G_Π is "accessible group" (see [14] and [17]), then group G_Π has context-free word problem. Using the result in D. E. Muller, P. E. Schupp [14], group G has the context-free word problem if and only if G has a free subgroup of finite index. So, if U is any universal *fp* group, it follows from Corrolary 1 that U has no free subgroups of finite index (since each of its subgroups of finite index is universal). In other words, none of the universal *fp* groups has property C. Using [3], it means that C is a Markov property. (Nontrivial algebraic property P of *fp* groups is a Markov property if there exists a group K which cannot be embedded in any group having P.) Now, using results of S. I. Adyan [1] and M. O. Rabin [16], it follows that C is unrecognizable. ■

Among algebraic properties of groups, a class of poly properties has been widely studied. Some questions about connection of this class with the class of Markov properties of groups, and more general, about connections with the class of unrecognizable properties, can be answered now.

Property P is poly-property, in notation $\mathsf{P}(P)$, if

$$\mathsf{P}(P)\longleftrightarrow(\forall G)(\forall N\lhd G)(P(N)\,\&\,P(G/N)\Rightarrow P(G)).$$

For example being "universal group" is poly-property, as well as properties being "finitely generated", "finitely presented", "solvable" etc.

Group G has Poly P property if G can be obtained from trivial group $\{1\}$ by finitely many extensions with groups having P, i.e.

$$\text{Poly}\,P(G)\longleftrightarrow(\exists\, n\in N)(\exists G_1,\dots,G_n)(G=G_0\lhd G_1\lhd\dots\lhd G_n=\{1\})\,\&$$
$$((\forall\, i<n)\,P(G_i/G_{i+1})).$$

Many among Poly-properties are Markov properties. For example, being "poly-Abelian" (i.e. solvable), then being "poly-cyclic" (this is subproperty of property to be "residually-finite"), being "poly-nilpotent", etc. But, there are poly-properties which are not Markov properties. For example, being "universal" group is not a Markov property.

So, the problem considered is: which among Poly-properties are Markov properties? The following theorem gives the answer for one subclass of the class of all Poly-properties.

Theorem 2. *If P is an SH property of fp groups, then Poly P is a Markov property, and therefore algorithmically unrecognizable.*

Proof. Suppose, on the contrary, that Poly P is not Markov. Then, there is an universal *fp* group U such that Poly $P(U)$. It means that there exists a sequence of subgroups

$$U=G_0\lhd G_1\lhd\dots\lhd G_{n-1}\lhd G_n=\{1\}$$

such that $G_i \lhd G_{i-1}$ and $P(G_{i-1}/G_i)$, $i=1,\ldots,n$. Since P is a property of *fp* groups, and being *fp* group is Poly-property, it means that G_i is *fp* group for every $i=1,\ldots,n$. So, we have: $P(U/G_1)$, $P(G_1/G_2),\ldots,P(G_{n-1})$. Since P is an SH property, because of Theorem 1, from $G_1 \lhd U$ and $P(U/G_1)$ it follows that G_1 is universal group. Similarly, from $G_2 \lhd G_1$ and $P(G_1/G_2)$ we can conclude that G_2 is universal also, etc. Finally, G_{n-2} is universal, $G_{n-1} \lhd G_{n-2}$ and $P(G_{n-2}/G_{n-1})$, which means that G_{n-1} is universal also. But, this is not possible, because $P(G_{n-1})$. So, none of universal groups has Poly$P$, i.e. Poly$P$ is a Markov property. ■

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