Mathematica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

Mathematica Balkanica

New Series Vol. 5, 1991, Fasc. 2

Transformation of a One-Dimensional Diffusion Process into a Wiener Process by Means of Random Change of Phase Space

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The article reports the conditions, under which a certain function ψ of the diffusion process $x(t)$ presents a Wiener process. The function ψ is random due to another diffusion process $y(t)$. Random change of time is also applied.

1. Formulation of the problem and main notation

Let n and m be fixed natural numbers, $r=n+m$, $T=\text{const}>0$, $J=[0, T]$, R_k^x is a k-dimensional Euclidean space or a certain one-connection domain of it with sufficiently smooth boundary, the points of R_k^x are designated as $x=(x^1,\dots,x^k)'$ with prime denoting transposition of a vector or matrix. We use (Ω, Σ, P) to denote a complete probability space, $F=\{F_t, t\in J\}$ is the σ-algebras flow in Σ, w_t is a standard separable Wiener process in the phase space $(R^r, \mathscr{B})$ adapted to the flow F, $R=(-\infty, \infty)$, $w_t=w_t(\omega)=w(t, \omega)=(w_t^1,\dots,w_t^r)$, $t\in J$, $\omega\in\Omega$. There are given n- and m-dimensional vector-columns $b^1(t, x)$, $b^2(t, x, y)$ as well as the matrices $\sigma^1(t, x)$, $\sigma^2(t, x, y)$ of $n\times r$ and $m\times r$ dimensions respectively, defined and taking real values when $x\in R_n^x$, $y\in R_m^y$, $t\in J$. The functions b^k, σ^k are such that with any fixed $x_0\in R_n^x$, $y_0\in R_m^y$ the system of Itô stochastic differential equations

$$\begin{aligned} &dx(t)=b^1(t, x(t))\,dt+\sigma^1(t, x(t))dw_t, \quad x(0)=x_0,\\ (1)\qquad &dy(t)=b^2(t, x(t), y(t))\,dt+\sigma^2(t, x(t), y(t))\,dw_t, \quad y(0)=y_0\end{aligned}$$

determines uniquely (up to stochastic equivalence) the strong solutions $x_t=x(t)=x(t, \omega)$, $y_t=y(t)=y(t, \omega)$, which form the diffusion process $z(t)=(x_t', y_t')'$ with values in $R_n^x\times R_m^y$. For the given continuous function $v(x, y)$ with values in $[\varepsilon, C]$, $x\in R_n^x$, $y\in R_m^y$, where $0<\varepsilon=\text{const}\leqq C=\text{const}<\infty$, we shall consider the random process $\tau(t)=\int_{[0, t]} v(x_s, y_s)\,ds$, denoting by $\zeta(\tau)$ such a function for which with $\tau=\tau_t$,

$$(2)\qquad t=\zeta(\tau)=\zeta(\tau, \omega), \quad \tau(\zeta_s)=s, \quad \zeta(0)=0$$

assuming henceforth that $T=\infty$, $J=R_+$. Suppose that

(3) $\bar{x}_\tau = x(\zeta(\tau)),\ \bar{y}_\tau = y(\zeta(\tau)),\ \bar{F}_\tau = F_{\zeta(\tau)},\ \bar{F} = \{\bar{F}_\tau,\ \tau \in J\},$

and $C(k, n)$ is the set of vector real-valued functions on $J \times R_n^x \times R_m^y$ defined and continuous together with k-order derivatives with respect to $t \in J$ and n-order derivatives with respect to x^i, y^j, $i = 1,\dots,n$, $j = 1,\dots,m$. The problem is to construct the function $\psi(t, x, y) \in C(1, 2)$ such that $\bar{\eta}(\tau) = \psi(\zeta(\tau), \bar{x}_\tau, \bar{y}_\tau)$ would be a n-dimensional Wiener process. Note, that the possibility of transforming a one-dimensional diffusion process into a Wiener process by using the absolutely continuous change of measure and random change of time is proved in [7]. The method of Wienerization of diffusion processes with the help of the non-random functions $v = v(t)$ and $\psi(t, x)$ is developed in [1]-[3], [5], [8]-[10]. The present article tackles the problem of Wienerization of the diffusion process $x(t)$ by using a random change of time. The problem is formulated for a multidimensional process, while the most exhaustive solution is offered so far only for $n = 1$.

In what follows we introduce the following notations:

$$\partial_t = \frac{\partial}{\partial t},\ \partial_i^x = \frac{\partial}{\partial x^i},\ \partial_k^y = \frac{\partial}{\partial y^k},\ z^i = x^i,\ z^{n+k} = y^k,$$

$$\partial_j^z = \frac{\partial}{\partial z^j},\ \partial_{i_1 i_2}^x = \partial_{i_1}^x \partial_{i_2}^x,\ \partial_{k_1 k_2}^y = \partial_{k_1}^y \partial_{k_2}^y,\ \partial_{jl}^z = \partial_j^z \partial_l^z,$$

$$z = (z^1, \dots, z^r),\ z_t = \begin{pmatrix} x_t \\ y_t \end{pmatrix},\ \bar{z}_\tau = \begin{pmatrix} \bar{x}_\tau \\ \bar{y}_\tau \end{pmatrix},$$

$$b(t, z) = \begin{pmatrix} b^1(t, x) \\ b^2(t, x, y) \end{pmatrix},\ \sigma(t, z) = \begin{pmatrix} \sigma^1(t, x) \\ \sigma^2(t, x, y) \end{pmatrix},$$

$$i,\ i_1,\ i_2 = 1,\dots,n,\ k,\ k_1,\ k_2 = 1,\dots,m,\ j,\ l = 1,\dots,r.$$

As a diffusion process, z_t is described by the Itô equation

(4) $$\partial z(t) = b(t, z(t))\,dt + \sigma(t, z(t))\,dw_t,\ z(0) = z_0,$$

where $z_0 = (x_0', y_0')'$. If we introduce the vector-column $\partial_z = (\partial_1^z, \dots, \partial_r^z)'$ and the matrix $a = 0{,}5\sigma\sigma'$, then

(5) $$A(t, z) = b'(t, z)\partial_z + [a(t, z)\partial_z]'\partial_z$$

will be the generating differential operator of the process $z(t)$.

Let $P_{s,z}(t, \Gamma)$ be a transition function for the process $z(t)$, E, $\mathsf{E}_{s,z}$ denote the expectation. If $\bar{z}_\tau$ here is a diffusion process, then its transition function $\bar{P}_{u,z}(\tau, \Gamma)$ and transfer vector $\bar{b}(\tau, z)$ are

$$\bar{P}_{s,z}(\tau, \Gamma) = \mathsf{E}[\mathsf{E}_{z,\zeta_u} P_{\zeta_u, z}(\zeta_\tau, \Gamma)],$$

$$\bar{b}(\tau, z) = \lim_{\Delta \downarrow 0} \mathsf{E}\,\mathsf{E}_{z,\zeta_\tau}\Big[\frac{1}{\Delta} \int_{|z'-z|<\delta} (z'-z) P_{\zeta_\tau, z}(\zeta_{\tau+\Delta}, \partial z')\Big];$$

a similar formula may be derived for the diffusion matrix $\bar{a}(\tau, z)$. If z_t is a time-homogeneous process, then the generating differential operators $\bar{\mathscr{A}}(z)$ of the process $\bar{z}_\tau$ and $\mathscr{A}(z)$ of the process z_t are connected by the relation

(6) $$\bar{\mathscr{A}}(z) = [v(z)]^{-1} \mathscr{A}(z).$$

2. Wienerization of a multidimensional process

We shall study the conditions of Wienerization of x_t $(v=1)$, having introduced additional notations:

$$\alpha=(\alpha^1,\dots,\alpha^r),\ \alpha=b'-0{,}5(\sigma'\partial_z)'\sigma',\ u=(u_i^k),$$

u is a matrix consisting of n rows and r columns,

$$i=1,\dots,r,\ k=1,\dots,n,\ \alpha^{*k}=(\alpha^{*1},\dots,\alpha^{*n}),\ I=(\delta_{kl})_1^n,$$

$$\delta_{kl}=0\,(k\neq l),\ 1\,(k=l),\ \alpha^{*k}=-\frac{1}{2}\sum_{1\leq i\leq r}\sigma_i^l\partial_l^z u_i^k,$$

identical Latin subscript and superscript of a monomial denote summing up within the range from 1 to r. The task is to construct a function $\psi(t,x,y)\in C(1,2)$, $\psi=(\psi^1,\dots,\psi^n)'$ such that

$$(7)\qquad [\partial_t+\mathscr{A}(t,z)]\psi=0,\ (\partial_z\psi')'a\partial_z\psi'=0{,}5I$$

([11], p. 298) and that at any fixed $t\in J$, $y\in R_m^y$ the system of equations $\eta^k=\psi^k(t,x,y)$, $k=1,\dots,n$, have the unique solution for x^i as

$$(8)\qquad x^i=\psi^i(t,\eta,y),\ i=1,\dots,n,\ \eta\in R^n,\ x\in R_n^x.$$

The conditions (7), (8) provide a possibility of transforming the diffusion process x_t into the Wiener process $\eta(t)$, and conversely, expressing x_t through the Wiener process η_t:

$$(9)\qquad \eta_t=\psi(t,x_t,y_t),\ x_t=\psi_1(t,\eta_t,y_t).$$

Accordingly ψ and ψ_1 may be considered as reciprocally inverse functions for any fixed $(t,y_t)\in J\times R_m^y$.

Henceforth we assume that everywhere $b\in C(0,1)$, $\sigma\in C(1,2)$, $|\sigma|\neq 0$, $\tilde\sigma=\sigma^{-1}$, $\tilde\sigma=(\tilde\sigma_i^k)_1^r$, $\mathscr{L}$ is a piecewise smooth curve in $J\times R_n^x\times R_m^y$ joining a fixed point $(0,x_0,y_0)$ to a variable point (t,x,y), with i being the number of a column and k – the number of a row of the matrix $\tilde\sigma$. The following Lemma is valid.

Lemma 1. *For the existence of a function $\psi\in C(1,2)$ realizing a random Wienerization of the process x_t it is necessary and sufficient that a certain matrix $u\in C(1,2)$ satisfy the system of differential equations*

$$(10)\qquad \partial_l^z(\tilde\sigma_k^i u_i^j)=\partial_k^z(\tilde\sigma_l^i u_i^j),\ \partial_t(\tilde\sigma'u')=\partial_z(\alpha^*-\alpha\tilde\sigma'u'),$$

with $uu'=I$, and that for any fixed $t\in J$, $y\in R_m^y$ the function

$$(11)\qquad \psi(t,z)=\int_{\mathscr{L}}(\alpha^*-\alpha\tilde\sigma'u')'\,dt+u\tilde\sigma\,dz$$

should map R_n^x in a one-to-one fashion into R^n (Condition A).

Proof. Without proving sufficiency we shall outline a scheme for proving necessity. If (7) and (8) hold, we, having introduced the notation $u=(\partial_z\psi')'\sigma$, obtain the system of equation for u and ψ:

$$(12)\qquad \partial_t\psi+\alpha^k\partial_k\psi=\alpha^*,\ (\partial_z\psi)'.\sigma=u,\ |u|=1.$$

Solving this system with respect to the derivatives gives

$$\partial_z \psi' = \tilde{\sigma}' u', \quad \partial_t \psi' = \alpha^* - \alpha \tilde{\sigma}' u'. \tag{13}$$

Forming the consistency conditions for this system, we get (10).

In what follows we introduce the assumptions: suppose a diffusion process $Q(t)$ adapted to the flow F admits stochastic Itô differential

$$\mathrm{d}\, Q(t) = Q_1\, \mathrm{d}t + Q_2\, \mathrm{d}w(t), \quad Q(0) = 0, \tag{14}$$

where $Q_1 = Q(t, Q)$ is an n-dimensional vector and $Q_2 = Q_2(t, Q)$ is a matrix of dimension $n \times r$. If under these conditions $Q_1 \equiv 0$, $Q_3 = Q_2 Q_2' = I$, then $Q(t)$ is a Wiener process. Moreover, if

$$Q_2 = (IO), \tag{15}$$

where O is the $n \times m$-matrix of zero elements, then $Q(t)$ is called a pure Wiener process. The latter is stochastically equivalent to the process $w(t)$. The processes of Wiener type which are not equivalent to any pure Wiener process are called mixtures (or mixed Wiener processes). It should be noted that with $\alpha^* = 0$ from (11) we get the following formula.

$$\psi(t, z) = \int_{\mathscr{L}} (-u\tilde{\sigma}\alpha')\, \mathrm{d}t + u\tilde{\sigma}\, \mathrm{d}z. \tag{16}$$

Corollary 1. *For the random pure Wienerization of the process x_t with the help of the function $\psi \in C(1, 2)$ it is necessary and sufficient that with $i = 1, \dots, n$, k, $l = 1, \dots, r$*

$$\partial_k \tilde{\sigma}^i_l - \partial_l \tilde{\sigma}^i_k = 0, \quad \partial_t \tilde{\sigma}^i_k + \partial_k (\alpha^j \tilde{\sigma}^i_j) = 0, \tag{17}$$

and that function $\psi = (\psi^1, \dots, \psi^n)'$ along with

$$\psi^i(t, x, y) = \int_{\mathscr{L}} (-\alpha^k \tilde{\sigma}^i_k)\, \mathrm{d}t + \tilde{\sigma}^i_k\, \mathrm{d}z^k \tag{18}$$

satisfy condition A, $i = 1, \dots, n$.

The proof is based on (11).

3. Lemmas on random Wienerization of a one-dimensional process

Suppose that $n = m = 1$, $r = 2$, $R_1(t)$, $R_2(t, x)$ are intervals in R, where $t \in J$, $x \in R_1(t)$, (R_i, B^i) is the phase space of the diffusion process x_t, $i = 1$, and y_t, $i = 2$, respectively. Assuming that the processes x_t, y_t are not time-homogeneous, we introduce the following notations:

$$z_0 = (x_0, y_0)', \ \partial_1 = \partial/\partial x, \ \partial_2 = \partial/\partial y, \ z = (x, y)',$$

$$z^1 = x, \ z^2 = y, \ \partial_z = (\partial_1, \partial_2)', \ z_t = (x_t, y_t)',$$

$$w_t = (w_t^1, w_t^2), \ v(z) = v(x, y),$$

$$b(t, z) = \begin{pmatrix} b^1(t, x) \\ b^2(t, x, y) \end{pmatrix}, \quad \sigma = \sigma(t, z) = \begin{pmatrix} \sigma_1^1(t, x)\, \sigma_2^1(t, x) \\ \sigma_1^2(t, x, y)\, \sigma_2^2(t, x, y) \end{pmatrix},$$

$$a=\frac{1}{2}\sigma\sigma', \ |\sigma|=\sigma_1^1\sigma_2^2-\sigma_1^2\sigma_2^1 \neq 0, \ \bar{z}_\tau=(\bar{x}_\tau, \ \bar{y}_\tau)',$$

$$\sigma_k=(\sigma_k^1, \ \sigma_k^2), \ \sigma^k=(\sigma_1^k, \ \sigma_2^k), \ \alpha=(\alpha^1, \ \alpha^2),$$

$$\alpha^k=b^k-\frac{1}{2}(\sigma_1^i\,\partial_i\sigma_1^k+\sigma_2^i\,\partial_i\sigma_2^k), \ k=1, \ 2.$$

The Itô equations for the processes x_t, y_t, z_t are

$$\begin{aligned} &dx(t)=b^1(t, \ x(t))\,dt+\sigma^1(t, \ x(t))\,dw_t, \ x(0)=x_0 \\ (19)\qquad &dy(t)=b^2(t, \ x(t), \ y(t))\,dt+\sigma^2(t, \ x(y), \ y(t))\,dw_t, \ y(0)=y_0, \\ &dz(t)=b(t, \ z(t))\,dt+\sigma(t, \ z(t))\,dw_t, \ z(0)=z_0. \end{aligned}$$

Lemma 2. *For the random Wienerization of the one-dimensional diffusion process x_t with the help of the function $v(z)\in C(2)$ and that of the process y_t, defined by the system of equations (19), it is necessary and sufficient that there exists a vector $u=(u_1, \ u_2)$ such that*

$$(20)\qquad \partial_1(\tilde{\sigma}_2^i u_i)=\partial_2(\tilde{\sigma}_1^i u_i), \ \partial_z(\alpha^*-\alpha\tilde{\sigma}'u')=\partial_t(\tilde{\sigma}'u'), \ |u|=\sqrt{v},$$

and that the function (11) *satisfies condition A. Thus* $\tilde{\sigma}=\sigma^{-1}$, $\tilde{\sigma}=(\tilde{\sigma}_i^k)_1^2$, $\alpha^*=-\frac{1}{2}\sum_{1\leq k\leq 2}\sigma_k^j\partial_j u_k$.

Subsequently we consider vector b and matrix σ in (19) as not depending on time. Introduce the notations:

$$B=(B^1, \ B^2), \ C=(C^1, \ C^2), \ B=2\alpha(\sigma')^{-1},$$

$$C^k=(-1)^{k-1}|\sigma|\,\partial_z'(|\sigma|^{-1}\sigma_{3-k}), \ k=1, \ 2.$$

Lemma 3. *For the Wienerization of $x(t)$ by means of the function $\psi\in C(1, 2)$ linearly depending on time, when $\mathscr{A}(t, z)=\mathscr{A}(z)$, $v=1$, it is necessary and sufficient that there exist a constant $q\in R$ and a vector $u(z)=(u_1(z), u_2(z))\in C(2)$, $|u|=1$ such that*

$$(21)\qquad \begin{aligned} &q+Bu'+(\sigma'\partial_z)'u'=0, \\ &\sigma_2^k\partial_k u_1-\sigma_1^k\partial_k u_2+Cu'=0. \end{aligned}$$

If this condition holds, the function ψ takes the form

$$(22)\qquad \psi(t, \ x, \ y)=\int_{\mathscr{L}}(q/2)\,dt+(u\sigma^{-1})\,dz,$$

provided that it satisfies A (the latter is implied in what follows).

Example 1. Let $|\sigma|\neq 0$, $\sigma_2^2\neq 0$, $B^1=C^1=0$. Then conditions (21) hold when $u_1=1$, $u_2=0$, $q=0$. By the formula (22) we obtain

$$(23)\qquad \psi(x, \ y)=\int_{\mathscr{L}}|\sigma|^{-1}(\sigma_2^2\,dx-\sigma_2^1\,dy),$$

where $\mathscr{L}$ is a piecewise smooth curve in $R_1^x\times R_1^y$ joining a point (x_0, y_0) to a point

(x, y). The condition $B^1 = C^1 = 0$ is necessary and sufficient for the pure Wienerization of the process x_t by means of the function ψ, with $\partial_t \psi = 0$.

Subsequently we regard x_t as a one-dimensional diffusion process defined by the Itô equation

$$dx(t) = b^1(x(t))\,dt + \sigma^1(x(t))\,dw_t, \quad x(0) = x_0. \tag{24}$$

It is necessary to transform x_t into the Wiener process $\bar{\eta}_\tau$, if v is a preassigned function, or to transform x_t into a pure Wiener process. These are the problems to be solved in the sequel.

4. Effective methods of random Wienerization

Together with (24) we regard as defined the equation

$$dy(t) = b^2(x(t), y(t))\,dt + \sigma^2(x(t), y(t))\,dw_t, \quad y(0) = y_0, \tag{25}$$

with vector b and matrix σ continuously differentiable twice and three times respectively, $b \in C(2)$, $\sigma \in C(3)$, $|\sigma| \neq 0$.
Let

$$d_{k0} = (-1)^{k-1}(\sigma_1^{3-k} C^q - \sigma_2^{3-k} B^2), \quad q = \text{const} \in R,$$

$$d_{kr} = (-1)^k [\sigma_{3-r}^{3-k}(B^1 + C^2) + (-1)^{r-1} \sigma_r^{3-k}(B^2 - C^1),$$

$$A_k(q, \gamma) = d_{k0} + (-1)^k (\sigma_1^{3-k} \gamma + \sigma_2^{3-k} \sqrt{1-\gamma^2})q + d_{k1} \gamma \sqrt{1-\gamma^2} + d_{k2} \gamma^2,$$

$$e_0 = \partial_2 d_{10} - \partial_1 d_{20} + B^2(B^2 - C^1) + C^2(B^1 + C^2),$$

$$e_k = -(\sigma_k^i \partial_i \ln |\sigma| + 2B^k), \quad k, r = 1, 2,$$

$$e_3 = \partial_2 d_{11} - \partial_1 d_{21} + 2(B^1 B^2 + C^1 C^2),$$

$$e_4 = \partial_2 d_{12} - \partial_1 d_{22} + (B^1)^2 - (B^2)^2 + (C^1)^2 - (C^2)^2.$$

Theorem 1. *For the random mixed Wienerization of the diffusion process x_t without a change of time ($v \equiv 1$) it is necessary and sufficient that for some q the equation*

$$e_4 \gamma^2 + e_3 \gamma \sqrt{1-\gamma^2} + q(e_1 \gamma + e_2 \sqrt{1-\gamma^2}) + e_0 + q^2 = 0 \tag{26}$$

has the solution $\gamma \in C(2)$, $|\gamma| \leqq 1$, $\sqrt{1-\gamma^2} \in C(2)$,

$$|\sigma|\, \partial_k \gamma = \sqrt{1-\gamma^2}\, A_k(q, \gamma), \quad k = 1, 2. \tag{27}$$

Under this condition ψ is given by (22), with $u_1 = \gamma$.

Proof. Starting from Corollary 2 we restrict the class of the functions ψ by functions which are linear in time. To prove the necessary, we assume here that the vector $u = (u_1, u_2)$ satisfies the system (21), $|u| = 1$. With $u_1 = \gamma$ and taking into account the equalities

$$u_2 = \sqrt{1-\gamma^2}, \quad (\sigma' \partial_z)' u' = [\sigma_1^k - \gamma(1-\gamma^2)^{-0.5} \sigma_2^k] \partial_k \gamma,$$

the set (21) may be presented as

$$(28)\qquad \begin{aligned}&[\sigma_1^k-\gamma(1-\gamma^2)^{-0,5}\sigma_2^k]\partial_k\gamma+Bu'+q=0,\\ &[\sigma_2^k+\gamma(1-\gamma^2)^{-0,5}\sigma_1^k]\partial_k\gamma+Cu'=0.\end{aligned}$$

This system, after being solved with respect to the derivatives, takes the form (27). The first equation in (27) is differentiated with respect to y, the second one — to x, then the obtained results are subtracted term by term. After elementary transformations we obtain

$$\sqrt{1-\gamma^2}\,[\mathscr{A}_2(q,\gamma)\mathscr{A}'_1(q,\gamma)-\mathscr{A}_1(q,\gamma)\mathscr{A}'_2(q,\gamma)]$$
$$+|\sigma|[\partial_2\mathscr{A}_1(q,\gamma)-\partial_1\mathscr{A}_2(q,\gamma)]=0.$$

Having classified the terms in the left-hand side of the equality according to the degrees of q, γ, $\sqrt{1-\gamma^2}$, we obtain (26), with prime denoting differentiation with respect to γ.

Sufficiency. Let Eq. (26) have the solution γ (with the necessary properties) satisfying the condition (27). Then (28) and (21) follow from (27). The only thing remaining is to verify condition (A).

In what follows we write out vectors B, C in coordinatewise form

$$B=2|\sigma|^{-1}(\sigma_2^2\alpha^1-\sigma_2^1\alpha^2,\ -\sigma_1^2\alpha^1+\sigma_1^1\alpha^2),$$
$$C^k=(-1)^{k-1}(\partial_i\sigma_{3-k}^i-\sigma_{3-k}^i\partial_i\ln|\sigma|),\quad k=1,2.$$

Example 2. Let $|\sigma|\neq 0$, $|C|\neq 0$, $i,k=1,2$,

$$\partial_i(|C|^{-1}C^k)=\partial_i[|C|^{-1}(B^1C^2-B^2C^1)]=0.$$

Then x_t may be transformed into a Wiener process with $v=1$, $u_1=\text{const}$, $|u_1|\leqq 1$, $u_2=\sqrt{1-u_1^2}$, if we construct ψ according to the formula (22), with $\mathscr{A}_1=\mathscr{A}_2=0$, $k=1,2$,

$$u_k=(-1)^{k-1}|C|^{-1}C^{3-k},\quad q=|C|^{-1}(B^2C^1-B^1C^2).$$

If for $a\neq 0$ we introduce

$$d_k(a,b,c)=(2a)^{-1}[-b+(-1)^k\sqrt{b^2-4ac}\,],\quad k=1,2,$$

then an important corollary is valid.

Corollary 2. *The necessary and sufficient condition for the mixed random Wienerization of the process x_t with the help of the function $\psi\in C(2)$, $\partial_t\psi=0$ is the validity of the inequalities (assuming that e_0, $e_3^2+e_4^2\neq 0$):*

$$(29)\qquad 0\leqq e_3^2-2e_0e_4<2(e_3^2+e_4^2),\quad 0\leqq e_3^2-4e_0(e_0+e_4),$$

and the equality (27) for $\gamma=\mp\sqrt{d_i}$; $\sqrt{d_i}$, $\sqrt{1-d_i}\in C(2)$, where $d_i=d_i(e_3^2+e_4^2,\ 2e_0e_4-e_3^2,\ e_0^2)$. Under this condition the function ψ is found by (22), with $q=0$, $u_1=\mp\sqrt{d_i}$, $u_2=\pm\sqrt{1-d_i}$, i.e.

$$(30)\qquad \psi(z)=\int_{\mathscr{L}}(u\sigma^{-1})\,dz.$$

For a more complicated method of random Wienerization we introduce the following notation:

$$(31)\quad \left.\begin{aligned} &D(q,\ \gamma)=e_4\gamma^2+e_3\gamma\sqrt{1-\gamma^2}+q(e_1\gamma+e_2\sqrt{1-\gamma^2})+e_0+q^2,\\ &\beta_k(q,\ \gamma)=|\sigma|\,\partial_k\gamma-\sqrt{1-\gamma^2}\,A_k(q,\ \gamma),\ k=1,\ 2,\end{aligned}\right\}$$

$$(32)\quad \left.\begin{aligned} &K_0=e_4-e_0+2e_1^{-1}e_2e_3,\ \Delta=K_0(e_1e_4+e_2e_3)^2\\ &-(e_3^2+e_4^2)[K_0(e_1^2+2e_4+e_2^2)+2e_0e_4-e_3^2]+2(e_3^2+e_4^2),\\ &K=(e_3^2+e_4^2)^{-1}[-\sqrt{K_0}(e_1e_4+e_2e_3)\mp\sqrt{\Delta}],\end{aligned}\right\}$$

$$\gamma=0{,}5\,[K-(\operatorname{sing}K)\sqrt{K^2-4}].$$

Theorem 2. *Suppose that (with $e_1\neq 0$, $e_0^2\geqq e_3^2+e_4^2$, $\Delta\geqq 0$) the following conditions are satisfied*

$$2e_0+2\sqrt{e_0^2-(e_3^2+e_4^2)}\leqq e_2^2,\ \partial_kK_0=0,\ \beta_k(\sqrt{K_0},\ \gamma)=0,$$

$$(33)\qquad K_0=\frac{1}{2}[e_2^2-2e_0\mp\sqrt{e_2^4+4(e_3^2+e_4^2+e_0e_2^2)}],$$

$$D(\sqrt{K_0},\ \gamma)=0,\ |K|\geqq 2,\ \gamma\in C(2).$$

Then a mixed Wienerization of the process x_t is possible with $q=\sqrt{K_0}$, $u_1=\gamma$, $v=1$.

P r o o f. From (33) we established that $K_0\geqq 0$, all the radical are non-negative and the functions K, γ are real. Since with $|K|>2$ we have

$$|\gamma|=0{,}5(|K|-\sqrt{K^2-4})\to 0,\ K\to\infty,$$

$$2\partial|\gamma|/\partial|K|=1-|K|\sqrt{(K^2-4)^{-1}}<0,$$

then $\max|\gamma|=1$ (it is achieved when $|K|=2$). If we suppose that $q=\sqrt{K_0}$ then it is easy to see that γ is the real solution of the equation

$$\begin{aligned}&(e_3^2+e_4^2)\gamma^4+2(e_1e_4+e_2e_3)q\gamma^3\\ &+[q^2(e_1^2+2e_4+e_2^2)+2e_0e_4-e_3^2]\gamma^2\\ &+2q(e_1e_4+e_2e_3)\gamma+(e_3^2+e_4^2)=0.\end{aligned}$$

Moreover, $\beta_k(q,\ \gamma)=0$, $D(q,\ \gamma)=0$, therefore, in virtue of Theorem 1 ψ may be constructed by the formula (22) with $u_1=\gamma$.

It should be noted that, if e_1 were equal to 0, then, with $e_0+e_4\leqq 0$ and $\partial_z(e_0+e_4)=0$, the following constant would satisfy Eq. (26): $\gamma=1$, $q=\sqrt{e_0+e_4}$.

5. Important particular cases

Let within the adopted notation (with $k=1,\ 2$)

$$\Delta_k=d_k(1,\ -e_1,\ e_0),\ d_k^*=d_k(1,\ 2e_0-e_1^2-e_2^2,\ e_0^2),$$

$$\gamma_k(q)=d_k((e_1^2+e_2^2)q^2,\ 2qe_1(q^2+e_0),\ (q^2+e_0)^2-(e_2q)^2).$$

Theorem 3. *Let* $e_3=e_4=0$, $e_1^2+e_2^2\neq 0$. *For the mixed random Wienerization of the process* x_t, $v=1$, *it is necessary and sufficient that for some* k, $r=1, 2$

$$(34)\qquad \begin{gathered} e_1^2-4e_0\geqq 0,\ e_1^2+e_2^2-2e_0\geqq 0,\ \partial_t\Delta_r=0,\ i=1,\ 2,\\ \beta_i(\mp\Delta_r,\ \gamma_k(\mp\Delta_r))=0,\ \max_z d_1^*\leqq\Delta_r^2\leqq\min_z d_2^*. \end{gathered}$$

P r o o f. It is important to note that under the given conditions Eq. (26) takes the form

$$(35)\qquad (e_1^2+e_2^2)q^2\gamma^2+2(q^2+e_0)e_1q\gamma+[(q^2+e_0)^2-(e_2q)^2]=0.$$

The discriminant of this equation is non-positive since γ is a real function, i.e.

$$(36)\qquad q^4+(2e_0-e_1^2-e_2^2)q^2+e_0^2\leqq 0.$$

Since the roots of the quadratic trinomial with respect to q^2 in the left-hand side of this inequality are real, its discriminant is non-positive: $e_1^2+e_2^2-4e_0\geqq 0$. It arises from $e_0^2\geqq 0$ that the roots of the trinomial in (36) are not of different signs and that q^2 lies between these roots: $d_1^*\leqq q^2\leqq d_2^*$. Therefore, both roots are non-negative, hence, the factor of q^2 in (36) is non-positive and thus the second inequality in (34) is proved. Moreover, both solution of γ in Eq. (35) are real and for some k we have $\gamma_k\in[-1, 1]$. This means that the following inequality is valid

$$(37)\qquad \begin{gathered} [e_1q(q^2+e_0)-q^2(e_1^2+e_2^2)]^2\leqq(e_2q^2)[q^2(e_1^2+e_2^2)-(e_0+q^2)^2]\\ \leqq[qe_1(q^2+e_0)+q^2(e_1^2+e_2^2)]^2,\\ (q^2-qe_1+e_0)^2\leqq 0\leqq(q^2+qe_1+e_0)^2 \end{gathered}$$

as well as the inequality inverse to it. It arises from (37) that $q=\Delta_r$, $\partial_k\Delta_r=0$, $k=1$, 2. The inequality inverse to (37) will give us $q=d_r(1,\ e_1,\ e_0)$, i.e., in any case, $q^2=(\Delta_r)^2$, thus the last double inequality in (34) is proved. Moreover, d_r and Δ_r are real and, therefore, the first inequality in (34) is also valid.

S u f f i c i e n c y. If the conditions (34) are satisfied, then it follows from the first inequality that Δ_r is real and from $\partial_z\Delta_r=0$ that it is a constant. It arises from the second inequality in (34) that d_k^* are real non-negative functions, $k=1, 2$. If $q^2=\Delta_r^2$ holds, then it follows from the last double inequality in (34) that with the chosen value of q the inequality (36) is valid. That is why both solutions of γ in Eq. (35) are real. As both (37) and the inverse inequality are valid, by performing the operations in an opposite order we establish that $\gamma=\gamma_k(\mp\Delta_r)\in[-1, 1]$.

Finally we shall prove that γ, with the corresponding sign before the square root, satisfies Eq. (26), i.e.

$$(38)\qquad q^2+(e_1\gamma+e_2\sqrt{1-\gamma^2})q+e_0=0.$$

If $q=\Delta_r=0$, then in virtue of (37) $e_0=0$, and therefore, the equality (26) is valid, it is an identity. Then γ is derived from the joint system (27), e.g., $\gamma=1$ is a solution. If $\Delta_r\neq 0$, $q\neq 0$, we have (i, $r=1$, 2, $i\neq r$):

$$\gamma=\gamma_r(q),\ \sqrt{1-\gamma^2}$$

$$=d_i\left(\frac{q}{2}(e_1^2+e_2^2),\ e_2(q^2+e_0),\ [(q^2+e_0)^2-(qe_1)^2](2q)^{-1}\right),$$

$$q^2+q(e_1\gamma+e_2\sqrt{1-\gamma^2})+e_0=\frac{1}{e_1^2+e_2^2}\Big\{q^2(e_1^2+e_2^2)$$

$$-e_1^2(q^2+e_0)\mp e_1e_2[q^2(e_1^2+e_2^2)-(q^2+e_0)^2]^{0,5}-e_2[e_2(q^2+e_0)$$

$$\mp e_1[q^2(e_1^2+e_2^2)-(q^2+e_0)^2]^{0,5}]+e_0(e_1^2+e_2^2)\Big\}=0.$$

Thus Theorem 3 is proved completely.

For the analysis of another special case we introduce the notation (regarding a and b as any real fixed numbers, $a\leqq b$):

$$\rho=[-\infty,\ \infty],\ ]a,b[=\rho-[a,\ b],$$

$$)a,\ b(=\rho-(a,\ b),\ )a,b[\ =\rho-(a,b],$$

$$]a,\ b(=\rho-[a,\ b),\ a,\ b\in\rho,$$

$$D^{kr}=\frac{1}{2}\{(-1)^k e_1+(-1)^r[e_1^2-4(e_0+e_4)]^{0,5}\},$$

$$D_*^{kr}=\inf_z D^{kr},\ D_{**}^{kr}=\sup_z D^{kr},$$

$$e=2[(e_1^2-4e_4)^{-1}e_0e_4]^{0,5},\ e_*=\inf_z e,$$

$$e_{**}=\sup_z e,\ M_1=[D_{**}^{21},\ D_{**}^{22}]\cap)D_*^{11},\ D_*^{12}(,$$

$$M_3=M_1\cap[-e_*^{0,5},\ e_*^{0,5}],\ M_2=[D_{**}^{21},\ D_{**}^{12}]\cap)D_*^{21},\ D_{**}^{22}(,$$

$$M_4=M_2\cap)-e_{**}^{0,5},\ e_{**}^{0,5}(,$$

$$M_5=M_1\cap)-e_{**}^{0,5},\ e_{**}^{0,5}(,\ \gamma_r(q)=d_r(e_4,\ qe_1,\ q^2+e_0),$$

$$\beta_{kr}(q)=\beta_k(q,\ \gamma_r),$$

$$A^1=\{e_1^2-4e_4\geqq 0,\ e_0e_4\leqq 0;$$

if $e_4>0$, then M_1 is nonempty, if $e_4<0$, then M_2 is nonempty; $e_1^2-4e_4\geqq 4e_0\}$, $A^2=\{4e_0\leqq e_1^2-4e_4<0,\ e_4<0,\ M_3$ is nonempty, $A^3=\{e_1^2-4e_4>0,\ e_0e_4>0$; if $e_4<0$, then M_4 is nonempty and if $e_4>0$, then M_5 is nonempty$\}$, $k,\ r=1,2$.

Theorem 4. *Let $e_2=e_3=0$, $e_4\neq 0$. Suppose that only one of the conditions A^i, $i=1, 2, 3$, holds as z varies in $R_1^x\times R_1^y$. For the mixed random Wienerization of the process x_t by means of the function (22) it is necessary and sufficient that $M_1\cup M_2$ (for A^1) or M_3 (for A^2) or $M_4\cup M_5$ (for A^3) contains a point q such that for some fixed $r=1, 2$ and for any $k=1, 2$ the equality $\beta_{kr}(q)=0$ holds. But if $e_1^2-4e_0\leqq 0$, $e_0e_4>0$, then Wienerization is impossible.*

Proof. Suppose that condition A^1, $e_4>0$, holds. Then the discriminant of the equation $e_4\gamma^2+qe_1\gamma+(q^2+e_0)=0$ is non-positive, i.e. for $q\in M_1$ we have

$$\Delta\equiv q^2(e_1^2-4e_4)-4e_0e_4\geqq 0.$$

Therefore, the solution γ of this equation is a real function. From $D^{21}\leqq q\leqq D^{22}$, $q\bar{\in}(D^{11},\ D^{22})$ we obtain the following double inequality:

(38*) $$q^2+qe_1+e_0+e_0 \leqq 0 \leqq q^2-qe_1+e_0+e_4.$$

Since

$$4e_4^2+4e_1e_4q \leqq -4e_4(q^2+e_0) \leqq 4e_4(e_4-e_1q),$$
$$(e_1q+2e_4)^2=(e_1q)^2-4e_4(q^2+e_0) \leqq (e_1q-2e_4)^2,$$
$$e_1q-2e_4 \leqq \mp[q^2(e_1^2-4e_4)-4e_0e_4]^{0,5} \leqq e_1q+2e_4,$$

we conclude that $|\gamma| \leqq 1$. For $e_4<0$, $q \in M_2$ the proof is the same with the inequality inverse to (38*) being applied.

If condition A^2 holds, then using (38*) for $q \in M_3$ we obtain the relation

$$q^2 \leqq e^2=4(e_1^2-4e_4)^{-1}e_0e_4, \quad \Delta \geqq 0, \quad |\gamma| \leqq 1.$$

Both (38*) and the inequality inverse to it are identically used for condition A^3.

6. Random pure Wienerization

Applying Lemma 2, we shall analyse now the case of Wienerization involving a random change of time (not necessarily $v=1$). For the existence of ψ realizing a pure random Wienerization by means of the function v it is necessary and sufficient that v alongside with previous conditions satisfy one more set of equations:

(39) $$\partial_k \ln\sqrt{v}=(-1)^k|\sigma|^{-1}(\sigma_2^{3-k}B^1-\sigma_1^{3-k}C^1+\sqrt{v^{-1}}.q\sigma_2^{3-k}),$$

where $k=1, 2$. The function ψ is derived from the formula

(40) $$\psi(t, x, y)=\int_{\mathscr{L}}(q/2)\,dt+|\sigma|^{-1}\sqrt{v}(\sigma_2^2\,dx-\sigma_2^1\,dy).$$

The equality (39) is derived from the system

$$\tilde\sigma_2^1\partial_1\ln\sqrt{v}-\tilde\sigma_1^1\partial_2\ln\sqrt{v}=\partial_2\tilde\sigma_1^1-\partial_1\tilde\sigma_1^1,$$
$$\sigma_1^1\partial_1\ln\sqrt{v}+\sigma_1^2\partial_2\ln\sqrt{v}=-(q\sqrt{v}+B^1),$$

which arises from the first two equalities in (20).

A very important problem is the following: find conditions on the sufficiently smooth functions α and σ, $|\sigma| \neq 0$, such that there exist the functions $v \in C(2)$ and $\psi \in C(1, 2)$ realizing a pure Wienerization of the process x_t involving the above-mentioned process y_t. For partial solution let $\mathscr{L}^0$ be a piecewise smooth curve in $R_1^x \times R_1^y$ joining a fixed point (x_0, y_0) to a variable point (x, y), and suppose that

$$e(x, y)=\int_{\mathscr{L}^0}|\sigma|^{-1}[(\sigma_1^2C^1-\sigma_2^2B^1)\,dx+(\sigma_2^1B^1-\sigma_1^1C^1)\,dy]$$

is a bounded function, $i=1, 2$, $l_1=\inf_z l(z)$, $l_2=\sup_z l(z)$, $\varepsilon_i=\exp(2l_i)$.

Theorem 5. *For the pure random Wienerization of the process x_t with the help of certain functions v, ψ, $\partial_t\psi=0$ it is necessary and sufficient that*

(41) $$\partial'_z[|\sigma|^{-1}(\sigma_1 C^1-\sigma_2 B^1)]=0.$$

If this identity holds, then

(42) $$v(z)=\exp[2l(z)],\ \psi(z)=\int_{\mathscr{L}^0}|\sigma|^{-1}e^l(\sigma_2^2\,dx-\sigma_2^1\,dy).$$

Proof. The necessity is established from (39) with $q=0$ if we differentiate the first equality with respect to y, the second one with respect to x and equate the obtained results. In order to prove the sufficiency we assume that with the notation already adopted the condition (41) is satisfied. Then the integral $l(z)$ should not depend on the curve, and therefore, this function is single-valued real, and so is $v(z)$ from (42), $0<\varepsilon_1\leqq v(z)\leqq\varepsilon_2<\infty$. As it follows from $\partial_2(\tilde\sigma_1^1\sqrt{v})=\partial_1(\sigma_2^1\sqrt{v})$, the curvilinear integral in the right-hand side of the second equality (42) does not depend on the curve $\mathscr{L}^0$, i.e. it defines the function $\psi(z)$ nonuniquely. The conditions (20) for the vector $u=(\sqrt{v},\ 0)$ are as follows:

(43) $$\sigma_1^k\partial_k\sqrt{v}+B^1\sqrt{v}=0,\ \sigma_2^k\partial_k\sqrt{v}+C^1\sqrt{v}=0.$$

Since $\sqrt{v}=\exp(l)$, $\partial_1\sqrt{v}=\exp(l)|\sigma|^{-1}(\sigma_1^2 C^1-\sigma_2^2 B^1)$, $\partial_2\sqrt{v}=\exp(l)\,|\sigma|^{-1}(\sigma_2^1 B^1-\sigma_1^1 C^1)$, we, substituting these expressions into (43), obtain identities.

From the second equality in (42) we obtain

$$\partial_1\psi=\sqrt{v}|\sigma|^{-1}\sigma_2^2,\ \partial_2\psi=-\sqrt{v}|\sigma|^{-1}\sigma_2^1,\ (\partial'_2\psi)\sigma=u.$$

Consequently the stochastic differential of $\psi(t,\ x_t,\ y_t)$ is

$$\partial\psi(x_t,\ y_t)=(\partial'_2\psi)\sigma dw_t=\sqrt{v}\,dw_t^1.$$

Moreover, if $\psi(x_0,\ y_0)=\eta(0)=0$, then $\eta(t)=\bar\eta(\tau)$ is a Wiener process, Q.E.D. Incidentally, in the case of pure Wienerization we can manage without the mixed Wiener process w_t^* (cf. Lemma 2).

Example 3. Let $R_1^x=R_1^y=R$, $\varphi(x)$ be an absolutely integrable over R continuous function satisfying the Lipschitz condition, and let

$$\sigma_1^2(x)=(x-1)\exp\Big[\int_0^x\varphi(u)\,du\Big],\ \sigma_2^2=1,\ \sigma_1^1(x)=x,$$

$$\sigma_2^1(x)=\exp\Big[-\int_0^x\varphi(u)\,du\Big],\ b^2=0,\ x_0=0,\ y_0=0,$$

$$b^1(x)=-0{,}5\varphi(x)\Big\{x^2+\exp\Big[-2\int_0^x\varphi(u)\,du\Big]\Big\}.$$

It is easy to prove that the system (24) ÷ (25) uniquely, up to stochastic equivalence [4], determines the diffusion process $z(t)=(x(t),\ y(t))'$ such that $P\{z(0)=(0,\ 0)'\}=1$. Analysing the possibility of the pure Wienerization of the diffusion process $x(t)$ with the process $y(t)$ involved, we apply Theorem 5. It is necessary to find α^1, α^2, etc., viz.

$$\alpha^1=-0{,}5x[1+x\varphi(x)],\ \alpha^2=2^{-1}(-x)e^{\int_0^x\varphi(u)\,du}[1+(x-1)\varphi(x)],$$

$$|\sigma|=1,\ C^1=\partial_1\sigma_2^1=-\varphi(x)\exp[-\int_0^x\varphi(u)\,du],$$

$$B^1=2[\alpha^1-\alpha^2\exp(-\int_0^x\varphi(u)\,du)]=-x\varphi(x),$$

$$\sigma_1^1C^1-\sigma_2^1B^1=0,\ \sigma_1^2C^1-\sigma_2^2B^1=\varphi(x),$$

i.e. condition (41) is satisfied. All our previous assumptions hold as well, hence

$$l=l(x)=\int_0^x\varphi(u)\,du,\ v(z)=\exp[2\int_0^x\varphi(u)\,du],$$

$$\psi(x,\ y)=\int_0^x\exp[\int_0^u\varphi(s)\,ds]\,du-y.$$

It is appropriate to mention here that $\partial_x\psi>0$ leads to the existence of a function ψ_1 inverse to ψ such that $\eta=\psi(x,\ y)\leftrightarrow x=\psi_1(\eta,\ y)$. Consequently we find the expression for the process $x(t)$:
$x(t)=\psi_1(\eta(t),\ y(t))$ where $\eta(t)=\bar{w}^1(\tau)$ is the first component of the Wiener process $w(\tau)$.

Also it is interesting to note that for another pure Wienerization, i.e. when $u_1=0,\ u_2=1$ with $\partial_t\psi=0,\ v=1$, it is necessary and sufficient that $B^2=C^2=0$.

7. Method of finding the inverse function

For applying the method of transformation of the diffusion process x_t one must very often know not only ψ but also its inverse function $\psi_1(t,\ \eta,\ y)$ such that for any fixed $t,\ y$ from J and R_1^y respectively and $\eta\in R,\ x\in R_1^x$ we have

(44) $$\eta=\psi(t,\ \psi_1(t,\ \eta,\ y),\ y),\ x=\psi_1(t,\ \psi(t,\ x,\ y),\ y).$$

On the other hand this function ψ_1 can be found by inverting ψ. But it is possible to derive a special system of differential equations for ψ_1 and to find it without knowing ψ. Let us consider some details of the problem, assuming that $v\equiv 1$.

Suppose that η_t is a one-dimensional Wiener process such that $x_t=\psi_1(t,\eta_t,y_t)$. Assuming that $\psi_1\in C(1,\ 2)$,

$$(u_1)^2+(u_2)^2=1,\ \alpha^3=-\frac{1}{2}\sum_{1\leq k\leq 2}\sigma_k^i\partial_iu_k,$$

with $\alpha^1,\ \alpha^2$ already found, we obtain the system of equation

(45) $$\partial_t\psi_1+\alpha^2\partial_y\psi_1+\alpha^3\partial_\eta\psi_1=\alpha^1,\ \sigma_k^2\partial_y\psi_1+u_k\partial_\eta\psi_1=\sigma_k^1,\ k=1,\ 2.$$

It is important to note that this system is consistent if and only if the consistency conditions for the system (10) are satisfied. Thus, in the case of pure Wienerization we have $\eta(t)=w^1(t),\ u_1=1,\ u_2=0,\ \alpha^3=0$. With $\sigma_2^2|\sigma|\neq 0,\ \partial_t\psi_1=0$ we obtain from (45)

(46) $$\partial_2\psi_1=(\sigma_2^2)^{-1}\sigma_2^1,\ \partial_3\psi_1=(\sigma_2^2)^{-1}|\sigma|,$$

where $\partial_3=\partial/\partial\eta$. Let d_y, d_η be the complete partial derivatives vectors with respect to y and η respectively, then the consistency condition for the system (45) takes the form

$$\text{(47)} \qquad d_\eta[(\sigma_2^2)^{-1}\sigma_2^1]=d_y[(\sigma_2^2)^{-1}|\sigma|], \quad \sigma_2^1\alpha=\sigma_2^2\alpha^1.$$

Since $d_\eta=(\partial_\eta\psi_1)\partial_1$, $d_y=\partial_y+(\partial_2\psi_1)\partial_1$, this condition (47) after transformations can easily be reduced to the form $B^1=C^1=0$. If this is observed, then by (46) we find the function $\psi_1(\eta, y)$ such that the diffusion process $x(t)$ is expressed in terms of the Wiener process $w^1(t)$ by the formula

$$\text{(48)} \qquad x(t)=\psi_1(w^1(t),\ y(t)).$$

The above problem is just another way of presenting Example 1. Identically, all the previous results concerning the function ψ can be formulated with reference to ψ_1.

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Received 21. 11. 1990