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On Viable Solutions of Feedback Controlled System

R. P. Ivanov

Presented by P. Kenderov

This paper considers a controlled system with a measurable set of controls. Necessary and sufficient conditions are given for the existence of measurable controls which guarantee the viability of the solutions of the differential system with respect to a given multi-function. Different aspects of the viability problem have been discussed by many authors (see for example [3]—[10]).

In the closed domain $D \subset R \times R^n$, where R is the line and R^n is the Euclidean space we shall consider the differential equation

(1) $$\dot{x} = f(t, x, u), \quad x(t_0) = x_0,$$

where $x \in R^n$, $f \in R^n$, $(t, x) \in D$, $u \in R^p$. We suppose the function $f(t, x, u)$ is defined on $(t, x, u) \in D \times P$, where P is a compact set and the control set is a measurable multifunction $P(t, x) \subset P$, i.e. $u \in P(t, x)$ on the set D. In this paper we assume that $f(t, x, u)$ is a measurable function with respect to t and it is continuous function with respect to (x, u). Finally $\| f(t, x, u) \| \leq m(t)$ on $(t, x) \in D$ and $u \in P$, where $m(t)$ is an integrable function.

For any measurable function $u(t, x) \in P(t, x)$ the solutions of the differential equation (see [1], [2])

(2) $$\dot{x} = f(t, x, u(t, x)), \quad x(t_*) = x_*,$$

are all absolutely continuous functions $x(t)$ which for a.a. t satisfy the differential inclusion

(3) $$\dot{x} \in \operatorname{co} f(t, x, U(t, x)), \quad x(t_*) = x_*,$$

where $\operatorname{co} A$ is the convex hull of the set A and

(4) $$U(t, x) = \{u \in P \mid u = \operatorname*{ess\,lim}_{y \to x} u(t, y)\}.$$

It means that for any set $N \subset R^n$ with Lebesque's measure equal to zero there exists a sequence $y_i \in D_t \setminus N$, $i = 1, 2, \ldots$ with

$$\lim_{k \to \infty} y_k = x \text{ for which } u = \lim_{k \to \infty} u(t, y_k),$$

where D_t is an intersection of the set D and the hyperplane $t = \text{const}$. Also we shall use the symbol "ess min" in above sense.

The multi-function $U(t, x)$ is not only upper semi-continuous with respect to x when t is fixed, but it is measurable with respect to t and to (t, x) as well as the superposition $U(t, x(t))$ is a measurable function for any continuous function $x(t)$, $x(t) \in D_t$ (see [2]).

Definition 1. The graph $GrW = \{(t, x) \in R \times R^n \mid x \in W(t)\}$ of the multi-function $W(t)$ on the interval $[t_0, T]$ is called stable if for any $t_* \in [t_0, T]$, $\varepsilon > 0$, $t_* + \varepsilon \leqq T$, $x_* \in W(t_*)$ there exists such a measurable control $u(t, x) \in P(t, x)$ so that all the solutions of the differential equation (2) satisfy the next inclusion $x(t_* + \varepsilon) \in W(t_* + \varepsilon)$.

We say that the function $f(t, x, u)$ satisfies the Lipschitz condition with respect to a variable x if there exists an integrable function $L(t)$ for which the following inequality

$$\|f(t, x, u) - f(t, y, u)\| \leqq L(t) \|x - y\|$$

holds. Here $(t, x) \in D$, $(t, y) \in D$, $u \in P$ and $\| . \|$ is the norm in the Euclidean space R^n. We also require that $W(t)$ be an upper semi-continuous multi-function throughout this paper.

Theorem 1. *Let there exist a neighbourhood G of the graph GrW, $GrW \subset G \subset D$ which satisfies the following conditions:*
1. *For any $(t, x) \in G$ the set*

$$Pr_t x = \operatorname{Arg} \min_{y \in W(t)} \|x - y\|,$$

consists of only one point.
2. *For any $(t, x) \in G$ and $y \in W(t)$ the next inequality*

$$\operatorname{ess} \min_{u \in P(t, x)} (x - y, f(t, y, u))$$
$$\leqq \operatorname{ess} \min_{u \in P(t, y)} (x - y, f(t, y, u)) + l(t) \|x - y\|,$$

is fulfilled, where $l(t)$ is an integrable function.

Let $f(t, x, u)$ satisfy the Lipschitz condition with respect to x with an integrable function $L(t)$. Then there exists a viable control $u(t, x) \in P(t, x)$ if and only if GrW is stable.

Proof. It is trivial to prove the necessity because the viability of the control $u(t, x)$ is a stronger condition than the stability of the graph GrW.

It is easy to see that the stability of GrW leads to the conclusion that the multi-function $W(t)$ is right side continuous. The condition 1 of the theorem leads to the conclusion that the function $Pr_t x(t)$, where $x(t) \in G_t$ is continuous, is right side continuous.

Let $x(t)$ satisfy the differential equation (3) with the following control function

$$u(t, x) = \operatorname{Arg} \operatorname{ess} \operatorname{lex} \min_{u \in P(t, x)} (x - Pr_t x, f(t, Pr_t x, u)). \tag{5}$$

Here the symbol "ess" is used in the above mentioned sense. We shall consider the increment of the function $\|x(\tau)-Pr_t x(\tau)\|^2$ on the interval $[t, s]$. Let $y(\tau)$, $t \leqq \tau \leqq s$ be any solution of the differential inclusion (3) which satisfies the following two conditions $y(t) \in Pr_t x(t)$ and $y(s) \in W(s)$. The existence of $y(\tau)$ with these properties is guaranteed from the stability of the graph GrW. Now we have

$$A=\|x(s)-Pr_s x(s)\|^2-\|x(t)-Pr_t x(t)\|^2$$
$$\leqq \|x(s)-y(s)\|^2-\|x(t)-y(t)\|^2$$
$$=2\int_t^s (x(\tau)-y(\tau),\ \dot{x}(\tau)-\dot{y}(\tau))\,d\tau \leqq 2\int_t^s (x(\tau)-y(\tau),\ \dot{x}(\tau))\,d\tau$$
$$-2\int_t^s \operatorname*{ess\,inf}_{u\in P(\tau,\ y(\tau))} (x(\tau)-y(\tau),\ f(\tau,\ y(\tau),\ u))\,d\tau.$$

According to the paper [2], for a.a. $\tau \in [t, s]$, we have

$$\dot{x}(\tau)=\sum_{i=1}^{n+1} a_i(\tau) f(\tau,\ x(\tau),\ u_i(\tau)),$$

where $a_i(\tau)$ and $u_i(\tau)$ are measurable functions, $\sum_{i=1}^{n+1} a_i(\tau)=1$, $a_i(\tau) \geqq 0$, $u_i(\tau) \in U(\tau,\ x(\tau))$, $i=1,\ 2, \ldots,\ (n+1)$. So we have

$$A \leqq 2\int_t^s \Big(x(\tau)-y(\tau),\ \sum_{i=1}^{n+1} a_i(\tau) f(\tau,\ x(\tau),\ u_i(\tau))\Big)\,d\tau$$
$$-2\int_t^s \operatorname*{ess\,inf}_{u\in P(\tau,\ y(\tau))} \Big(x(\tau)-y(\tau),\ \sum_{i=1}^{n+1} a_i(\tau) f(\tau,\ y(\tau),\ u)\Big)\,d\tau$$
$$\leqq 2\int_t^s L(\tau)\,\|x(\tau)-y(\tau)\|\,d\tau+2\int_t^s \Big(x(\tau)-y(\tau),\ \sum_{i=1}^{n+1} a_i(\tau) f(\tau,\ y(\tau),\ u_i(\tau))\Big)\,d\tau$$
$$-2\int_t^s \operatorname*{ess\,inf}_{u\in P(\tau,\ y(\tau))} \Big(x(\tau)-y(\tau),\ \sum_{i=1}^{n+1} a_i(\tau) f(\tau,\ y(\tau),\ u)\Big)\,d\tau.$$

Note that

$$\int_t^s (x(\tau)-y(\tau),\ a_i(\tau) f(\tau,\ y(\tau),\ u_i(\tau)))\,d\tau$$
$$\leqq \int_t^s (x(\tau)-Pr_\tau x(\tau)+Pr_\tau x(\tau)-y(\tau),\ a_i(\tau) f(\tau,\ y(\tau),\ u_i(\tau)))\,d\tau$$
$$\leqq \int_t^s (x(\tau)-Pr_\tau x(\tau),\ a_i(\tau) f(\tau,\ Pr_\tau x(\tau),\ u_i(\tau)))\,d\tau$$
$$+\int_t^s \|x(\tau)-Pr_\tau x(\tau)\|\, a_i(\tau)\, L(\tau)\, \|y(\tau)-Pr_\tau x(\tau)\|\,d\tau$$

$$+\int_t^s \|Pr_\tau x(\tau)-y(\tau)\|\, a_i(\tau)\, m(\tau)\,d\tau$$

$$\leqq \int_t^s \operatorname*{ess\,min}_{u\in P(\tau,\,x(\tau))} (x(\tau)-\mathrm{Pr}_\tau x(\tau),\ a_i(\tau)f(\tau,\ Pr_\tau x(\tau),\ u))\,d\tau + o(s-t)$$

$$\leqq \int_t^s \operatorname*{ess\,min}_{u\in P(\tau,\,x(\tau))} (x(\tau)-y(\tau),\ a(\tau)f(\tau,\ y(\tau),\ u))\,d\tau + o(s-t).$$

According to the condition 2 of the theorem, we obtain

$$A\leqq 2\int_t^s \operatorname*{ess\,min}_{u\in P(\tau,\,x(\tau))} (x(\tau)-y(\tau),\ f(\tau,\ y(\tau),\ u))\,d\tau + o(s-t)$$

$$-2\int_t^s \operatorname*{ess\,min}_{u\in P(\tau,\,y(\tau))} (x(\tau)-y(\tau),\ f(\tau,\ y(\tau),\ u))\,d\tau$$

$$+2\int_t^s L(\tau)\,\|x(\tau)-y(\tau)\|^2\,d\tau$$

$$\leqq 2\int_t^s (L(\tau)+l(\tau))\,\|x(\tau)-y(\tau)\|^2\,d\tau + o(s-t)$$

$$=2\int_t^s (L(\tau)+l(\tau))\,\|x(\tau)-Pr_\tau x(\tau)\|^2\,d\tau + o(s-t).$$

Suppose that for $A>\varepsilon>0$ the interval $[t,\,s]$ can be partitioned on numbers of intervals. An analogous estimate as above can be written for every partitioning interval. Let the partitioning points be r_i, $i=0,\,1,\ldots,\,k$, $r_0=t$, $r_k=s$ and r be the length of any partitioning interval. Summarizing all inequalities, we obtain

$$\|x(s)-Pr_s x(s)\|^2 \leqq 2\int_t^s (L(\tau)+l(\tau))\,\|x(\tau)-Pr_\tau x(\tau)\|^2\,d\tau$$

$$+\|x(t)-Pr_t x(t)\|^2 + O(r).$$

According to Gronwall's inequality (see [6]) we have

$$\|x(s)-Pr_s x(s)\|^2 \leqq (\|x(t)-Pr_t x(t)\|^2 + O(r))$$

$$\times\Big(1+\int_t^s 2(L(\tau)+l(\tau))\exp\Big(\int_\tau^s 2(L(\sigma)+l(\sigma))\,d\sigma\Big)d\tau\Big).$$

If t is the first moment when the trajectory leaves $W(\tau)$ then we can write

$$\|x(s)-Pr_s x(s)\| \leqq O(r)\Big(1+\int_t^s (L(\tau)+l(\tau))\exp\Big(\int_\tau^s (L(\sigma)+l(\sigma))\,d\sigma\Big)dg\Big).$$

According to the integrability of the functions $L(t)$ and $l(t)$ the inequalities $\|x(s)-Pr\,x(s)\|^2 \geqq \varepsilon>0$ contradict the above mentioned inequality for all sufficiently small r. So, every solution $x(t)$ of the differential equation (2) with the control $u(t, x)$ from (5) can not leave $W(t)$. The proof is completed.

Some viability criteria for the constant sets are formulated using the contingent cone. In the case when the multi-function $W(t)$ can be represented as an intersection of the hyperplane $t=\text{const}$ and the solutions of some differential inclusion we shall formulate a sufficient condition for the existence of a viable control. We denote the interior of the domain D with $\operatorname{int} D$.

Theorem 2. *Let $W(t)$ be the intersection of all solutions of the differential inclusion*

$$\dot{x} \in Y(t, x), \quad x(t_0)=x_0, \tag{6}$$

and the hyperplane $t=\text{const}$. Let the multi-function $Y(t, x)$ with compact and convex values be measurable with respect to (t, x) and upper semi-continuous with respect to x on $\operatorname{int} D$. If

$$f(t, x, P(t, x)) \cap Y(t, x) \neq 0 \tag{7}$$

then there exist a viable measurable control $u(t, x) \in P(t, x)$.

P r o o f. Let $u(t, x) \in P(t, x)$ be any measurable selection of (7). Filippov's lemma (see [2]) guarantees the existence of such measurable selection. Since the multi-function $Y(t, x)$ with convex values is upper semi-continuous we have

$$\operatorname{co} f(t, x, U(t, x)) \subset Y(t, x).$$

So, we obtain that every solution $x(t)$ of the differential inclusion (3) satisfies $x(t) \in W(t)$. The proof is completed.

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Institute of Mathematics
Bulgarian Academy of Sciences
P.O.B. 373
1090 Sofia
BULGARIA

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