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π_1 -Semigroups

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Partial results on π_1 -semigroups have been obtained by the author in [1]. In the present paper the classification of π_1 -semigroups will be completed, and some characterizations will be established.

In the paper [1] the author introduced the notation of the π_n -semigroup as a finite semigroup S with $|\pi(S)|=n$, where the set $\pi(S)=\{|T|>1/2|S|: T$ forms a proper subsemigroup of $S\}$. Determining all of L-semigroups (Lagrange semigroups) was the earliest work on π_n-semigroups, and it was completely motivated by the consideration to Lagrange's Theorem – one of the most fundamental theorems of the theory of finite groups – in the theory of finite semigroups. The question of determining all of L-semigroups was first raised in [2] and entirely resolved in [3]. It follows from the characterization, which was established in [3], that L-semigroup and π_0 -semigroup are the same one concept. In [1] the author have determined the structure of the π_1 -semigroup with one-side identity, and characterized some π_1 -semigroups without one-side identity. The present paper will be used to resolve the remain problem about π_1 -semigroups, that is, to determine the types of π_1 -semigroups without one-side identity.

The notations and terminologies are taken from [1, 4].

We first deal with a special situation of π_1 -semigroups, utilising the results of [1]:

Theorem 1. *For a finite non-simple semigroup* S, $\pi(S)=\{|S|-1\}$ *if and only if* S *is one of the following types*:

1) $|S|=3$ or 4; 2) $S=G\cup\{x\}$, $x=x^2$;

3) $S=G\cup\{x\}$, $x^2=x^{2+n}\in G$, $n\in N$, *where* G *is a finite group admitting no subgroup* R *of index* 2 *such that* $\langle x\rangle - R=\{x\}$.

Proof. We need only prove the essentiality, and assume $|S|\geqq 5$.

By the condition $\pi(S)=\{|S|-1\}$. We may suppose that G is a subsemigroup of S of order equal to $|S|-1$ and $S-G=\{x\}$. Evidently, $\pi(G)=\emptyset$ *or* $\{1/2|S|\}$. Since $1/2|S|<1/2|G|+1$, there must be the equation $\pi(G)=\emptyset$ by Th. 4.3 of [1] and so G is an L-semigroup by the result of [3].

G must be a group. Otherwise, we may suppose G has no right identity and there exist two idempotents e and f of G such that $G=Ge\cup Gf$ by the result of [3]. If $S^1x=S$, then $S=S^1x=(S^1x)^1x=Sx^2\cup\{x\}$, this shows $G\subseteq Sx^2$ and so

$G=Gx^2$ and $x=x^2$, hence $Se\cup\{x\}$ forms a subsemigroup of S of order $1/2|G|+1=1/2(|S|+1)$, a contradictions; if $S^1x\neq S$, then either $Se\cup S^1x=Ge\cup\{x\}$ or $Sf\cup S^1x=Gf\cup\{x\}$, hence S must contains a proper subsemigroup $Se\cup S^1x$ or $Sf\cup S^1x$ of order $1/2|G|+1=1/2(|S|+1)$, a contradiction. So G forms a group.

Let R be a subgroup of G of index 2, then $\langle x\rangle-R\neq\{x\}$. In fact, if $\langle x\rangle-R=\{x\}$, then $R\cup\{x\}$ forms a subsemigroup of S and its order is $1/2|G|+1=1/2(|S|+1)$, a contradiction.

S is of the type 3) if $x\neq x^2$. Since $\langle x\rangle-\{x\}=\langle x\rangle\cap G$ forms a subgroup of G, we have the identity $x^{n+2}=x^2$, $n\in N$ for the monogonic semigroup $\langle x\rangle$.

This completes the proof.

Example 1. Let $S=\langle a,x;\ a^7=a,\ x^2=x^4,\ x^3=a^3\rangle$. Then S forms a semigroup of order 7 with $\pi(S)=\{|S|-1\}$, and it is worth to indicate that $S-\{x\}$ contains a subgroup $\langle a^2\rangle$ of index 2.

To determine the types of π_1-semigroups without one-side identity, we need consult some properties of the finite semigroup $S=Se\cup Sf$, where $ef=f$ and $fe=e$. It will be seen that S may have a more "complex" structure.

Theorem 2. *Let $S=Se\cup Sf$, where $ef=f$ and $fe=e$, be a finite semigroup. Then the following conditions are equivalent*:

1) *$Sxy=Sy$ for any two elements x, y of $E(S)\cap(L_e\cup L_f)$*;

2) *$T=\bigcup_{x\in E(S)\cap(L_e\cup L_f)}\mathrm{Tor}(x)$ forms a simple subsemigroup of S and $S-T$ the maximal ideal of S.*

Proof. By the given condition $\mathrm{Tor}(x)\subseteq Sx$ for any element x of the set $E(S)\cap(L_e\cup L_f)$, hence $\mathrm{Tor}(x)=H_x$ forms a subgroup of S by Th. 2.1 of [1].

2) implies 1): It is evident.

1) implies 2): By Th. 2.1 of [1] we need only prove the conclusion for the case of $e\neq f$. If there exists an element p of the set $P=Se-\bigcup_{x\in E(S)\cap L_e}\mathrm{Tor}(x)$ such that $pf\in\mathrm{Tor}(y)$ for some $y\in E(S)\cap L_f$, then there exists a number $n\in N$ such that $(pf)^n=y$, and so, by the condition 1), $Sy=Sey=Se(pf)^n=Sep(fp)^{n-1}f$, hence $Sy=Pf$ by Th. 2.1 of [1]. Further, we have $Se=Sye=(Pf)e=P(fe)=P$, this is a contradiction. Thus Pf must be contained in $Q=Sf-\bigcup_{x\in E(S)\cap L_f}\mathrm{Tor}(x)$. By the same way we can prove $Qe=P$, and so $Pf=Q$, $Qe=P$, $(Se-P)f=Sf-Q$, $(Sf-Q)e=Se-P$. Hence $T=S-P\cup Q$ forms a simple subsemigroup of S, and $S-T$ forms the maximal ideal of S by Th. 2.1 of [1].

Example 2. The semigroup S presented with the following Cayley table will be of the form $S=Se\cup Sf$, $ef=f$ and $fe=e$. Although $|E(S)\cap L_e|=|E(S)\cap L_f|$, yet S does not satisfy the condition as required in Th. 2.1 Here $e=2$ and $f=6$. (See [5]: NR. 55, or see [6]).

	1	2	3	4	5	6	7
1	1	4	5	4	5	1	5
2	5	2	6	5	5	6	2
3	3	7	5	7	5	3	5
4	5	4	1	5	5	1	4
5	5	5	5	5	5	5	5
6	6	2	5	2	5	6	5
7	5	7	3	5	5	3	7

Lemma 1. *Le $S=Se\cup Sf$ where $ef=f$ and $fe=e$ be a finite non-simple semigroup. If $r<3|S|/4$ for any number $r\in\pi(S)$, then $X=\{|E(S)\cap L_x|:x=e, f\}=\{1\}$, or $\{2\}$, or $\{2, 3\}$.*

Proof. (1) $\operatorname{Max}X\leqq 3$ and $\operatorname{Min}X\leqq 2$.

Let t be an element of $E(S)\cap L_e$ and z an element of S. By the given condition $eS=fS$ and $z=ze$ or zf, hence $zS=(ze)S$ or $(zf)S=z(eS)$ or $z(fS)=z(eS)=z(et)=(ze)(tS)$ and so $|zS|\leqq|tS|$. This shows that $tS=zS$ if $tS\subseteq zS$. Thus xS must be a maximal left principle ideal of S for any element x of $E(S)\cap(L_e\cup L_f)$.

Clearly, by the given condition we have $S=\bigcup_{x\in S} xS$. If $\operatorname{Max}X\geqq 4$, then, by the preceding result, $n\geqq 4$ if there exist the elements $x_1, x_2, \ldots, x_n$ of S such that $S=x_1S\cup x_2S\cup\ldots\cup x_nS$, and so $\bigcup_{y\in S-R_h} yS$ must forms a proper subsemigroup of S of order $\geqq 3|S|/4$, h an element of $E(S)\cap(L_e\cup L_f)$ such that $|E(S)\cap L_h|\geqq 4$. In fact, $S-R_h=\bigcup_{y\in S-R_h} yS$ and $|R_h|\leqq|S|/4$ by $|E(S)\cap L_h|\geqq 4$. This is a contradiction. So $\operatorname{Max}X=3$.

Now it remains to prove $\operatorname{Min}X\neq 3$: otherwise, $X=\{3\}$, if S satisfies the condition as required in Th. 2, then either $S-\operatorname{Tor}(e)-\operatorname{Tor}(f)$ or $T=\bigcup_{x\in E(S)\cap(L_e\cup L_f)}\operatorname{Tor}(x)$ forms a proper subsemigroup of S of order $\geqq 3|S|/4$; if S does not the condition, then the order of the set $\{xS:x\in S\}$ is greater than 4, and so S must admit a proper subsemigroup of order $\geqq 3|S|/4$ by the same way as the proof of $\operatorname{Max}X\leqq 3$, a contradiction. Therefore $\operatorname{Min}X=2$.

(2) $\operatorname{Max}X=1$ if $\operatorname{Min}X=1$.

Otherwise, let $|E(S)\cap L_f|>|E(S)\cap L_e|=1$. By $|E(S)\cap L_f|\geqq 2$ we have $|\operatorname{Tor}(e)|\leqq 1/4|S|$, and so $Se\cup(Se-\operatorname{Tor}(e))f$ forms a proper subsemigroup of order $\geqq 3|S|/4$ since $Se-\operatorname{Tor}(e)$ forms the maximal ideal of Se by Th. 2.1 of [1], a contradiction.

Thus the conclusion holds as required.

Example 3. In example 2 the semigroup S clearly satisfies the demand of Le 1 and its $X=\{2\}$. Now we give an example with the following Cayley table for $X=\{2, 3\}$, and it's easy to check that $\pi(S)=\{4, 5\}$. Here $e=1$ and $f=6$. (See [6])

*	1	2	3	4	5	6	7
1	1	1	4	4	5	5	4
2	2	2	4	4	6	6	4
3	3	3	4	4	7	7	4
4	4	4	4	4	4	4	4
5	1	1	1	4	5	5	5
6	2	2	2	4	6	6	6
7	3	3	3	4	7	7	7

Theorem 3. *Let* $S=Sa\cup Sb\cup Sc$ *where* $a, b, c\in S$ *be a finite non-simple semigroup and* $S\neq Sx\cup Sy$ *for any* $x, y\in S$. *Then* $|\pi(S)|=1$ *if and only if* $S=\cup(T, P, 3, 1)$ *where* T *is a G-monoid of order* $2|G|$ *and* G *a finite group admitting no subgroup of index* 2.

Proof. We need only prove the essentiality.

Step 1. There exist $e, f, h\in E(S)$ such that $S=Se\cup Sf\cup Sh$ and $|Se|=|Sf|=|Sh|$.

1) We assume $|Sa|\geqq|Sb|\geqq|Sc|$. By the given condition the element a must be contained in Sa, hence there exists an element e of $E(S)$ such that $a=ea$. Clearly, $|Se|=|Sa|$ and so $Se=Sa$, or Sb, or Sc. This shows that $S=Se\cup Sw\cup Sz$ for some $e\in E(S)$, $w, z\in S$, and we have the inequation $|Se|\geqq|Sw|\geqq|Sz|$. If $|Se|\neq|Sw|$, then both $T=Sw\cup Sz\cup(Se-\bigcup_{x\in E(S)\cap L_e}\mathrm{Tor}(x))$ and $T\cup\{e\}$ form two subsemigroups of order greater than $1/2|S|$. Evidently, $e\notin T$ and so $T\cup\{e\}$ by the condition $|\pi(S)=1$, hence $\pi(S)=\{|S|-1\}$, this is a contradiction to the assumption $S=Sa\cup Sb\cup Sc$ by Th 1. Hence there exist $w, z\in S$ and $e\in E(S)$ such that $S=Se\cup Sw\cup Sz$ and $|Se|=|Sw|\geqq|Sz|$.

2) There exist $e,f\in E(S)$ and $x\in S$ such that $S=Se\cup Sf\cup Sx$ and $|Se|=|Sf|=|Sx|$. At first, we prove the required result is true if $S=Se\cup Sf\cup Sx$ for some $e, f\in E(S)$, $x\in S$ and $|Se|=|Sf|\geqq|Sx|$: otherwise, $|Sf|>|Sx|$. Clearly, for any two $t, u\in E(S)\cap(L_e\cup L_f)$, $|St\cup Sx|>1/2|S|$ and S contains two subsemigroups of order greater than $1/2|S|$: $St\cup Sx\cup Stu\cup Sxu$ and $St\cup Sx\cup Stu\cup Sxu\cup\{u\}$. By $|\pi(S)|=1$ and Th 1 we have $S=St\cup Sx\cup Stu\cup Sxu$, hence $Su=Stu$. This shows the semigroup $Se\cup Sf$ satisfies the condition 1) of Th. 2. By Th. 2 and the assumption $|Sf|>|Sx|$ it is easy to verify that S admits the subsemigroups: $T=\bigcup_{x\in E(S)\cap(L_e\cup L_f)}\mathrm{Tor}(x)$, $S-T$, $(S-T)\cup\{e\}$. Since $|T|<|Se\cup Sf|<|S|$, $\pi(S)=\{1/2|S|+1\}$ or $\{|S|-1\}$ by $|\pi(S)|=1$. Based on Th. 4.3 of [1] and Th. 1, this is a contradiction to the given condition $S=Sa\cup Sb\cup Sc$ and $S\neq Sx\cup Sy$ for any $x, y\in S$.

Now we prove $S=Se\cup Sf\cup Sx$ for some $e,f\in E(S)$, $x\in S$ and $|Se|=|Sf|\geqq|Sx|$: in 1) we have showed that $S=Se\cup Sw\cup Sz$ for some $e\in E(S)$, $w, z\in S$ and $|Se|=|Sw|\geqq|Sz|$. By the given condition w must be contained in Sw, hence there exists $t\in E(S)$ such that $w=tw$, and so $St=Se$, or Sw, or Sz. If $St=Sw$ or Sz, the required result has been proved; if $St=Se$, $Sw=Sew$. Clearly, $|Sw\cup Sz|>1/2\,|S|$, and so $S=Swe\cup Sze\cup Sw\cup Sz$ by the condition $|\pi(S)|=1$, hence $Se=Swe$ or Sze, if $Se=Swe$, then $Sw=S(ew)^n$ for any $n\in N$ and so the required result is true; if $Se=Sze$, we consider Sz by the same way, there exists $h\in E(S)$ such that $z=hz$ and $Sh=Se$, or Sw, or Sz: if $Sh=Se$, then $Sz=S(ez)^n$ for any $n\in N$ and so the required result is also true; otherwise, the required result has been proved.

3) There exist $e,f,h\in E(S)$ such that $S=Se\cup Sf\cup Sh$ and $|Se|=|Sf|=|Sh|$. By the given condition x must be contained in Sx, hence there exists $t\in E(S)$ such that $x=tx$ and $St=Se$, or Sf, or Sx. If $St=Sx$, the required result has been proved; now we consider the case of $St=Sf$ or Sx: at first, we indicate two points: (1) there exists $g\in E(S)\cap(L_e\cup L_f)$ such that $Sxg=Sg$ (otherwise, $Suv=Sv$ for any two $u,v\in E(S)\cap(L_e\cup L_f)$, this will derive a contradiction. The proof is the same as 2)); (2) $St=Sgt$ or Sxt (otherwise, S contains two subsemigroups $Sg\cup Sx\cup Sgt\cup Sxt$ and $Sg\cup Sx\cup Sgt\cup Sxt\cup\{t\}$, this will derive a contradiction to $|\pi(S)|=1$). If $g\in L_t$, then $Sx=(St)x=(Sg)x=(S(xg))x=(Sx)(gx)=\ldots=S(gx)^n$ for any $n\in N$ and so $Sx=Sh$, h the idempotent of $\langle gx\rangle$; if $g\notin L_t$, then $Sx=(St)x=(Sgt)x$ or $(Sxt)x=\ldots=S(gtx)^n$ or $S(tx)^n$ for any $n\in N$ and so $Sx=Sh$, h the idempotent of $\langle gtx\rangle$ or $\langle tx\rangle$.

Step 2. For any two $x,y\in\{e,f,h\}$ there exist $t\in E(S)\cap L_x$ and $u\in E(S)\cap L_y$ such that $tu=u$ and $ut=t$.

It is enough to prove that there exist $k\in E(S)\cap L_x$, $l\in E(S)\cap L_y$ such that $Skl=Sl$ or $Slk=Sk$ for any two $x,y\in\{e,f,h\}$: in fact, if $Skl=Sl$, then, by the given condition, there exists $v\in E(Sk)$ such that $l=vl$, hence t and u satisfy the demand if we let $t=lv$ and $u=l$.

For any $x\in E(S)\cap L_e$, $y\in E(S)\cap L_f$, $z\in E(S)\cap L_h$ it is easy to prove that $Sx=Syx$ or Szx, $Sy=Sxy$ or Szy, $Sz=Sxz$ or Syz; for example, since $T=Sx\cup Sy\cup Sxz\cup Syz$ and $T\cup\{z\}$ form two subsemigroups of S of order greater than $1/2\,|S|$, $Sz=Sxz$ or Syz by $|\pi(S)=1$, the given condition and Th. 1. Clearly, we need only consider the case of $Sx=Syx$ or Szx, $Sy=Sxy$ and $Sz=Sxz$: by the preceding result and the given condition the subsemigroups $Sx\cup Sy$ and $Sx\cup Sz$ satisfy the condition of Le 1, hence $Y=\{\,|E(S)\cap L_a|: a=e,f,h\}=\{1\}$, or $\{2\}$, $\{2,3\}$. If $Y=\{1\}$, then $ef=f$, $fe=e$, $eh=h$, and $he=e$ by the assumption and the preceding result, and so $h=eh=(fe)h=f(eh)=fh$ and $f=ef=(he)f=h(ef)=hf$, hence the required result is true; otherwise, we can assume $\{f,p\}\subseteq E(S)\cap L_f$, $\{h,q\}\subseteq E(S)\cap L_h$, and, by the same way as last section there exist e_1, e_2, e_3, $e_4\in E(S)\cap L_e$ such that

$$\begin{cases}e_1f=f,\\ fe_1=e_1,\end{cases}\quad\begin{cases}e_2=p,\\ pe_2=e_2,\end{cases}\quad\begin{cases}e_3h=h,\\ he_3=e_3,\end{cases}\quad\begin{cases}e_4q=q,\\ qe_4=e_4,\end{cases}$$

and clearly $e_1\neq e_2$, $e_3\neq e_4$; since $|E(S)\cap L_e|\leqq 3$, we may assume $e_1=e_3$, and so $h=e_1h=(fe_1)h=f(e_1h)=fh$, $f=e_1f=(he_1)f=h(e_1f)=hf$, hence the required result is also true by the preceding result.

Step 3. $Y=\{1\}$, where $Y=\{\,|E(S)\cap L_a|: a=e,f,h\}$.

Let $\pi(S)=\{r\}$, then $r\geqq 2\,|S|/3+1$ by the results of [1]. If the conclusion does not hold, by step 2 and Le 1 there must be $Y=\{2\}$ or $\{2, 3\}$, and $|E(S)\cap L_e|=|E(S)\cap L_f|=2$ if we might as well assume $|E(S)\cap L_h|=\text{Max } Y$. It is easy to prove that xS forms a maximal left principle ideal of S for any $x\in E(S)\cap(L_e\cup L_f\cup L_h)$, hence $|X|=2$ or 3 by the assumption and $|\pi(S)|=1$, where $X=\{xS: x\in E(S)\cap(L_e\cup L_f\cup L_h)\}$. Now we derive a contradiction, dividing the argument into two cases:

1) the case of $|X|=3$. By step 2 we may assume

$$\begin{cases} ef=f, \\ fe=e, \end{cases} \qquad \begin{cases} f_1 h=h, \\ hf_1=f_1, \end{cases} \qquad \begin{cases} e_1 h_1=h_1, \\ h_1 e_1=e_1, \end{cases}$$

where $e_1\in E(S)\cap L_e$, $f_1\in E(S)\cap L_f$, $h_1\in E(S)\cap L_h$. If $Sef_1=Sf_1$, then there exists $e_2\in E(S)\cap L_e$ such that $f_1=e_2 f_1$ and so

$$\begin{cases} e_3 f_1=f_1, \\ f_1 e_3=e_3, \end{cases} \qquad \begin{cases} f_1 h=h, \\ hf_1=f_1, \end{cases}$$

where $e_3=f_1 e_2$. It follows from $|X|=3$ and $\pi(S)=\{r\}$ that $S=f_1 S\cup wS\cup zS$ where $w, z\in S$ and $wS\cup zS$ forms a subsemigroup of order r, hence $3|\text{Tor}(e)|=|\bigcup_{t=e_3, f, h}\text{Tor}(t)|\leqq|S-(wS\cup zS)|=|S|-r<|S|/3$, and so $9|\text{Tor}(e)|<|S|$; if $Sef_1\neq Sf_1$, then $Sef_1\subseteq T=Sf-\bigcup_{t\in E(S)\cap L_f}\text{Tor}(t)$ and so $\text{Tor}(f_1)\subseteq Kf$, where $K=Se-\bigcup_{t\in E(S)\cap L_e}\text{Tor}(t)$ (otherwise, $\text{Tor}(f_1)\cap Kf=\emptyset$, hence $Kf=T$, by $Sf=Sef$ there exists $t\in Se-K$ such that $tf=f_1$, this shows that $Sf_1=Sgf_1$ where g is the idempotent of $\langle t\rangle$, and so $Sef_1=Sgf_1=Sf_1$, a contradiction). Thus $(Se-K)f\cup \text{Tor}(f_1)\geqq 3\ \text{Tor}(e)$; on the other hand, $(Se-K)f\cap(Se\cup Sh)=\emptyset$, therefore $3|\text{Tor}(e)|\leqq(|Se-K)f\cup\text{Tor}(f_1)|\leqq|S-(Se\cup Sh)|=|S|-r<|S|/3$, that is, $9|\text{Tor}(e) <|S|$. This shows that $9|\text{Tor}(e)|<|S|$ if $|X|=3$.

At the final, we derive a contradiction: since $(Se-K)f\cap(Se\cup Sh)=\emptyset$ and $(Se-K)h_1\cap(Se\cup Sf)=\emptyset$, $Se\cup Kf\cup Kh_1=S-((Se-K)f\cup(Se-K)h_1)$ and so $|Se\cup Kf\cup Kh_1|=|S|-4|\text{Tor}(e)|>1/2|S|$ by the preceding result. Clearly, $Se\cup Kf\cup Kh_1$ forms a subsemigroup of S since K is an ideal of Se by Th. 2.1 of [1], hence $|Se\cup Kf\cup Kh_1|=r$ by $\pi(S)=\{r\}$, that is, $r=|S|-4|\text{Tor}(e)|$; on the other hand, $2|\text{Tor}\ (e)|=|(Se-K)f|=|(Se\cup Sf)-(Se\cup Kf)|=|Se\cup Sf|-|Se\cup Kf|\geqq r-1/2|S|$, hence $r\leqq 2|S|/3$. This is a contradiction.

2) the case of $|X|=2$. At this time, we may assume $L_t\cap E(S)=\{t, t_1\}$, $t=e, f, h$ and $eS=fS=hS$, $e_1 S=f_1 S=h_1 S$. Clearly, $Sxy=Sy$ for any two $x, y\in E(S)\cap(L_e\cup L_f\cup L_h)$. By Th. 2 it is easy to derive a contradiction to $|\pi(S)|=1$.

By 1) and 2) we have showed $Y=\{1\}$.

Step 3. The conclusion holds as required.

By Step 2-3 and Th. 2 it is easy to verify that $\text{Tor}(e)\cup\text{Tor}(f)\cup\text{Tor}(h)$ forms a simple subsemigroup of S and $S-\text{Tor}(e)\cup\text{Tor}(f)\cup\text{Tor}(h)$ forms the maximal ideal of S, hence $|\text{Tor}(e)\cup\text{Tor}(f)\cup\text{Tor}(h)|=r$ by $\pi(S)=\{r\}$ and $r>2|S|/3$, and so

$S=eS$. (since $\mathrm{Tor}(e) \cup \mathrm{Tor}(f) \cup \mathrm{Tor}(h)$ is properly contained in eS). So the conclusion holds as required by Th. 3.1 of [1].

Now we can determine the types of π_1-semigroups without one-side identity in the following

Theorem 4. *For a finite non-simple semigroup S without one-side identity, $|\pi(S)|=1$ if and only if S is one of the following:*

1) $|S|=3$, *or* 4;

2) $S=G \cup \{x\}$, $x^2=x^{n+2} \in G$, $n \in N$, *where G is a finite group admitting no subgroup R of index* 2 *such that* $R \cup \{x\} \supseteq \langle x \rangle$;

3) *S contains $\mathcal{M}[G; I, J, P]$, where G is a finite group admitting no subgroup of index* 3, $|I|=|J|=2$ *and* $|S|=6|G|$, *as a maximal subsemigroup.*

Proof. We need only prove the essentiality and let $|S| \geqq 5$.

Clearly, the result have been proved in Th. 1 if $\pi(S)=\{|S|-1\}$. Now we may assume that $\pi(S)=\{r\}$, $r<|S|-1$.

Step 1. $S=Sa \cup Sb$ for some $a, b \in S$.

Otherwise, if $S=Sa \cup Sb \cup Sc$ for some $a, b, c \in S$, S must have left identity by the former theorem, a contradiction; if $S \neq Sa \cup Sb \cup Sc$ for any three, $a, b, c \in S$, then we may assume $S=\bigcup_{i=1}^{n} Sa_i$ by the condition $\pi(S)=\{r\}$, $r<|S|-1$ and this is the most short decomposition, $n \geqq 4$. Now we assume that $|Sa_1-\bigcup_{i=2}^{n} Sa_i|$ is the minimal number and $|Sa_2-\bigcup_{i \neq 2}^{n} Sa_i|$ is the second, then by the assumption we have

$$|S|>|\bigcup_{i=2}^{n} Sa_i \cup I|>|\bigcup_{i=3}^{n} Sa_i \cup I|>1/2\,|S|,$$

where I is the minimal ideal of S. Clearly, it follows from the inequation that $|\pi(S)|=2$, a contradiction.

Thus there exist $a, b \in E(S)$ such that $S=Se \cup Sb$.

Step 2. There exist $e, f \in E(S)$ such that $S=Se \cup Sf$.

By step 1 we let $|Sa| \geqq |Sb|$. Evidently, $a \in Sa$ and so there exists $e \in E(S)$ such that $a=ea$. For the idempotent e we have $Se=Sa$ or Sb, this shows that there exist $e \in E(S)$ and $x \in S$ such that $S=Se \cup Sx$ and $|Se| \geqq |Sx|$.

Now we prove $|Sx| \geqq 1/2\,|S|$: otherwise, $|Se|=r$ and $|Sx|<1/2\,|S|$. If Se is simple, then Se is contained in the minimal ideal I of S and so $x \in Sx^2$ by the given condition, hence $Sx=Sx^2$. This shows the monogonic semigroup $\langle x \rangle$ forms a group. Clearly, $I \cup E(\langle x \rangle)$ forms a subsemigroup of order $r+1$, a contradiction; if Se is non-simple, then $K=\bigcup_{y \in E(S) \cap L_e} \mathrm{Tor}(y)$ is properly contained in Se and so $|S-K| \geqq 1/2\,|S|$, hence $r=|S|-1$ or $1/2\,|S|+1$ since both $S-K$ and $(S-K) \cup \{e\}$ are closed, this is also a contradiction.

Finally, we prove the required result: by the same reason there exists $t \in E(S)$

such that $x=tx$. If $St=Sx$, the conclusion has been proved; if $St \neq Sx$, then $St \subseteq Se$ (in fact, if $t \in Sx$, then $St=Sx$, and so $St=Sx$ since $Stx=Sx$), and so $St=Se$ since St and $St \cup \{e\}$ form two subsemigroups of order $\geqq |St| \geqq |Sx| \geqq 1/2\,|S|$; on the other hand, $Sx \cup Sxt$ and $Sx \cup (Sx)^1 t$ are two subsemigroups, and so $St=Sxt$ by $|Sx| \geqq 1/2\,|S|$, $t \in Sx$ and the condition $\pi(S)=\{r\}$, $r \leqq |S|-2$, thus $Sx=S(tx)^n=Sf$ where f is the idempotent of $\langle tx \rangle$. So the conclusion holds as required.

Step 3. S must be of the case 3).

See Th. 4.2 of [1].

This completes the proof.

So far, the classification of π_1-semigroup has been completed, and now we conclude with the following

Theorem 5. *For a finite semigroup S, $\pi(S)=\{2\,|S|/3\}=\{1+1/2\,|S|\}$ if and only if S is a Z_3-monoid of order 6, or $S=\langle a, b;\ a^4=a,\ b^4=b,\ ab=ba=a \rangle$, or S contains an L-semigroup of order $2\,|S|/3=4$ as a maximal subsemigroup.*

Proof. It is easy to verify the direct part. For the converse, if S has no identity, S must contains an L-semigroup of order 4 as a maximal subsemigroup by Th. 4 and Th. 4.3 of [1]; otherwise, S must be a Z_3-monoid of order 6, or $S=\langle a, b;\ a^4=a,\ b^4=b,\ ab=ba=a \rangle$ by Th. 3.2 of [1].

Theorem 6. *Let S be a finite semigroup with $\pi(S)=\{r\}$. Then $2\,|S|3<r<3\,|S|/4$ if and only if S is a G-monoid of order n, where G is a finite group admitting no subgroup of index 2 or 3 and $8n<12\,|G|<9n$.*

Proof. We need only prove the essentiality. By Th. 4 and Th. 3.3 of [1] S must be a monoid, and so the conclusion holds as required by Th. 3.2 of [1].

Theorem 7. *For a finite non-simple semigroup S, $\pi(S)=\{3\,|S|/4\}$ if and only if $|S|=4$, or S contains a subsemigroup $T \cong \mathscr{M}[G;\ I,\ J;\ P]$ of order $3\,|S|/4$, where G is a finite group admitting no subgroup of index 2 and $(|I|, |J|)=(1, 1)$, or $(1, 3)$, or $(3, 1)$, and $H \cup (S-T)$ forms a H-monoid for any $\mathscr{H}$-class H in T.*

Proof. We need only prove the essentiality. By Th. 4 S must have one-side identity if $|S| \neq 4$, hence the conclusion holds as required by Th. 3.2, Th. 3.3 of [1].

Theorem 8. *Let S be a finite semigroup with $\pi(S)=\{r\}$. Then $3|S|/4<r<|S|-1$ if and only if S is a G-monoid of order n, where G is a finite group admitting no subgroup of index 2 and $3n<4\,|G|<4(n-1)$.*

Proof. (As the proof of Th. 6).

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