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Constructions of Hyperfields

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In this article the constructions of certain hyperfields are introduced. These hyperfields have the property to be generated by the difference of every non zero element from itself, and so they can be characterized as monogene hyperfields. The isomorphism of such hyperfields to the quotient hyperfields is also studied and it leads to the genesis of a new problem in the theory of fields.

Introduction

The notion of the hyperfield was introduced by M. Krasner in [2]. A hyperfield is an algebraic structure $(H, +, .)$ where $(x, y) \to x.y$ is an internal composition of H (i.e. for every $x, y \in H$, $x.y \in H$) and $(x, y) \to x+y$ is a hypercomposition of H (i.e. for every $x, y \in H$, $x+y \subseteq H$). This structure satisfies the following axioms:

I. $(H, +)$ is a canonical hypergroup (see [8]), i.e. the next axioms are valid:

i) $x+y=y+x$ for every $x, y \in H$;

ii) $(x+y)+\omega=x+(y+\omega)$ for every $x, y, \omega \in H$;

iii) There is an element $0 \in H$ such that $x+0=x$ for every $x \in H$;

iv) For every $x \in H$, there is one and only one $x' \in H$ such that $0 \in x+x'$. x' is denoted $-x$ and it is called the opposite of x. We write $x-y$ instead of $x+(-y)$;

v) $\omega \in x+y$ implies that $y \in \omega - x$.

II. $H=H^* \cup \{0\}$, where $(H^*, .)$ is a multiplicative group.

III. The distributive axiom holds, i.e.

$$\omega.(x+y)=\omega.x+\omega.y, \quad (x+y).\omega=x.\omega+y.\omega.$$

Remark. If x, y are subsets of H, then $X.Y$ signifies the set $\{x.y \mid (x, y) \in X+Y\}$, and $x.Y$, $X.y$ have the same meaning as $\{x\}.Y$, $X.\{y\}$ respectively. Also $X \times Y$ signifies the union $\cup_{(x.y) \in X \times Y}(x+y)$, and $x+Y$, $X+y$ have the same meaning as $\{x\}+Y$, $X+\{y\}$ respectively.

In [2] M. Krasner uses a hyperfield which is constructed in the following way: Let K be a valuated field whose valuation is $|\cdot|$, let $\rho \geqq 0$ be a semi-real number [1] of species 0 or $-$, and let π_ρ be the equivalence relation in K which is defined as follows:

$$\alpha \neq 0, \ \beta \equiv \alpha \longleftrightarrow \left|\frac{\beta}{\alpha}-1\right| \leqq \rho \longleftrightarrow |\beta-\alpha| \leqq \rho\,|\alpha|$$

$$\beta \equiv 0 \longleftrightarrow \beta = 0.$$

The classes mod π_p are cycles $C_\xi = C(\xi, p|\xi|)$ with center $\xi \in K$ and radius $p|\xi|$. The set of these classes $K^\wedge = K/\pi_p$ become a hyperfield if we define the product of two elements to be their setwise product and their sum to be the set of the classes which are contained in their setwise sum. This hyperfield was named by M. Krasner residual hyperfield. Next M. Krasner in [3], generalized this construction, by presenting the quotient hyperfields. The construction of these hyperfields is as follows: Let F be a field and let G be a normal subgroup of its multiplicative group. Then the set of the classes xG, $x \in F$ becomes a hyperfield, if we define their product and their sum as we did in the previous method. In the end of this section we prove two propositions that give conditions under which the sum of two elements contains the participating elements, or not.

Proposition 1. *In a hyperfield H the sum $x+y$ of every two elements $x, y \neq 0$ contains these two elements if and only if, the difference $x-x$ equals to H, for every $x \neq 0$.*

Proof. Let us suppose that $x \in x+y$ for every $x, y \in H$. Then for every $y \in H$ we have, $y \in x-x$, so $H \subseteq x-x$. But $x-x \subseteq H$, thus $x-x=H$. Conversely, if $x-x=H$ for every $x \neq 0$, then for every $x, y \in H$ we shall have $y \in x-x$, thus $x \in x+y$.

Proposition 2. *In a hyperfield H the sum $x+y$ of every two elements $x, y \neq 0$ does not contain these two elements if and only if, the difference $x-x$ equals to $\{x, -x, 0\}$ for every $x \neq 0$.*

Proof. $w \in x-x$ if and only if $x \in w+x$. So if $x, y \notin x+y$, then we have $x-x \subseteq \{x, -x, 0\}$. Now if for some $x \in H^*$, holds $x-x=\{0\}$, then for any $y \in H^*$ we shall have: $x.(y-y)= x.y-x.y=(x-x).y=0.y=0$. Thus

$$x.(y-y)=\{x.t \mid t \in y-y\}=\{0\}.$$

So for every $t \in y-y$ we have $x.t=0$, from where it derives that $t=0$, and therefore $y-y=\{0\}$. Now if for every $x \in H$, the relation $x-x=\{0\}$ is valid, then H is a field. Indeed, for every a, $b \in x+y$ we have:

$$a-b \subseteq (x+y)-(x+y)=(x-x)-(y-y)=\{0\}$$

so $a=b$. Thus x or $-x$ must belong to $x-x$. But then, because of the distributivity, $x-x=\{x, -x, 0\}$. Conversely now. If $x-x=\{x, -x, 0\}$ for every $x \in H$, then for every x, $y \in H^*$, with $y \neq \pm x$, holds: $y \notin x-x$, so $x \notin x+y$. Similarly $y \in x+y$, and so the proposition.

Construction I

Let G be a multiplicative group and let $(H^*, .)$ be its direct product with the multiplicative group $(1, -1)$. We consider an element 0 with the property

$$\alpha.0=0.\alpha=0 \text{ for every } \alpha \in H^* \cup \{0\}.$$

We define in $H=H^* \cup \{0\}$ a hypercomposition "+" as follows:

- $(x, i)+(y, j)=\{(x, i), (y, i)\}$
  if $(x, i) \neq (y, j)$, $i, j \in \{1, -1\}$,

- $(x, 1)+(x, -1)=H$,
- $(x, i)+0=0+(x, i)=(x, i)$
- for every $x \in G$. $i \in \{1, -1\}$.

The structure $(H, +, .)$ is a hyperfield. Indeed, the hypercomposition "+" is commutative and the 0 is its neutral element. Also $(x, -1)$ is the opposite of $(x, 1)$ because:

$$0 \in (x, 1)+(x, -1)=H.$$

Next, we shall denote the arbitrary element (x, i) of H by $x^\wedge$, and $(x, -i)$ by $-x^\wedge$. So, for the proof of the associativity we distinguish the following cases:
i) if $x^\wedge$, $y^\wedge$, $z^\wedge$ are different from 0 and each one different from $-x^\wedge$, $-y^\wedge$, $-z^\wedge$, then

$$x^\wedge+(y^\wedge+z^\wedge)=x^\wedge+\{y^\wedge, z^\wedge\}=(x^\wedge+y^\wedge)\cup(x^\wedge+z^\wedge)$$
$$=\{x^\wedge, y^\wedge\}\cup\{x^\wedge, z^\wedge\}=\{x^\wedge, y^\wedge, z^\wedge\}$$

similarly, $(x^\wedge+y^\wedge)+z^\wedge=\{x^\wedge, y^\wedge, z^\wedge\}$;
ii) if $x^\wedge$, $y^\wedge$, $z^\wedge$ are different from 0, but one of them equals to one of $-x^\wedge$, $-y^\wedge$, $-z^\wedge$, e.g. $z^\wedge=-x^\wedge$, we have

$$x^\wedge+(y^\wedge+z^\wedge)\supseteq x^\wedge+(-x^\wedge)=H$$

and $$(x^\wedge+y^\wedge)+z^\wedge\supseteq x^\wedge+(-x^\wedge)=H;$$
iii) if one of $x^\wedge$, $y^\wedge$, $z^\wedge$ is 0, the associativity holds. Now let us verify the axiom I.v. Suppose that $z^\wedge \in x^\wedge+y^\wedge$, then:
i) If $y^\wedge \neq -x^\wedge, 0$ then $x^\wedge+y^\wedge=\{x^\wedge, y^\wedge\}$, thus $z^\wedge=x^\wedge$ or $z^\wedge=y^\wedge$. So we have:

$$x^\wedge \in \{z^\wedge, -y^\wedge\}=z^\wedge+(-y^\wedge)$$

or $$x^\wedge \in H=y^\wedge+(-y^\wedge);$$
ii) if $y^\wedge=-x^\wedge$, then $x^\wedge+y^\wedge=H$ and for every $z^\wedge \in H$ we have:
$$x^\wedge=-y^\wedge \in z^\wedge+(-y^\wedge);$$
iii) if $y^\wedge=0$, then $x^\wedge+y^\wedge=x^\wedge$ and so $x^\wedge=x^\wedge-0=x^\wedge-y^\wedge$. Now for the verification of the distributive axiom we distinguish the cases:
i) if $x^\wedge \neq 0$ and $y^\wedge \neq -z^\wedge, 0$ then:

$$x^\wedge.(y^\wedge+z^\wedge)=x^\wedge.\{y^\wedge, z^\wedge\}=\{x^\wedge.y^\wedge, x^\wedge.z^\wedge\}$$

and since $x^\wedge.y^\wedge \neq 0$, $x^\wedge.y^\wedge \neq -x^\wedge.z^\wedge$, we have:
$$x^\wedge.y^\wedge+x^\wedge.z^\wedge=\{x^\wedge.y^\wedge, x^\wedge.z^\wedge\};$$

ii) if $x^\wedge \neq 0$ and $y^\wedge \neq -z^\wedge$ we have:

$$x^\wedge.(y^\wedge+z^\wedge)=x^\wedge.[y^\wedge+(-y^\wedge)]=x^\wedge.H=H$$

and $$x^\wedge.y^\wedge+x^\wedge.z^\wedge=x^\wedge.y^\wedge+(-x^\wedge.y^\wedge)=H.$$

Finally if $x^\wedge=0$ or $y^\wedge=0$ or $z^\wedge=0$, the distributivity also holds.
Another hyperfield can be constructed as follows: Consider a multiplicative group G, and a bilaterally absorbing element 0 (i.e. $0.x=x.0=0$ for every $x \in G \cup \{0\}$) and endow the set $K=G \cup \{0\}$ with a hypercomposition "+" defined as follows:

- $x+y=\{x, y\}$ for every $x, y \in G$ with $y \neq x$,

- $x+0=0+x=x$ for every $x \in K$,
- $x+x=K$ for every $x \neq 0$.

Then $(K, +, .)$ is a hyperfield.
Also the following construction gives us a hyperfield: Let $(K, +, .)$ be a field. We define in K a hypercomposition "+" as follows:

- $x+y=\{x, y\}$ for every $x, y \in K^*$ with $x \neq y$,
- $x+0=0+x=x$ for every $x \in K$,
- $x+(-x)=K$ for every $x \in K^*$.

Then the structure $(K, +, .)$ is a hyperfield.
The proposition that follows has been proved by A. Nakassis.

Proposition 3. *There are quotient hyperfields in which $x+y=\{x, y\}$, for every two elements x, y with $x \neq -y$ and $x, y \neq 0$.*

Proof. Let K be an ordered field, L an ordered additive group and L_0 a subgroup of L such that, for every $\lambda \in L$ should exist $\lambda_0 \in L_0$ such that $\lambda < \lambda_0$ (e. g. $L=R$ and $L_0=Q$ or Z). Next we construct the set KL as follows:

$$KL=\{q(\theta) \mid q(\theta)=\frac{\sum_{i=1}^{k} A_i \theta^{\alpha_i}}{\sum_{j=1}^{t} B_j \theta^{\beta_j}}\},$$

where:
a) the A_i and the B_j belong to K,
b) the α_i and the β_j belong to L,
c) for the α_i and the β_j holds:

$$\alpha_1 \geqq \alpha_2 \geqq \ldots \geqq \alpha_k \text{ and } \beta_1 \geqq \beta_2 \geqq \ldots \geqq \beta_t,$$

d) $A_i \neq 0$, $i=1, 2, \ldots, k$ unless $q(\theta)$ is the zero element of KL. In this case $k=0$, while in all the other cases $k \geqq 1$,
e) $t \geqq 1$, $B_j \neq 0$ for $j=1, 2, \ldots, t$ and $B_1=1$.
KL become a field if we define the addition and the multiplication as in the case of the rational functions. Next we consider the subgroup G of KL^*, which is defined as follows:

$$G=\{\pi(\theta) \mid \pi(\theta)=\frac{\sum_{i=1}^{k} A_i \theta^{\alpha_i}}{\sum_{j=1}^{t} B_j \theta^{\beta_j}},\ A_1>0 \text{ and } \alpha_1-\beta_1 \in L_0\}.$$

Now if we choose one element α from every class of L/L_0 then, KL/G is the set of $\pm\theta^{\alpha}G$. The sum of two elements of KL/G has the form $(\pm\theta^{\alpha})G+(\pm\theta^{\beta})G$. An arbitrary element of this sum will have the form:

$$[\pm\theta^{\alpha}\pi_1(\theta)]+[\pm\theta^{\beta}\pi_2(\theta)]$$

(suppose that when we have "$-$" in one of the above, then $\beta \neq \alpha$). One can always assume that $\pi_1(\theta)$ and $\pi_2(\theta)$ have the same denominator, and so, if α and β belong to different classes of L/L_0, then the highest exponent of $\theta^\alpha \pi_1(\theta)$ will never be equal to the highest exponent of $\theta^\beta \pi_2(\theta)$, and therefore the result:

$$[\pm\theta^\alpha \pi_1(\theta)] + [\pm\theta^\beta \pi_2(\theta)]$$

will belong to the one or the other term of this sum. On the other hand, if $\alpha=\beta$, then:

$$\theta^\alpha G + \theta^\alpha G = \theta^\alpha G.$$

Construction II

Let $(H, +, .)$ be a hyperfield. We define in H a hypercomposition "#" as follows:

- $x\# \; y=(x+y)\cup\{x, y\}$ if $y\neq -x$, $x, y\neq 0$,
- $x\# \; (-x)=H$ for every $x\in H^*$,
- $x\# \; 0=0\# \; x=x$ for every $x\in H$.

Then the structure $(H, \#, .)$ is a hyperfield.
Indeed it is obvious that the new hypercomposition is commutative and that 0 is its neutral element. Also the opposite of x in $(H, \#, .)$ is the same with the opposite of x in $(H, +, .)$, since, if $0\in x\# \; x'$ and $x'\neq -x$, then we shall have

$$0\in(x\# \; x')\cup\{x, x'\} \text{ and } 0\notin x\# \; x'$$

which gives $x=0$ and $x'=0$.
Next let us prove the axiom I.v. This axiom obviously holds when $y=0$. If $y=-x$ then $x\# \; y=H$, so for every $w\in H$ we must have $x\in w\# \; x$, which is true. Finally if $y\neq -x$, 0, then from the relation $w\in x\# \; y$ derives that $w\in(x+y)\cup\{x,y\}$, so $w\in x+y$ or $w\in\{x, y\}$. If $w\in x+y$, then $x\in w+(-y)\subseteq w\# \; (-y)$. If $w\in\{x, y\}$, then $w=x$ or $w=y$. If $w=x$ then $x\in x\# \; (-y)$ for every $y\in H$ and if $w=y$ then $w\# \; (-y)=H$, which contains x. Thus the axiom I.v. holds.
Now for the associativity we have: If $-x\in y\# \; w$, then $-w\in x\# \; y$ and so: $x\# \; (y\# \; w)=H=(x\# \; y)\# \; w$. If $y=0$ the proof is obvious. Finally, if $y\neq 0$ and $-x\notin y\# \; w$, then: $-w\notin x\# \; y$, $y\neq -w$, 0, $-x\notin y+w$, $-w\notin x+y$, $x\neq -y$ and $x\neq -w$. Thus we have:

$$\begin{aligned} x\# \; (y\# \; w) &= x\# \; [(y+w)\cup\{y, w\}] \\ &= [x\# \; (y+w)]\cup(x\# \; y)\cup(x\# \; w) \\ &= x+(y+w)\cup\{x\}\cup(y+w) \\ &\quad \cup(x+y)\cup\{x, y\}\cup(x+w)\cup\{x, w\} \\ &= x+(y+w)\cup\{x, y, w\} \\ &\quad \cup(x+y)\cup(x+w)\cup(y+w). \end{aligned}$$

In a similar way we have that:

$$(x\# \; y)\# \; w = x+(y+w)\cup\{x, y, w\}$$

$$\cup (x+y) \cup (x+w) \cup (y+w).$$

Finally for the proof of the distributive axiom we have:
If $x \neq -y$, then:

$$\begin{aligned}(x\# \; y).w &= [(x+y) \cup \{x, \; y\}].w \\ &= (x+y).w \cup \{x.w, \; y.w\} \\ &= (x.w+y.w) \cup \{x.w, \; y.w\} \\ &= (x.w\# \; y.w (x.w \neq -y \; .w).\end{aligned}$$

If $x = -y$ then

$$(x\# \; y).w = H.w = H = x.w\# \; (-x).w = x.w\# \; y.w.$$

Thus $(x\# \; y).w = x.w\# \; y.w$ for every x, y, $w \in H$. Similarly we can prove that: $x.(y\# \; w) = x.y\# \; x.w$. The multiplicative axioms are still valid and so $(H, \#, .)$ is a hyperfield. The construction we have just developed shows that every hyperfield can give a new hyperfield, which can be generated as the difference of every non zero element from itself.

Remark. If $(H, +, .)$ is field, then $x\# \; y = \{x, \; y, \; x+y\}$ when $x \neq -y$, x, $y \neq 0$.

Example. Let us consider the quotient hyperfield $(C/R^+, +, .)$ where C is the field of complex numbers and R^+ the set of the positive real numbers. The elements of this hyperfield are the rays of the convex field with origin the point (0, 0). We see that the sum of two elements aR^+, bR^+ of C/R^+ with $aR^+ \neq bR^+$ gives the interior rays xR^+ of the angle which is created from these two elements of C/R^+, while the sum of two opposite elements gives these two elements and the 0. Now let us apply the above construction to $(C/R^+, +, .)$. We observe that the sum $aR^+ + bR^+$ of two elements aR^+, bR^+ with $aR^+ \neq -bR^+$ is the set of all the rays that are contained in the angle which is constructed from these two elements (i. e. including the two sides of this angle), while the sum of two opposite elements is the whole C/R^+.

Proposition 4. *Let $H = F/G$ be a quotient hyperfield of a field F with a multiplicative subgroup G of F^*. If we apply the above construction on F/G then the new hyperfield which derives is isomorphic to a quotient hyperfield.*

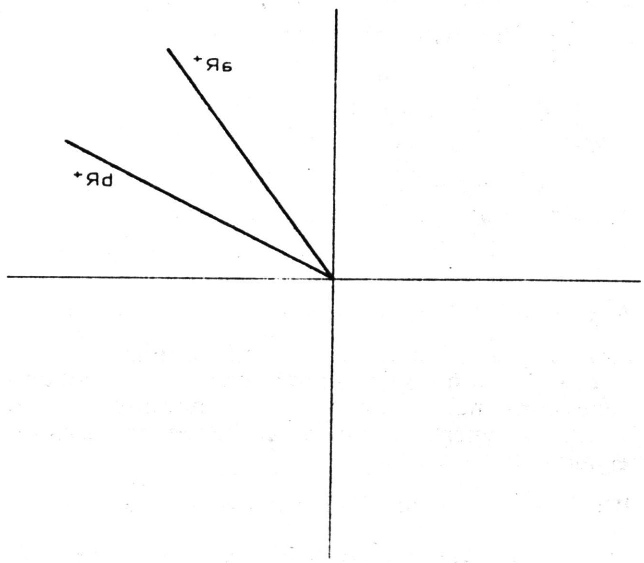
Proof. This proof is based on the same idea with the proof of Proposition 3. So we consider the field of the quotients $F(x)$ of polynomials' ring over F. We can suppose that in all the rational functions the coefficient of the highest power of the denominator's polynomial equals to 1 (if not, we can get it with a simple division). Now for the group G of F there exists a corresponding group Q in $F(x)$, which is defined as follows:

$$Q = \{\pi(x) \in F(x) \text{ with } \alpha_\mu \in G\}$$

where α_μ is the coefficient of the numerator's highest power. Next we get the hyperfield $H^\sim = F(x)/Q$ and we consider the function $\varphi: H \to H^\sim$, defined as follows: $\varphi(\alpha G) = \alpha Q$, for every $\alpha \in H$. This function is injective, because, if $\alpha G \neq \beta G$ then

$$\alpha Q = \{\pi(x) \in F(x) \text{ with } \alpha_\mu \in \alpha G\}$$

$$\beta Q = \{\pi(x) \in F(x) \text{ with } \alpha_\mu \in \beta G\}$$



thus $\alpha Q \neq \beta Q$. Moreover, let $\pi(x) \in F(x)$, then if α_μ is the coefficient of its highest power, then $(1/\alpha_\mu)\pi(x)$ belongs to Q, and so $\pi(x)$ belongs to $\alpha_\mu Q$, and therefore φ is also surjective. Now let $\pi_1(x)$, $\pi_2(x)$ belong to Q and

$$\pi_1(x) = \frac{\sum_{i=1}^{k} \alpha_i x^i}{\sum_{j=1}^{\lambda} \beta_j x^j}$$

$$\pi_2(x) = \frac{\sum_{i=1}^{k} \alpha'_i x^i}{\sum_{j=1}^{\lambda} \beta_j x^j}$$

(we suppose that the rational functions have the same denominator and if not we make them such).

Now we consider the sum $\alpha Q + \beta Q$ with $\alpha Q \neq -\beta Q$. Then

i) if $k > k'$, then the highest power of x in $\alpha\pi_1(x) + \beta\pi_2(x)$ will have coefficient $\alpha\alpha'_k$, thus $\alpha\pi_1(x) + \beta\pi_2(x)$ will belong to αQ, while the highest power of x in $\alpha\pi_2(x) + \beta\pi_1(x)$ will have coefficient $\beta\alpha_k$, and so $\alpha\pi_2(x) + \beta\pi_1(x)$ will belong to βQ.

ii) if $k = k'$, then the coefficient of the highest power of $\alpha\pi_1(x) + \beta\pi_2(x)$ will be $\alpha\alpha_k + \beta\alpha'_k$ and when the α_k, α'_k get, one after the other, all the values in G, then this coefficient will get all the possible values in $\alpha G + \beta G$.

A problem in the theory of fields

The problem which arises after the above constructions of the monogene hyperfields and from the propositions that have been proved is:

Are there monogene hyperfields that are non quotient hyperfields?

But this problem leads to another problem in the theory of fields. Indeed: Let H be a monogene hyperfield, and let us suppose that H is isomorphic to some quotient hyperfield K/G. Then in K/G the equality $xG-xG=K/G$ holds for every $xG \neq 0$. Thus $G-G=K/G$, and so for K the equality $G-G=K$ must be valid. And the question arises:

Which fields can be written as a difference of a subgroup of their multiplicative group from itself, and which are these subgroups?

In [4] an answer to this problem is given for the case of some finite fields. We remark that in [5], [6], [7], [9] one can find theorems which prove the existence of non quotient hyperfields.

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