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Non-Zero Solutions for a Homogeneous Semilinear Elliptic Equation

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In the present paper the existence of non-zero solutions in $H_0^1(\Omega)$ for the equation $\Delta u+\alpha|u|+\lambda u=0$ is studied. The main result is that if μ_k is an eigenvalue of the Laplace operator, then there exists an interval for α, depending on the adjacent eigenvalues only such that there are at least two continuous functions $\lambda^-(\alpha)\leqq\lambda^+(\alpha)$ such that the equation has non-zero solutions, i.e. that bifurcation occurs from any eigenvalue. Thus results concerning simple eigenvalues only are extended.

The present paper treats the existence of nontrivial solutions in $H_0^1(\Omega)$ of the equation

$$(1)\qquad \Delta u+\alpha|u|+\lambda u=0$$

in a bounded region $\Omega\subset R^n$. Sometimes this equation is written in the form

$$(2)\qquad \Delta u+\lambda^+u^+-\lambda^-u^-=0,$$

where $u^+=\max(u, 0)$, $u^-=-\min(u, 0)$, the relation between the parameters α, λ, λ^+ and λ^- being obvious. In a number of works the set of $(\lambda^+, \lambda^-)\in R^2$ for which the equation (2) has non-trivial solutions is called resonance set. Adopting this terminology, we call also resonance set the set of those $(\alpha, \lambda)\in R^2$, for which (1) has non-trivial solutions. The study of this set is far from complete. On the other hand, the condition (α, λ) n o t in the resonance set makes part of the hypotheses of a number of theorems (cf. [1]). Thus the structure of this set is of certain interest. Some partial results are contained in [2] and [3]. In particular in [3] it is proved that if μ_k is a simple eigenvalue for the Laplace operator, then in the notations of (1) there exist e x a c t l y two continuous functions $\lambda_1(\alpha)\leqq\lambda_2(\alpha)$ defined for $|\alpha|<\min((\mu_{k+1}-\mu_k)/2, (\mu_k-\mu_{k-1})/2)$, such that the equation (1) has non-trivial solutions for the pairs $(\alpha, \lambda_1(\alpha))$, $(\alpha, \lambda_2(\alpha))$, i.e. these pairs belong to the resonance set of (1). Moreover, $\lim_{\alpha\to 0}\lambda_i(\alpha)=\mu_k$ $(i=1, 2)$ holds.

In the present paper, dropping the assumption that μ_k is simple, we obtain a similar result in the sense that there exist a t l e a s t two continuous functions $\lambda_1(\alpha)\leqq\lambda_2(\alpha)$ defined for the same values of α, such that the pairs $(\alpha, \lambda_1(\alpha))$ and $(\alpha, \lambda_2(\alpha))$ belong to the resonance set of (1).

The approach we use is suggested by the one used in [4] and [5] for the study of the bifurcations from eigenvalues of smooth potential operators.

In the sequal we use the following notations. For a bounded region $\Omega \subset R^n$, $H_0^1(\Omega)$ is the standard Sobolev space with scalar product $\int_\Omega \nabla u . \nabla v \, dx$ and the corresponding norm. The pairing between $H_0^1(\Omega)$ and its dual $H^{-1}(\Omega)$ is denoted by $\langle\cdot, \cdot\rangle$. Also $(\cdot, \cdot)$ and $\|\cdot\|$ are the usual scalar product and norm in $L^2(\Omega)$.

Let

$$0<\mu_1<\mu_2<\dots \tag{3}$$

be the eigenvalues of the Laplace operator in Ω listed in growing order (i.e. the corresponding eigenspace is not necessarily of dimension one).

Our main result is the following theorem:

Theorem. *Let μ_k $(k \geq 2)$ be an eigenvalue for the Laplace operator in a bounded region $\Omega \subset R^n$. Then for $|\alpha| < \min\{(\mu_{k+1}-\mu_k)/2, (\mu_k-\mu_{k-1})/2\}$ there exist at least two continuous functions $\lambda^-(\alpha) \leq \lambda^+(\alpha)$ with $\lambda^\pm(0)=\mu_k$ such that the equation* (1) *has non-trivial solutions for the pairs $(\alpha, \lambda^-(\alpha))$ and $(\alpha, \lambda^+(\alpha))$. Moreover, if $\int_\Omega |u|u \, dx \not\equiv 0$ on* $\operatorname{Ker}(\Delta+\mu_k)$, *then $\lambda^-(\alpha)<\lambda^+(\alpha)$ for α in a neighbourhood of* 0, $\alpha \neq 0$.

Remark. Since μ_1 is simple and the corresponding eigenfunctions are of constant sign this case does not present any difficulty.

Proof. Let $W=\operatorname{Ker}(\Delta+\mu_k)$ and let

$$D(\Delta)=\{u \in H_0^1(\Omega) : \Delta u \in L^2(\Omega)\}.$$

As it is well known, under some regularity conditions on $\partial\Omega$ we have $D(\Delta)=H^2(\Omega) \cap H_0^1(\Omega)$. Let us note also that any solution $u \in H_0^1(\Omega)$ of (1) in fact is in $D(\Delta)$ and thus without loss of generality we can consider the equation (1) for $u \in D(\Delta)$ only. Furthermore let

$$V=\{u \in L^2(\Omega) : k \perp W \text{ in } L^2(\Omega)\}$$

with the norm induced by $L^2(\Omega)$.

For any $\lambda \in R$ we can consider $\Delta+\lambda$ as an unbounded operator in $L^2(\Omega)$. It is easy to see that

$$\Delta+\lambda : V \cap D(\Delta) \to V.$$

Moreover, for $\lambda \in (\mu_{k-1}, \mu_{k+1})$ this operator is invertible on V and the norm of its inverse $(\Delta+\lambda)^{-1}$ as operator from V into V is

$$\|(\Delta+\alpha)^{-1}\| = \max\{(\lambda-\mu_{k-1})^{-1}, (\mu_{k+1}-\lambda)^{-1}\}. \tag{4}$$

Let P and Q be the orthogonal projections of $L^2(\Omega)$ on W and V respectively so that every $u \in L^2(\Omega)$ is decomposed as $u=v+w$ with $w=Pu$, $v=Qu$. Now it is clear that (1) is equivalent to the following system of equations

$$\Delta v+\lambda v+\alpha Q|v+w|=0, \tag{5}$$

$$(\lambda-\mu_k)w+\alpha P|v+w|=0. \tag{6}$$

Let

$$m=\min\{\mu_{k+1}-\mu_k, \mu_k-\mu_{k-1}\}, \tag{7}$$

$$\lambda \in [\mu_k-m/2, \mu_k+m/2], \tag{8}$$

$$|\alpha|<m/2. \tag{9}$$

Under these assumptions (5) has unique solution for every $w \in W$. Indeed, (5) is equivalent to

(10) $$v = -\alpha(\Delta + \lambda)^{-1} Q\,|v + w|.$$

From (4) and (7)-(9) it follows that the right-hand side is a Lipschitz continuous function on V with respect to the $L^2(\Omega)$ norm with a constant

(11) $$q = |\alpha|\;\|(\Delta + \lambda)^{-1}\| < 1.$$

Now the existence and the uniqueness of the solution follow from the contracting mapping principle. Let us denote this solution, which depends on α and λ also by

(12) $$v = \varphi(w, \alpha, \lambda).$$

To solve the equation (1) now it remains to find values of α and λ (related between them) for which there exist non-zero solutions of the equation

(13) $$(\lambda - \mu_k)\,v + \alpha P|\,w + \varphi(w, \alpha, \lambda)| = 0$$

arising now from (6). (Evidently $w = 0$ implies $\varphi(0, \alpha, \lambda) = 0$ for all α and λ, hence $w = 0$ is solution for all α, λ.)

The part of the proof that follows is very similar to the argument in [1], Prop. 2.1, so we only sketch it here. Let us consider the functional

(14) $$I_{\alpha,\lambda}(u) = \frac{1}{2}\int_\Omega \{|\nabla u|^2 - \alpha\,|u|u - \lambda u^2\}\,dx$$

defined for $u \in H_0^1(\Omega)$. As it is well known this functional is differentiable in $H_0^1(\Omega)$, its derivative being

(15) $$DI_{\alpha,\lambda}(u) = -\Delta u - \alpha\,|u| - \lambda u.$$

It is easy to see that the partial derivatives of $I_{\alpha,\lambda}$ with respect to the subspaces W and

$$V \cap H_0^1(\Omega) = \{v \in H_0^1(\Omega) : v \perp W \text{ in } L^2(\Omega)\}$$

denoted by $D_W I_{\alpha,\lambda}$ and $D_V I_{\alpha,\lambda}$ respectively exist and are equal to

(16) $$D_V I_{\alpha,\lambda}(u) = -\Delta v - \alpha Q\,|u| - \lambda v$$

and

(17) $$D_W I_{\alpha,\lambda}(u) = -\Delta w - \alpha P\,|u| - \lambda w = (\mu_k - \lambda)w - \alpha P\,|u|$$

all derivatives being elements of $H^{-1}(\Omega)$.

Now (1) and (15) show that we are looking for the non-trivial critical points of $I_{\alpha,\lambda}$, this critical points belonging in fact to $D(\Delta)$. It is not difficult to see now, even though the proof is tedious, that (5) and (16) imply (following for instance the argument in [1], Prop. 2.1 with $s = 0$, $a = (\alpha + \lambda)/2$, $b = (\alpha - \lambda)/2$ in the notations adopted there) that the solutions of (5) obtained above are in fact saddle points for $I_{\alpha,\lambda}$ considered on the subspace V for w fixed. This means that $I_{\alpha,\lambda}$ for fixed w is convex-concave on certain subspaces of V (cf. [1], [6], Lemma 2.2). Hence we can apply Lemma 2.3 in [6] according to which the functional on W

(18) $$J_{\alpha,\lambda}(w) = I_{\alpha,\lambda}(w + \varphi(w, \alpha, \lambda))$$

is differentiable even though the function $\varphi(w, \alpha, \lambda)$ defined in (12) is only Lipschitz continuous. Moreover, in the above notations the equality

$$DJ_{\alpha,\lambda}(w)=D_W I_{\alpha,\lambda}(w+\varphi(w, \alpha, \lambda)) \tag{19}$$

holds. Now (13), (17) and (19) imply that it is sufficient to find the non-trivial critical points of the functional (18) in order to solve the problem, i. e. if we find a $w_0 \neq 0$ such that $DJ_{\alpha,\lambda}(w_0)=0$ then $u_0=w_0+\varphi(w_0, \alpha, \lambda)$ is a non-zero solution of (1) (cf. again [1], Prop. 2.1).

In the sequal instead of the differential $DJ_{\alpha,\lambda}$ we shall make use of the gradient $\nabla J_{\alpha,\lambda}\in W$ which is defined in the usual manner when W is considered as a Hilbert space with a scalar product induced from $L^2(\Omega)$. It is easy to see that we have

$$\nabla J_{\alpha,\lambda}(w)=(\mu_k-\lambda)w-\alpha P|w+\varphi(w, \alpha, \lambda)| \tag{20}$$

too in this case since the right-hand side in (20) belongs to $L^2(\Omega)$.

Let us suppose now that dim $W>1$. Then the set $S=\{w\in W: \|w\|=1\}$ is a sphere and as it is well known if w_0 with $\|w_0\|=1$ is a critical point of $J_{\alpha,\lambda}$ on S, then

$$\nabla J_{\alpha,\lambda}(w_0)=\tau w_0 \tag{21}$$

for some real number τ (the Lagrange multiplyer). In the case *dim* $W=1$ we cannot apply the same argument, but now S reduces to two points and the equivalent of (21) obviously holds for both. We shall turn our attention to the one dimensional case later.

We proceed as follows. For every pair α, λ the functional $J_{\alpha,\lambda}$ has at least two critical points on S (minimum and maximum) and we have the analogue of (21) for both. Then we look for values of α and λ such that the corresponding number τ be zero. First we consider the maximum. In more detail let $w(\alpha, \lambda)$ be a point in which the maximum of $J_{\alpha,\lambda}$ on S is attained, i. e. we have

$$J_{\alpha,\lambda}(w(\alpha, \lambda))\geqq J_{\alpha,\lambda}(w) \ \forall \ \|w\|=1.$$

According to (21), we have

$$\nabla J_{\alpha,\lambda}(w(\alpha, \lambda))=\tau(\alpha, \lambda)\, w(\alpha, \lambda).$$

The definition of $\tau(\alpha, \lambda)$ seems ambiguous, but it is a well defined function of α and λ even though the point $w(\alpha, \lambda)$ may not be uniquely determined. Indeed, let us note that the functional $J_{\alpha,\lambda}$ is homogeneouos of second order and hence its gradient $\nabla J_{\alpha,\lambda}$ is homogeneous of first order. For the proof cf. for instance [1], Lemma 3.2. Since $\|w(\alpha, \lambda)\|=1$ this implies

$$\tau(\alpha, \lambda)=(\nabla J_{\alpha,\lambda}(w(\alpha, \lambda)), w(\alpha, \lambda))=2J_{\alpha,\lambda}(w(\alpha, \lambda))=2\max_{\|w\|=1} J_{\alpha,\lambda}(w) \tag{22}$$

and the last term is a well defined function of α and λ. Let us note also that because of the homogeneity there is no loss of generality considering critical values on S only.

Before proceeding further on we need study in more detail the continuity properties of the function $\varphi(w, \alpha, \lambda)$. We claim that $\varphi(w, \alpha, \lambda)$ is Lipschitz function of any of its arguments when α varies in a compact subset of $(-m/2, m/2)$ and (8) holds. Let

(23) $$|\alpha| \leq m/2-\varepsilon$$

for some $\varepsilon>0$. Then as is easily seen from (11) we have $q \leq c(\varepsilon)<1$, where the constant $c(\varepsilon)$ does not depend on α. The function $\varphi(w, \alpha, \lambda)$ is a Lipschitz function of w with a constant $q/(1-q) \leq C(\varepsilon)$ for some other $C(\varepsilon)$ independent of α and λ, i.e. for all α and λ satisfying (23) and (8) respectively we have

(24) $$\|\varphi(w_1, \alpha, \lambda)-\varphi(w_2, \alpha, \lambda)\| \leq C(\varepsilon)\|w_1-w_2\|.$$

In the sequal we denote by $C(\varepsilon)$ various constants of this type. Let λ and μ satisfy (8). Putting for brevity $v_\lambda=\varphi(w, \alpha, \lambda)$ and $v_\mu=\varphi(w, \alpha, \mu)$ we have (cf. (10))

$$v_\lambda=-\alpha(\Delta+\lambda)^{-1} Q|w+v_\lambda|, \quad v_\mu=-\alpha(\Delta+\mu)^{-1} Q|w+v_\mu|$$

whence

(25) $$\begin{aligned} v_\mu-v_\lambda= & \alpha\{(\Delta+\lambda)^{-1}-(\Delta+\mu)^{-1}\} Q|w+v_\lambda| \\ & +\alpha(\Delta+\mu)^{-1} Q\{|w+v_\lambda|-|w+v_\mu|\}. \end{aligned}$$

Now (25) and the so-called resolvent formula

$$(\Delta+\lambda)^{-1}-(\Delta+\mu)^{-1}=(\mu-\lambda)(\Delta+\lambda)^{-1}(\Delta+\mu)^{-1}$$

imply

$$\begin{aligned} v_\mu-v_\lambda= & \alpha(\mu-\lambda)(\Delta+\lambda)^{-1}(\Delta+\mu)^{-1} Q|w+v_\lambda| \\ & +\alpha(\Delta+\mu)^{-1} Q\{|w+v_\lambda|-|w+v_\mu|\}. \end{aligned}$$

Now (11) implies

$$\|v_\mu-v_\lambda\| \leq|\mu-\lambda| q \frac{m}{2}\|Q|w+v_\lambda|\|+q\|v_\mu-v_\lambda\|$$

whence

$$\|v_\mu-v_\lambda\| \leq C(\varepsilon)\|w+v_\lambda\||\mu-\lambda|.$$

From $\varphi(0, \alpha, \lambda)=0$ and (24) it follows that $\|w+v_\lambda\| \leq C(\varepsilon)\|w\|$ whence

(26) $$\|\varphi(w, \alpha, \lambda)-\varphi(w, \alpha, \mu)\| \leq C(\varepsilon)\|w\||\mu-\lambda|.$$

In a similar way is obtained also

$$\|\varphi(w, \alpha, \lambda)-\varphi(w, \beta, \lambda)\| \leq C(\varepsilon)\|w\||\alpha-\beta|$$

for all α and β with (23) and λ satisfying (8).

Thus we have proved that the function φ is continuous function into V with respect to the $L^2(\Omega)$ norm in V. Since (5) can be rewritten as

$$-\Delta \varphi(w, \alpha, \lambda)=\lambda \varphi(w, \alpha, \lambda)+\alpha Q|w+\varphi(w, \alpha, \lambda)|$$

and the right-hand side is continuous function in $L^2(\Omega)$, we can use the fact that the operator $-\Delta$ is invertible on $H^{-1}(\Omega)$ its inverse $(-\Delta)^{-1}$ being continuous from $H^{-1}(\Omega)$ into $H_0^1(\Omega)$ and hence from $L^2(\Omega)$ into $H_0^1(\Omega)$ and assert that the function φ is continuous with respect to the norm of $H_0^1(\Omega)$ too. Argument similar to the above shows that for α satisfying (23) and λ as in (8), the function φ is a Lipschitz function of λ into $H_0^1(\Omega)$, i.e. inequality of the form

(27) $\int_\Omega |\nabla(\varphi(w,\ \alpha,\ \lambda)-\varphi(w,\ \alpha,\ \mu))|^2\,dx \leqq C(\varepsilon)\,|\lambda-\mu|^2\,\|w\|^2$

holds.

Let us denote

(28) $F(\alpha,\ \lambda,\ w)=J_{\alpha,\lambda}(w)=I_{\alpha,\lambda}(w+\varphi(w,\ \alpha,\ \lambda)).$

Next we prove that the function F has partial derivative with respect to λ. To this end let λ and $\lambda+\varkappa$ satisfy (8). If

$$\delta\varphi=\varphi(w,\ \alpha,\ \lambda+\varkappa)-\varphi(w,\ \alpha,\ \lambda)$$

elementary computations show that we have the representation

$$\text{(29)}\quad F(\alpha,\ \lambda+\varkappa,\ w)-F(\alpha,\ \lambda,\ w)=-\frac{\varkappa}{2}\int_\Omega (w+\varphi(w,\ \alpha,\ \lambda))^2\,dx$$

$$+\int_\Omega \nabla(w+\varphi(w,\ \alpha,\ \lambda))\nabla\delta\varphi\,dx+\frac{1}{2}\int_\Omega|\nabla\delta\varphi|^2\,dx$$

$$-\alpha\int_\Omega |w+\varphi(w,\ \alpha,\ \lambda)|\,\delta\varphi\,dx+o(\|\delta\varphi|)$$

$$-\lambda\int_\Omega (w+\varphi(w,\ \alpha,\ \lambda))\,\delta\varphi\,dx-\frac{\lambda}{2}\int_\Omega(\delta\varphi)^2\,dx$$

$$-\varkappa\int_\Omega (w+\varphi(w,\ \alpha,\ \lambda))\,\delta\varphi\,dx-\frac{\varkappa}{2}\int_\Omega(\delta\varphi)^2\,dx$$

$$=-\int_\Omega\{\Delta(w+\varphi(w,\ \alpha,\ \lambda))+\alpha|w+\varphi(w,\ \alpha,\ \lambda)|+\lambda(w+\varphi(w,\ \alpha,\ \lambda))\}\,\delta\varphi\,dx$$

$$-\frac{\varkappa}{2}\int_\Omega(w+\varphi(w,\ \alpha,\ \lambda))^2\,dx+\frac{1}{2}\int_\Omega|\nabla\delta\varphi|^2\,dx-\frac{(\lambda+x)}{2}\|\delta\varphi\|^2+o(\|\delta\varphi\|).$$

The first integral in fact is

$$-\int_\Omega(\Delta w+\alpha P|w+\varphi(w,\ \alpha,\ \lambda)|+\lambda w)\,\delta\varphi\,dx$$

$$-\int_\Omega(\Delta\varphi(w,\ \alpha,\ \lambda)+\alpha Q(w+\varphi(w,\ \alpha,\ \lambda)|+\lambda\varphi(w,\ \alpha,\ \lambda))\,\delta\varphi\,dx=0$$

since $\varphi(w,\alpha,\lambda)$ by definition satisfies the equation (5) and $\delta\varphi$ is orthogonal to W. From (26) and (27) it follows that the last terms in (29) are $o(\varkappa)$. Thus the function $F(\alpha,\ \lambda,\ w)$ is differentiable with respect to λ and

$$\text{(30)}\quad F'_\lambda(\alpha,\ \lambda,\ w)=-\frac{1}{2}\int_\Omega(w+\varphi(w,\ \alpha,\ \lambda))^2\,dx.$$

Let us now return to the function $\tau(\alpha,\lambda)$ defined by (22). We claim that it is continuous. Under the notations of (22) and (28) we have

$$\tau(\alpha,\ \lambda)=2\max_{\|w\|=1}F(\alpha,\ \lambda,\ w).$$

On the compact set $\{\|w\|=1\}\times[\mu_k-m/2,\ \mu_k+m/2]\times[-m/2+\varepsilon,\ m/2-\varepsilon]$ the function F is uniformely continuous, i. e. in particular for every $\eta>0$ there exists $\delta>0$ such that for every $|\lambda'-\lambda''|<\delta$, $|\alpha'-\alpha''|<\delta$ we have

$$|F(\alpha', \lambda', w)-F(\alpha'', \lambda'', w)|<\eta \tag{31}$$

for all $\|w\|=1$. Let α_0, λ_0 be fixed and let $|\alpha-\alpha_0|<\delta$, $|\lambda-\lambda_0|<\delta$. According to the definition there exists w_0 such that

$$\tau(\alpha_0, \lambda_0)=2F(\alpha_0, \lambda_0, w_0). \tag{32}$$

Then

$$\tau(\alpha, \lambda)=2\max_{\|w\|=1} F(\alpha, \lambda, w)\geqq 2F(\alpha, \lambda, w_0) \tag{33}$$
$$>2F(\alpha_0, \lambda_0, w_0)-2\eta=\tau(\alpha_0, \lambda_0)-2\eta.$$

On the other hand, (31) implies $F(\alpha, \lambda, w)<F(\alpha_0, \lambda_0, w)+\eta$ for all $\|w\|=1$ and since $F(\alpha_0, \lambda_0, w)\leqq F(\alpha_0, \lambda_0, w)$, from (32) we obtain

$$F(\alpha, \lambda, w)<\frac{1}{2}\tau(\alpha_0, \lambda_0)+\eta$$

for all $\|w\|=1$ whence also

$$\tau(\alpha, \lambda)=2\max_{\|w\|=1} F(\alpha, \lambda, w)\leqq\tau(\alpha_0, \lambda_0)+2\eta. \tag{34}$$

Now the assertion follows from (33) and (34).

Next we prove that $\tau(\alpha, \lambda)$ is strictly decreasing function of λ. Indeed, (30) with $\|w\|=1$ implies for fixed w and α that

$$F'_{\alpha}(\alpha, \lambda, w)=-1-\|\varphi(w, \alpha, \lambda)\|^2<0,$$

i. e. for fixed α and w, $\lambda'<\lambda''$ implies

$$F(\alpha, \lambda', w)>F(\alpha, \lambda'', w).$$

Now we have

$$\tau(\alpha, \lambda')=2F(\alpha, \lambda', w'), \quad \tau(\alpha, \lambda'')=2F(\alpha, \lambda'', w'')$$

whence

$$\tau(\alpha, \lambda'')=2F(\alpha, \lambda'', w'')<2F(\alpha, \lambda', w'')\leqq 2F(\alpha, \lambda', w')=\tau(\alpha, \lambda').$$

Now we can return to the main problem, i. e. the study of the set of points (α, λ) such that $\tau(\alpha, \lambda)=0$. First we show that

$$\tau(\alpha, \mu_k+m/2)\neq 0, \quad \tau(\alpha, \mu_k-m/2)\neq 0. \tag{35}$$

Let $Q_i=\{(\alpha, \lambda): \mu_i+|\alpha|<\lambda<\mu_{i+1}-|\alpha|\}$, $i=1, 2, \dots$, $Q_0=\{(\alpha, \lambda): \lambda<\mu_1-|\alpha|\}$. It is well known and can be easily seen that the equation $\Delta u+\alpha|u|+\lambda u=0$ has zero as an unique solution for $(\alpha, \lambda)\in Q_i$, $i=0, 1, 2, \dots$ Indeed, the right-hand side of

$$u=-\alpha(\Delta+\lambda)^{-1}|u| \tag{36}$$

is a Lipschitz function with a constant $q<1$ for such (α, λ), and so the equation (36) has unique solution and $u=0$ is a solution.

Since for every (α_0, λ_0) such that $\tau(\alpha_0, \lambda_0)=0$ the function $u=\bar{w}+\varphi(\bar{w}, \alpha, \lambda)$ where $\bar{w}$ is such that $\tau(\alpha_0, \lambda_0)=2\max F(\alpha_0, \lambda_0, \bar{w})$ is a solution of (36) which is different from zero since $\|\bar{w}\|=1$, we have that this $(\alpha_0, \lambda_0)\notin Q_k \cup Q_{k-1}$. Since obviously $(\alpha, \mu_k-m/2)\in Q_{k-1}$ $(\alpha, \mu_k+m/2)\in Q_k$ for $|\alpha|<m/2$, this proves (35).

Now (35) implies that the functions $\tau(\alpha, \mu_k+m/2)$ and $\tau(\alpha, \mu_k-m/2)$ have constant sign. To determine this sign we calculate it for $\alpha=0$. Now (10) and (12) imply $\varphi(w, 0, \lambda)\equiv 0$ and we can calculate from (18) and (28) the value of $F(0, \lambda, w)$ for $\|w\|=1$ and obtain

$$F(0, \lambda, w)=\frac{1}{2}\int_{\Omega}(|\nabla w|^2-\lambda w^2)\,dx=\frac{1}{2}(\mu_k-\lambda),$$

whence

$$\tau(0, \mu_k+m/2)=2F(0, \mu_k+m/2, w)=-m/2$$

$$\tau(0, \mu_k-m/2)=2F(0, \mu_k-m/2, w)=m/2.$$

Thus the functions in (35) are of different sign. All this allows to conclude that for every $|\alpha|\leqq m/2-\varepsilon$ there exists an unique $\lambda(\alpha)$ such that $\tau(\alpha, \lambda(\alpha))=0$. It remains to prove that the function $\lambda(\alpha)$ thus obtained is continuous. To this end let $\alpha_\nu\to\alpha_0$. Since the sequence $\lambda(\alpha_\nu)$ is bounded, we can choose a subsequence $\lambda(\alpha_{\nu_k})\to\bar{\lambda}$. For this subsequence we have

$$\tau(\alpha_0, \bar{\lambda})=\lim_{k\to\infty}\tau(\alpha_{\nu_k}, \lambda(\alpha_{\nu_k}))=0.$$

Since $\lambda(\alpha_0)$ is the unique solution of $\tau(\alpha_0, \lambda)=0$, we must have $\bar{\lambda}=\lambda(\alpha_0)$, i.e. the function $\lambda(\alpha)$ is continuous. At last the fact that $(\alpha, \lambda(\alpha))\notin Q_{k-1}\cup Q_k$ implies

$$\mu_k-|\alpha|\leqq\lambda(\alpha)\leqq\mu_k+|\alpha|$$

whence $\lambda(0)=\mu_k$.

Since the argument holds for every $\varepsilon>0$, the result is in fact valid for $|\alpha|<m/2$.

A similar argument holds if instead of $\max F(\alpha, \lambda, w)$ we consider $\min F(\alpha, \lambda, w)$. Let $\tau^+(\alpha, \lambda)=\tau(\alpha, \lambda)=2\max_{\|w\|=1} F(\alpha, \lambda, w)$ and $\tau^-(\alpha, \lambda)=2\min F(\alpha, \lambda, w)$. Let $\lambda^+(\alpha)=\lambda(\alpha)$ and let $\lambda^-(\alpha)$ be its analogue determined by $\tau^-(\alpha, \lambda)=0$. Obviously $\tau^-(\alpha, \lambda)\leqq\tau^+(\alpha, \lambda)$ whence $0=\tau^-(\alpha, \lambda^-(\alpha))\leqq\tau^+(\alpha, \lambda^-(\alpha))$. Now $\tau^+(\alpha, \lambda^-(\alpha))\geqq 0$, $\tau^+(\alpha, \lambda^+(\alpha))=0$ and the fact that the function $\tau^+(\alpha, \lambda)$ is strictly decreasing imply $\lambda^+(\alpha)\geqq\lambda^-(\alpha)$.

At last, let for some α, $\lambda^-(\alpha)=\lambda^+(\alpha)$. According to the definitions this means that for this α and $\lambda(\alpha)=\lambda^+(\alpha)=\lambda^-(\alpha)$ we have

$$F(\alpha, \lambda(\alpha), w)\equiv 0 \tag{37}$$

for all $\|w\|=1$. On the other hand, from (20), the homogeneity of $J_{\alpha,\lambda}$ and (28) it follows that

(38) $$2F(\alpha, \lambda, w)=(\nabla J_{\alpha,\lambda}(w), w)=(\mu_k-\lambda)-\alpha\int_\Omega |w+\varphi(w, \alpha, \lambda)|\, w\, dx.$$

Now (37) and (38) imply

$$\frac{\mu_k-\lambda(\alpha)}{\alpha}=\int_\Omega |w+\varphi(w, \alpha, \lambda)|\, w\, dx$$

for $\alpha \neq 0$ and all $\|w\|=1$, i.e. $\int_\Omega |w+\varphi(w, \alpha, \lambda)|\, w\, dx$ has constant sign for all w. Since the function $\varphi(w, \alpha, \lambda)$ is continuous and $\varphi(w, 0, \lambda)\equiv 0$, clearly this cannot be the case if $\int_\Omega |w|\, w\, dx \not\equiv 0$ and $\alpha \neq 0$ is small enough.

This completes the proof of the theorem.

Remark. Throughout the proof of the theorem we have not used in an essential way the assumption $dim\, W>1$. (See the remark following (21).) If on the contrary $dim\, W=1$, everything is much easier. Indeed, $W=\{\lambda e_0, \lambda\in R\}$ in this case and now we have that $\tau^+(\alpha, \lambda)$ is equal either to $F(e_0, \alpha, \lambda)$ or to $F(-e_0, \alpha, \lambda)$. This fact only simplifies much of the argument as regards the study of the function $\tau^+(\alpha, \lambda)$. Moreover, since in this case the solutions of (1) thus obtained are of the form $u(\alpha, \lambda)=e_0+\varphi(e_0, \alpha, \lambda)$ or $-e_0+\varphi(-e_0, \alpha, \lambda)$ we can affirm that to the continuous functions $\lambda^+(\alpha)$ and $\lambda(\alpha)$ (which provide the only possible elements of the resonance set in this case) correspond continuous curves $u(\alpha, \lambda^{\pm}(\alpha))$.

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