Mathematica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

Mathematica Balkanica

New Series Vol. 5, 1991, Fasc. 4

Best Approximation in Linear Spaces Endowed with Subinner Products

Sever S. Dragomir

Presented by P. Kenderov

Some characterizations of S-best approximation element, of S-proximinal, S-semitchebychefian and S-tchebychefian linear subspaces in a linear space endowed with a subinner product are given.

1. Introduction

Let E be a linear space over real or complex number field K. A mapping $(,)_S$ of $E\times E$ into K will be called subinner product on E if the following conditions (P1)–(P3) are satisfied:

(P1) $\quad (x, x)_S \neq 0$ if $x \neq 0$;

(P2) $\quad (\lambda x, y)_S = \lambda (x, y)_S$ and $(x, \lambda y)_S = \bar{\lambda}(x, y)_S$ for all $\lambda \in$ K and x, y in E;

(P3) $\quad (x+y, z)_S = (x, z)_S + (y, z)_S$ for all x, y, z in E.

This concept is a natural generalization of inner product, of semi-inner product in the sense of G. Lumer [5], of semi-inner product in the sense of R. A. Tapia defined on smooth normed spaces [8] and of R-semi-inner product which was introduced in paper [2].

Definition 1. Let E be a linear space and $(,)_S$ be a subinner product on it. The element $x\in E$ is said to be orthogonal over $y\in E$ in the sense of subinner product or S-orthogonal, for short, if $(y, x)_S = 0$. We denote $x \perp_S y$.

The following properties of S-orthogonality are obvious from the above definition:

(i) $\quad x \perp_S y$, $x \perp_S z$ imply $x \perp_S (y+z)$;

(ii) $\quad x \perp_S y$, $\lambda \in$ K imply $x \perp_S \lambda y$ and $\lambda x \perp_S y$.

Now, let G be a nonvoid subset of E. Then

$$G^{\perp_S} := \{y \in E \mid y \perp_S x \text{ for all } x \in G\},$$

will be called the orthogonal complement of G in the sense of subinner product or S-orthogonal complement of G, for short. We also remark that: $0 \in G^{\perp_S}$, $G \cap G^{\perp_S} \subseteq \{0\}$ and $x \in G^{\perp_S}$, $\alpha \in K$ imply $\alpha x \in G^{\perp_S}$.

The above orthogonality extend usual orthogonality in inner product spaces, orthogonality in the sense of J. R. Giles [4] and R. A. Tapia [8] and R-orthogonality which was introduced in [2].

2. Characterizations of best approximation element in the sense of subinner product

Now, we recall some concepts and results in best approximation theory in normed linear spaces that will be used in the sequel.

Let E be a normed space and x, y two elements in E. The vector x is called orthogonal in the sense of Birkhoff over y if $\|x+\lambda y\| \geqq \|x\|$ for all $\lambda \in K$. We denote this $x \perp_B y$.

If G is a nondense linear subspace in E and:

$$\mathcal{P}_G(x_0) := \{g_0 \in G \mid \|x_0 - g_0\| = \inf_{g \in G} \|g - x_0\|\},$$

denotes the set of best approximation element referring to $x_0 \in E \setminus \bar{G}$ in G, then the following simple characterization lemma in terms of Birkhoff's orthogonality holds (see Lemma 1.14, from [7]):

Lemma 1. *Let E, G, x_0 be as above and $g_0 \in G$. Then $g_0 \in \mathcal{P}_G(x_0)$ if and only if $x_0 - g_0 \perp_B G$.*

For other characterizations of best approximation element in normed linear spaces see the monography [7] and the recent papers [1] and [3].

The following result is also valid (see for example [3], Lemma 1.1):

Lemma 2. *Let E be a smooth normed linear space $[\,,]$ be the semi-inner product in the sense of Lumer which generates its norm and x, y two elements in E. Then $x \mathrm{L} y$ (i.e., $[y, x]=0$) if and only if $x \perp_B y$.*

In virtue of this fact we can introduce the following concept.

Definition 2. Let E be a linear space, $(\,,)_S$ be a subinner product on E, G be a proper linear subspace in E, $x_0 \in E \setminus G$ and $g_0 \in G$. The vector g_0 is called the best approximation element of x_0 in G in the sense of subinner product or S-best approximation element of x_0, for short, if $x_0 - g_0 \perp_S G$. We denote $g_0 \in \mathcal{P}_G^S(x_0)$.

The following simple characterization holds.

Proposition 1. *Let E, $(\,,)_S$, G, x_0 and g_0 be as above. Then $g_0 \in \mathcal{P}_G^S(x_0)$ iff there exists an element $w_0 \in G^{\perp_S}$ such that:*

$$x_0 = g_0 + w_0. \tag{1}$$

The proof is obvious from the definition of S-best approximation element and we omit the details.

From the above proposition we have the following corollary.

Corollary. *If E, $(\,,)_S$, G, x_0 and g_0 are as above, then the following statements are equivalent:*

(i) $\mathscr{P}_G^S(x_0)$ *contains at least one [at most one (a unique)] element;*

(ii) *There exists at least one [at most one (a unique)] element $g_0 \in G$ and at least one [at most one (a unique)] element $w_0 \in G^{\perp S}$ such that* (1) *holds.*

The following result is important in the sequel.

Proposition 2. *Let E, $(\,,)_S$ be as above and f be a non-zero linear functional on E, $x_0 \in E \setminus \mathrm{Ker}(f)$ and $g_0 \in \mathrm{Ker}(f)$. Then the following statements are equivalent:*

(i) $g_0 \in \mathscr{P}^S_{\mathrm{Ker}(f)}(x_0)$;

(ii) *The following representation holds:*

(2) $f(x) = f(x_0)(x, (x_0 - g_0)/(x_0 - g_0)_S^2)_S$ *for all* $x \in E$,

where $(x_0 - g_0)_S^2$ *denotes* $(x_0 - g_0, x_0 - g_0)_S$.

Proof. Let $g_0 \in \mathscr{P}^S{}_{\mathrm{Ker}(f)}(x_0)$ and put $w_0 := x_0 - g_0 \neq 0$. Then $w_0 \in \mathrm{Ker}(f)^{\perp S}$. Since $f(x)w_0 - f(w_0)x \in \mathrm{Ker}(f)$ for all $x \in E$, hence $(f(x)w_0 - f(w_0)x, w_0)_S = 0$ what implies:

$$f(x)(w_0, w_0)_S = f(w_0)(x, w_0)_S \text{ for all } x \in E.$$

Because $(w_0)_S^2 \neq 0$, we obtain the desired representation.

Conversely, if (2) is valid and since $f(x_0) \neq 0$, then $x_0 - g_0 \perp_S \mathrm{Ker}(f)$, i.e., $g_0 \in \mathscr{P}^S_{\mathrm{Ker}(f)}(x_0)$.

The next corollary is also valid.

Corollary. *Let f and x_0 be as above. Then the following statements are equivalent:*

(i) $\mathscr{P}^S_{\mathrm{Ker}(f)}(x_0)$ *contains at least one [at most one (a unique)] element;*

(ii) *There exists at least one [at most one (a unique)] element $g_0 \in \mathrm{Ker}(f)$ such that the representation* (2) *holds.*

By the use of Proposition 2 we can prove the second characterization of S-best approximation element.

Proposition 3. *Let G be a linear subspace in E, $(\,,)_S$ be a subinner product on E, $x_0 \in E \setminus G$ and $g_0 \in G$. Then $g_0 \in \mathscr{P}_G^S(x_0)$ if and only if for all linear functional defined on $G \oplus \mathrm{Sp}(x_0)$ such that $\mathrm{Ker}(f) = G$, the following representation holds:*

(3) $f(x) = f(x_0)(x, (x_0 - g_0)/(x_0 - g_0)_S^2)_S$ *for all* $x \in G \oplus \mathrm{Sp}(x_0)$.

The following result is valid too.

Corollary. *Let G and x_0 be as above. Then $\mathscr{P}_G^S(x_0)$ contains at least one [at most one (a unique)] element if and only if for all linear functional defined on $G \oplus \mathrm{Sp}(x_0)$ such that $\mathrm{Ker}(f) = G$ there exists at least one [at most one (a unique)] element $g_0 \in G$ such that* (3) *holds.*

3. Characterization of semitchebychefian, proximinal and tchebychefian subspaces in the sense of subinner product

Firstly, we recall these concepts in the classic sense.

A proper linear subspace G in normed linear space E is called proximinal [semitchebychefian (tchebychefian)] in E if for every $x_0 \in E$ the set $\mathscr{P}_G(x_0)$ contains at least one [at most one (a unique)] element.

For some characterizations of proximinal [semitchebychefian (tchebychefian)] subspaces in a normed space see the monography [7] and the recent papers [1] and [3] where further references are given.

As in the case of normed spaces, we can introduce the following classes of linear subspaces.

Definition 3. Let E be a linear space and $(,)_S$ be a subinner product on it. The linear subspace G, $G \neq E$, will be called proximinal [semitchebychefian (tchebychefian)] in the sense of subinner product, or S-proximinal [S-semitchebychefian (S-tchebychefian)], for short, if $\mathscr{P}_G^S(x_0)$ contains at least one [at most one (a unique)] element for all x_0 in E.

The following theorem of characterization holds.

Theorem 1. *Let G be a linear subspace in E and* $(,)_S$ *be a subinner product on it. Then G is S-semitchebychefian [S-proximinal (S-tchebychefian)] if and only if for all* $x \in E$ *there exists at most one [at least one (a unique)] element* $x' \in G$ *and at most one [at least one (a unique)] element* $x'' \in G^{\perp S}$ *so that:*

$$x = x' + x'', \tag{4}$$

and we denote that: $E = G \boxplus G^{\perp S}\ [E = G + G^{\perp S}\ (E = G \oplus G^{\perp S})]$.

The proof is obvious from the definition of semitchebychefian [proximinal (tchebychefian)] linear subspaces in the sense of subinner product and from Corollary of Proposition 1. We shall omit the details.

The following proposition contains an example of S-proximinal linear subspaces in linear spaces endowed with a subinner product.

Proposition 4. *Let E and* $(,)_S$ *be as above. Then every finite-dimensional linear subspace in E is S-proximinal.*

Proof. Let G_n be a n-dimensional linear subspace in E and $x_0 \in E \setminus G_n$. Put $G_{n+1} := G_n \oplus \mathrm{Sp}(x_0)$. Then G_n can be regarded as a hyperplane in G_{n+1}.

On the other hand, let $\{x_2, \ldots, x_{n+1}\}$ be a base in G_n and $x_1 \in G_{n+1} \setminus G_n$ such that $\{x_1, x_2, \ldots, x_{n+1}\}$ is also a base in G_{n+1}. We construct the vectors (as in the case of inner product spaces):

$$e_1 = x_1/(x_1)_S,\ e_2 = x_2 - (x_2, e_1)_S e_1, \ldots,\ e_{n+1} = x_{n+1} - \sum_{i=1}^{n} (x_{n+1}, e_i)_S e_i.$$

It is easy to see that:

$$(e_2, e_1)_S = (e_3, e_1)_S = \ldots = (e_{n+1}, e_1)_S = 0$$

and since:

$$x_1=(x_1)_S e_1, \; x_2=(x_2, e_1)_S e_1+e_2, \ldots, \; x_{n+1}=\sum_{i=1}^{n}(x_{n+1} e_i)_S e_i+e_{n+1},$$

we have $\{e_1, e_2, \ldots, e_{n+1}\}$ is a base in G_{n+1} and $\{e_2, \ldots, e_{n+1}\}$ is also a base in G_n. Then $(u, e_1)_S=0$ for all $u\in G_n$ and since $e_1=\lambda_0 x_0+u_0$ with $\lambda_0\in K\setminus\{0\}$ and $u_0\in G_n$ we obtain: $(u, x_0-v_0)_S=0$ for all $u\in G_n$, where $v_0:=-1/\lambda_0\ u_0\in G_n$, i.e., $x_0-v_0 \perp_S G_n$, what is equivalent to $v_0\in\mathscr{P}^S_G(x_0)$ and the proposition is proven.

Corollary. *Let E and* $(\,,)_S$ *be as above. Then for all G a finite-dimensional linear subspace in it, we have the decomposition*:

$$E=G+G^{\perp_S}. \tag{5}$$

The following theorem establish a connection between proximinal [semitchebychefian (tchebychefian)] linear subspaces in the sense of subinner product and the representation of linear functional on a linear space endowed with a subinner product.

Firstly, we prove the next lemma.

Lemma 3. *Let H be a hyperplane containing the null element and* $(\,,)_S$ *be a subinner product on it. Then H is S-proximinal if and only if there exists a nonzero element u in E such that* $u \perp_S H$.

Proof. If H is S-proximinal and $x_0\in E\setminus H$ then there exists an element $g_0\in H$ such that $g_0\in\mathscr{P}^S_H(x_0)$ and putting $u:=x_0-g_0$ we have $u\perp_S H$ and $u\neq 0$.

Conversely, assume that $x_0\in E\setminus H$, $u\in E$, $u\perp_S H$ and $u\neq 0$ and let f be a nonzero linear functional on X such that $H=\mathrm{Ker}(f)$. If we choose $g_0:=x_0-(f(x_0)/f(u)u)$ $(f(u)\neq 0)$ so we have $g_0\in\mathrm{Ker}(f)$ and since:

$$(y, x_0-g_0)_S=(f(x_0)/f(u))\,(y, u)_S=0 \text{ for all } y\in H,$$

we deduce that $g_0\in\mathscr{P}^S_H(x_0)$, i.e., H is S-proximinal.

Now, we can give the main result of this section.

Theorem 2. *Let f be a nonzero linear functional on linear space E and* $(\,,)_S$ *be a subinner product on it. Then the following statements are equivalent*:

(i) $\mathrm{Ker}(f)$ *is S-proximinal [S-semitchebychefian (S-tchebychefian)]*;

(ii) *There exists at least one [at most one (a unique)] element* $u_f\in E$, $(u_f)_S=1$ *such that the following representation holds*:

$$f(x)=f(u_f)(x, u_f)_S \textit{ for all } x \textit{ in } E. \tag{6}$$

Proof. "(i) → (ii)" a. Let $\mathrm{Ker}(f)$ be S-proximinal. Then by Lemma 3 there exists $w_0\in E\setminus\mathrm{Ker}(f)$ such that $w_0\perp_S\mathrm{Ker}(f)$. By an argument similar to that in the proof of Proposition 2 we have:

$$f(x)=f(w_0)(x, w_0/(w_0)^2_S)_S \text{ for all } x \text{ in } E.$$

Put $u_f:=w_0/(w_0)_S$, then we obtain the representation (6).

"(ii) → (i)". a. Suppose that $u_f \in E$, $(u_f)_S = 1$ and u_f verifies (6). Then $u_f \perp_S \mathrm{Ker}(f)$ and by Lemma 3 it follows that Ker(f) is S-proximinal.

"(i) → (ii)". b. Assume that Ker(f) is S-semitchebychefian and suppose, by absurd, that there exists two distinct elements $u_f, v_f \in E$, $(u_f)_S = (v_f)_S = 1$ such that they satisfy (6). Then $u_f, v_f \in \mathrm{Ker}(f)^{\perp S}$. Now, let $x \in E \backslash \mathrm{Ker}(f)$ and put:

$$y_0 := x - f(x)u_f / f(u_f) \text{ and } y_0' := x - f(x)v_f / f(v_f).$$

Then $f(y_0) = f(y_0') = 0$, i.e., $y_0, y_0' \in \mathrm{Ker}(f)$.

On the other hand, for all $y \in \mathrm{Ker}(f)$, we have

$$(y, x - y_0)_S = (\overline{f(x)/f(u_f)})(y, u_f)_S = 0$$

and a similar relation for y_0'. Consequently $x - y_0$, $x - y_0' \perp_S \mathrm{Ker}(f)$ i.e., $y_0, y_0' \in \mathscr{P}^S_{\mathrm{Ker}(f)}(x_0)$. Now, if we assume that $y_0 = y_0'$ we derive $u_f / f(u_f) = v_f / f(v_f)$ and since $f(u_f) = f(v_f)$ one gets $u_f = v_f$. Thus $y_0 \neq y_0$ and since $y_0, y_0' \in \mathscr{P}^S_{\mathrm{Ker}(f)}(x_0)$ we obtain a contradiction to the fact that Ker(f) is S-semitchebychefian and the implication is proven.

"(ii) → (i)". b. Assume that (6) holds for a unique element $u_f \in E$, $(u_f)_S = 1$ and suppose, by absurd, that there exists $x_0 \in E \backslash \mathrm{Ker}(f)$ and two distinct elements g_0 and g_0' in $\mathscr{P}^S_{\mathrm{Ker}(f)}(x_0)$. As above, we obtain:

(7) $$f(x) = f(x_0)(x, (x_0 - g_0)/(x_0 - g_0)_S^2)_S \text{ for all } x \text{ in } E,$$

and a similar representation for g_0'. Put:

$$u_f := (x_0 - g_0)/(x_0 - g_0)_S \text{ and } v_f := (x_0 - g_0')/(x_0 - g_0')_S.$$

Then $(u_f)_S = (v_f)_S = 1$ and u_f, v_f satisfy (6). Now, if we assume that $u_f = v_f$, we derive $(x_0 - g_0)/(x_0 - g_0)_S = (x_0 - g_0')/(x_0 - g_0')_S$ and since $(x_0 - g_0)_S = (x - g_0')_S$ (from (7)) we obtain $g_0 = g_0'$. Consequently, there exists two distinct elements $u_f, v_f \in E$, $(u_f)_S = (v_f)_S = 1$ and they satisfy (6), what produce a contradiction and the proof is finished.

"(i) ⟷ (ii)". c. The statement: Ker(f) is S-tchebychefian iff there exists a unique element $u_f \in E$, $(u_f)_S = 1$ such that (6) holds, follows by the above arguments.

The next corollary contains a characterization of S-proximinal [S-semitchebychefian (S-tchebychefian)] linear subspaces in terms of linear functionals.

Corollary 1. *let G be a linear subspace in linear space E, $G \neq E$ and $(,)_S$ be a subinner product on it. Then the following statements are equivalent:*

(i) *G is S-proximinal [S-semitchebychefian (S-tchebychefian)];*

(ii) *For all $x_0 \in E \backslash G$ and for any f a nonzero linear functional on $G \oplus \mathrm{Sp}(x_0)$ such that* Ker$(f) = G$, *there exists at least one [at most one (a unique)] element $u_{x_0,f} \in G \oplus \mathrm{Sp}(x_0)$, $(u_{x_0,f})_S = 1$ with the property:*

$$f(x) = f(u_{x_0,f})(x, u_{x_0,f})_s \textit{ for all } x \in G \oplus \mathrm{Sp}(x_0).$$

The proof follows by the previous theorem for the linear space $E_{x_0} := G \oplus \mathrm{Sp}(x_0)$. We shall omit the details.

Corollary 2. *Let E and $(,)_S$ be as above and G be a finite-dimensional linear* *subspace in E. Then for all nonzero linear functional on G there exists at least one element* $u_{G,f}$ *in G,* $(u_{G,f})_S=1$ *such that:*

$$f(x)=f(u_G, f)\ (x, u_{G,f})_S \text{ for all } x\in G.$$

Further on, we shall give some applications of the above results in the case of smooth normed linear spaces.

4. Applications to smooth normed spaces

Let E be a linear space over real or complex number field K. A mapping $[,]$ of $E\times E$ into K is a semi-inner product in the sense of Lumer or L-semi-inner-product, for short, if the following conditions (P1)—(P4) are satisfied (see [5] or [4]):

(P1) $[x, x]\geqq 0$ for all $x\in E$ and $[x, x]=0$ implies $x=0$;
(P2) $[\lambda x, y]=\lambda[x, y]$ and $[x, \lambda y]=\bar{\lambda}[x, y]$ for all $\lambda\in$ K and x, y in E;
(P3) $[x+y, z]=[x, z]+[y, z]$ for all x, y, z in E;
(P4) $|[x, y]|^2\leqq[x, x][y, y]$ for all x, y in E.

It is easy to see that the mapping $E\ni x\to[x, x]^{1/2}\in \mathrm{R}_+$ is a norm on E and if E is a normed space, then every L-semi-inner product on E which generates the norm is of the form:

$$[x, y]=\langle \tilde{J}(y), x\rangle \text{ for all } x, y\in E,$$

where $\tilde{J}$ is a section of normalized dual mapping [6]. It is also known that a normed linear space E is smooth iff there exists a unique L-semi-inner product which generates the norm or if and only if there exists a continuous L-semi-inner product which generates the norm, i.e., a L-semi-inner product satisfying condition:

$$\lim_{t\to 0}\mathrm{Re}[y, x+ty]=\mathrm{Re}[y, x] \text{ for all } x, y\in E \text{ (see [4]).}$$

On the other hand, in a smooth normed space the Giles' orthogonality is equivalent to Birkhoff's orthogonality, i.e.,

$$[y, x]=0 \text{ iff } \|x+\lambda y\|\geqq\|x\| \text{ for all } \lambda\in \mathrm{K},$$

and since a L-semi-inner product is a subinner product, we have the following results.

Theorem 3. *Let E be a smooth normed space,* $[,]$ *be the L-semi-inner product which generates its norm, G be a nondense linear subspace in E,* $x_0\in E\backslash G$ *and* $g_0\in G$*. Then the following statements are equivalent:*

(i) $g_0\in\mathcal{P}_G(x_0)$;
(ii) *There exists an element* $g_0'\in G^L:=\{y\in E\mid[g, y]=0$ for all $g\in G\}$ *such that:*

$$x_0=g_0+g_0'; \tag{8}$$

(iii) *For all linear functional* $f \in (G \oplus \mathrm{Sp}(x_0))^*$ *such that* $G = \mathrm{Ker}(f)$ *the following representation holds*:

(9) $$f(x) = [x, \overline{f(x_0)}(x_0 - g_0)/\|x_0 - g_0\|^2] \text{ for all } x \in G \oplus \mathrm{Sp}(x_0).$$

The proof is obvious from Proposition 1 and Proposition 3 and we omit it.

Corollary. *Let* E, G, x_0 *and* $[,]$ *be as above. Then the following statements are equivalent*:

(i) $\mathscr{P}_G(x_0)$ *contains at least one [at most one (a unique)] element*;

(ii) *There exists at least one [at most one (a unique)] element* g_0 *in* G *and at least one [at most one (a unique)] element* g_0' *in* G^L *such that* (8) *holds*;

(iii) *For all linear and continuous functional* f *defined on* $G \oplus \mathrm{Sp}(x_0)$ *such that* $\mathrm{Ker}(f) = G$ *there exists at least one [at most one (a unique)] element* $g_0 \in G$ *with the property* (9).

Finally, we have:

Theorem 4. *Let* G *be a [(closed)] linear subspace in smooth normed space* E ($G \neq E$). *Then the following sentences are equivalent*:

(i) G *is semitchebychefian [proximinal (tchebychefian)]*;

(ii) *The following decomposition holds*:

$$E = G \boxplus G^L \; [E = G + G^L (E = G \oplus G^L)];$$

(iii) *For all* $x_0 \in E \setminus G$ *and for any* f *a nonzero continuous linear functional on* $G \oplus \mathrm{Sp}(x_0)$ *such that* $\mathrm{Ker}(f) = G$, *there exists at most one [at least one (a unique)] element* $u_{x_0,f} \in G \oplus \mathrm{Sp}(x_0)$, $\|u_{x_0,f}\| = 1$ *with the property that*:

$$f(x) = f(u_{x_0,f})[x, u_{x_0,f}] \text{ for all } x \in G \oplus \mathrm{Sp}(x_0).$$

The proof follows by Theorem 1 and Theorem 2 and we shall omit the details.

References

1. S. S. Dragomir. A Characterization of Best Approximation Element in Real Normed Linear Spaces (Romanian). *Stud. Cerc. Mat.*, **36**, 1987, 497-506.
2. S. S. Dragomir. Linear and Continuous Functionals on Complete R-Semi-inner product Spaces (I), (II. *Itin. Sem. on Func. Eq., Approx. Conv., Cluj*, 1987, 127-134; 1988, 163-168.
3. S. S. Dragomir. Representation of Continuous Linear Functionals on Smooth Normed Linear Spaces. *L'Anal. Num. Théor. L'Approx.* 17, 1988, 125-132.
4. J. R. Giles. Classes of Semi-Inner-Product Spaces. *Trans. Amer. Math. Soc.*, **129**, 1867, 436-446.
5. G. Lumer. Semi-Inner-Product Spaces. *Trans Amer. Math. Soc.*, **100**, 1961, 29-43.
6. I. Rosca. Semi-Produits Scalaires et Represéntation du Type Riesz pour les Fonctionnélles Linéaires et Bornées. *C. R. Acad. Sci. Paris*, **283 (19)**, 1976.
7. I. Singer. Best Approximation in Normed Linear Spaces by Elements of Linear Subspaces (Romanian). Ed. Acad., Bucureşti, 1967.
8. R. A. Tapia. A Characterization of Inner Product Spaces. *Proc. Amer. Math. Soc.*, **41**, 1973, 569-574.

Strada Trandafirilor 60
Bl. 34 Sc. D Ap. 9
1600 Băile Herculane
Jud. Caraş-Severin
ROMANIA

Received 04.03.1990