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Splines and Numerical Solutions with an Accuracy $O(h^3)$ for a Hyperbolic Differential-Integral Equation

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The numerical solutions with an accuracy $O(h^3)$ for a simple case of von Foerster-Gurtin-MacCamy model are obtained by splines. The existence, uniqueness and convergence theorems are proved.

1. Introduction

In this paper we are interested in finding numerical solutions for the following problem (see [1])

$$\frac{\partial u(x,\ t)}{\partial t}+\frac{\partial u(x,\ t)}{\partial x}=-d(x,\ t,\ P(t))u(x,\ t), \tag{1}$$

$$u(x,\ 0)=\varphi(x),\ 0\leqq x\leqq A, \tag{2}$$

$$u(0,\ t)=\psi(t),\ 0\leqq t\leqq B, \tag{3}$$

$$P(t)=\int_0^A u(x,\ t)\,dx. \tag{4}$$

Here $u(x,\ t)$ is an unknown function; d, φ, ψ, are given functions. The problem (1)–(4) is a simple case of von Foerster-Gurtin-McCamy model, describing the age structure in time of a population (see, for example [2], [3]). In that model instead of (3) one has an equation

$$u(0,\ t)=\int_0^A b(x,\ t,\ P(t))u(x,\ t)\,dx.$$

The numerical solutions with an accuracy $O(h)$ for these models were considered in [4].

For numerical solutions of (1)-(4) with an accuracy $O(h^3)$ we shall need the following assumptions:

1) The problem (1)-(4) has a smooth solution $u(x,\ t)$ and $u(A,\ t)=0$ for all $t\geqq 0$. This condition is a well-known condition for the age structure of a population (see, for example [3]). We note that the solution $u(x,\ t)$ satisfies the inequality (see [1])

(5) $$0 \leqq u(x, t) \leqq C_1 = \max\left(\max_{0 \leqq x \leqq A} \varphi(x), \max_{0 \leqq t \leqq B} \psi(t) \right).$$

2) The function $d(x, t, P)$ has a form (see [5])

$$d(x, t, P) = d_1(x) + d_2(t) + d_3(P),$$

where $d_k(x) \geqq 0$, $d_k(\cdot)$ are differentiable and $d_1'(x)$ is a continuous function for $0 \leqq x < A$. If $d_1'(x) \to \infty$ when $x \to A$, then there exists a number A_1, $0 < A_1 < A$ such that

(6) $$d_1(x) \leqq \frac{C_2}{(A-x)^\alpha}, \quad 0 < C_2 < \infty, \quad 0 < \alpha < 2,$$

(7) $$|d_1(x)| \geqq C_3 d_1^2(x), \quad A_1 \leqq x \leqq A, \quad 0 < C_3 < 1,$$

$d_2'(t)$ is a continuous function for $0 \leqq t \leqq B$.

$d_3'(P)$ is differentiable and $d_3''(P)$ is a continuous function on $[0, A_2]$, where

(8) $$A_2 \geqq \frac{10}{3} C_1 A.$$

3) $d(x, t, P) \geqq C_4 > 0$.

The paper consists of 5 parts. After the introduction in the 2nd part the nonlinear system of equations for numerical solutions with the accuracy $O(h^3)$ will be obtained. In the 3rd, 4th, and 5th parts the existence, uniqueness, and convergence theorems will be proved.

2. Numerical solutions of the problem (1)-(4)

For construction of numerical solutions we take a positive integer number N. Let $h = A/N$ and

$$x_i = x_{i-1} + h, \quad i = 1, 2, \ldots, N, \quad x_0 = 0.$$

We shall determine numerical solutions $U_{i,j}$ on every line $t = t_j$.

$$t_j = t_{j-1} + h, \quad j = 1, 2, \ldots, \quad t_0 = 0,$$

at the points (x_i, t_j).

On the line $t = 0$, from (2) we obtain

$$U_{i,0} = \varphi(x_i), \quad i = 0, 1, \ldots, N.$$

Now we suppose that, on the line t_{j-1}, $j \geqq 1$ the values $U_i = U_{i,j-1}$ — the approximations for $u(x_i, t_{j-1})$ — are given. Let $S(x)$ be a cubic spline, interoplating U_i and let $P = \int_0^A S(x)\,dx$ be an approximation for $P(t_{j-1})$. Then we shall determine $\hat{U}_i = U_{i,j}$ — the approximations for $u(x_i, t_j)$ on the line t_j.

From (3) and assumption 1) we have

$$\hat{U}_0 = \psi(t_j), \quad \hat{U}_N = 0.$$

Now denote

(9) $$\bar{d}(\tau) \equiv u(x+\tau,\ t+\tau),\ \bar{d}(\tau) \equiv d(x+\tau,\ t+\tau,\ P(t+\tau)).$$

Then (see [2]), formula (2.5))

(10) $$\frac{d\bar{u}}{d\tau} + \bar{d}(\tau)\bar{u}\ (\tau) = 0,$$

i. e.

(11) $$\bar{u}\ (h) - \bar{u}\ (0) = -\int_0^h \bar{d}(\tau)\bar{u}\ (\tau)\, d\tau.$$

Using now quadratic formulas for the integral in (11) we can get numerical solutions with different accuracies. For example, in the case

$$\int_0^A \bar{d}\ (\tau)\bar{u}\ (\tau)\, d\tau \approx -\frac{h}{2}[\bar{d}\ (0)\bar{u}\ (0) + \bar{d}(h)\bar{u}\ (h)]$$

we get numerical solutions with the accuracy $O(h^2)$. The existence, uniqueness and convergence of these solutions were studied in [8].

In this paper we shall consider the numerical solutions with an accuracy $O(h^3)$. For this purpose, let $f(\tau) = \bar{d}(\tau)\bar{u}\ (\tau)$ and let $G(\tau)$ be a quadratic Hermite spline (see, for example [7], pp. 304) for $f(\tau)$ on $[0,\ h]$, i. e. $G(\tau)$ is a quadratic polynomial and

$$G(0) = f(0),\ g(h) = f(h),$$
$$G'(0) = f'(0).$$

Replacing $\bar{d}(\tau)\bar{u}\ (\tau)$ in (11) by $G(\tau)$ one can get that

$$\int_0^h \bar{d}(\tau)\bar{u}\ (\tau)\, d\tau = \int_0^h f(\tau)\, d\tau \approx \int_0^h G(\tau)\, d\tau$$
$$= \frac{h}{3}[2f(0) + f(h)] + \frac{h^2}{6} f'\ (0).$$

In the formula (12) we have to calculate $f'\ (\tau)$. We note that $u(x, t)$ is the solution of (1). Then from (9) we obtain equalities

$$f'(\tau) = [d'_1(\cdot) + d'_2(\cdot) + d'_3(\cdot)\frac{d}{dt}P(\cdot)$$
$$-d^2(.,.,.)](x+\tau,\ t+\tau,\ P(t+\tau))u(x+\tau,\ t+\tau),$$

$$\frac{d}{dt}P(t) = \frac{d}{dt}\int_0^A u(x,\ t)\, dx = \int_0^A \frac{\partial}{\partial t}u(x,\ t)\, dx = -\int_0^A [\frac{\partial}{\partial x}u(x,\ t)$$

(3) $$+ d(x,\ t,\ P(t))u(x,\ t)]\, dx = \psi(t) - \int_0^A d(x,\ t,\ P(t))u(x,\ t)\, dx.$$

So $\hat{U}_i$, $i = 1,\ 2, \ldots,\ N-1$, can be determined by the following equations:

(14) $$\hat{U}_i = U_{i-1}[1 - \frac{2h}{3}d(x_{i-1}, t_{j-1}, P) - \frac{h^2}{6}F_{i-1,j-1}(P, P')] - \frac{h}{3}d(x_i,\ t_j, \hat{P})\hat{U}_i,$$

where

$$F_{i-1,j-1}(P, P') \equiv [d'_1(\cdot)+d'_2(\cdot)+P'd'_3(\cdot)-d^2(.,.,.)](x_{i-1}, t_{j-1}, P),$$

$$P \equiv \max(0, \textstyle\int_0^A S(x)\,dx), \quad P' \equiv \psi(t_{j-1})-\int_0^A d(x, t_{j-1}, P)S(x)\,dx,$$

$$(15) \qquad \hat{P} \equiv \max(0, \textstyle\int_0^A \hat{S}(x)\,dx),$$

$\hat{S}(x)$ — a cubic spline (see, for example [6]), interpolating $\hat{U}_i$, i.e.

$$(16) \qquad \hat{S}(x_i)=\hat{U}_i, \quad i=0, 1,\dots, N.$$

We can use the following boundary conditions for $\hat{S}(x)$

At the point x_N, from assumption 1) we have

$$(17) \qquad \hat{S}'(x_N)=0.$$

At the point x_0, we require continuity (see [6]) of $\hat{S}'''(x)$ at the point x_1, i.e.

$$(18) \qquad \hat{S}'''(x_1-0)=S'''(x_1+0).$$

For given $\hat{U}_i$, $i=0, 1,\dots, N$, there exists unique $\hat{S}(s)$. So $\hat{U}_i$ are implicitly determined by the system of equations (14)-(18). We note that on $[x_{N-1}, A]$ $S(x)$ is a polynomial and $S(A)=S'(A)=0$. From (6) it follows that the integral in P' takes a finite value.

3. Existence of the numerical solutions

For the proof of the existence of $\hat{U}_i$ we shall suppose that the given U_i, $i=0, 1,\dots, N$, satisfy the inequalities

$$(19) \qquad 0 \leqq U_i \leqq C_1,$$

where C_1 is determined in (5). We shall need the following lemmas:

Lemma 1. *Let $\hat{S}(x)$ be a cubic spline satisfying* (16), (17), (18). *Then*

$$(20) \qquad |\hat{P}| \equiv |\textstyle\int_0^A \hat{S}(x)\,dx| \leqq C_5\,\|\hat{S}(x_i)\|, \quad C_5 \equiv \dfrac{10A}{3}$$

where

$$\|U_i\| \equiv \max_{i=0,1,\dots N} |U_i|.$$

Proof. On $[x_{i-1}, x_i]$ the spline $\hat{S}(x)$ has a representation (see [6], p. 98)

$$\hat{S}(x)=m_{i-1}\frac{(x_i-x)^2(x-x_{i-1})}{h^2}-m_i\frac{(x-x_{i-1})^2(x_i-x)}{h^2}$$

$$(21) + S(x_{i-1})\frac{(x_i-x)^2[2(x-x_{i-1})+h]}{H^3}+S(x_i)\frac{(x-x_{i-1})^2[2(x_i-x)+h]}{h^3},$$

where m_i, $i=0, 1,\dots, N$, satisfy a system of equations

$$m_0=m_2-2\frac{\hat{S}(x_2)-2\hat{S}(x_1)+\hat{S}(x_0)}{h},$$

(22) $$2m_1+m_2=\delta_1\equiv\frac{1}{2h}(\hat{S}(x_2)-\hat{S}(x_0))+\frac{2}{h}(\hat{S}(x_2)-\hat{S}(x_1)),$$

(23) $$m_{i-1}+4m_i+m_{i+1}=\delta_1\equiv\frac{3}{h}(\hat{S}(x_{i+1})-\hat{S}(x_{i-1}))\quad i=2,3,\dots,N-1,$$

$$m_N=0.$$

We note that (22)-(23) is a diagonal dominant system. Then

$$|m_i|\leqq\max_{i=1,2,\cdots N-1}|\delta_i|\leqq\frac{6}{h}\|\hat{S}(x_i)\|$$

and therefore

$$|m_0|\leqq\frac{14}{h}\|\hat{S}(x_i)\|,$$

i.e.

(24) $$\|m_i\|\leqq\frac{14}{h}\|\hat{S}(x_i)\|.$$

From (21) and (24) it follows that

$$|P|=|\int_0^A \hat{S}(x)\,dx|\leqq\sum_{i=1}^{N}\int_{x_{i-1}}^{x_i}|\hat{S}(x)|\,dx$$

$$\leqq\frac{\|m_i\|}{h^2}\sum_{i=1}^{N}\int_{x_{i-1}}^{N}[(x_i-x)^2(x-x_{i-1})+(x-x_{i-1})^2(x_i-x)]\,dx$$

$$+\frac{\|\hat{S}(x_i)\|}{h^3}\sum_{i=1}^{N}\int_{x_{i-1}}^{x_i}\{(x_i-x)^2[2(x-x_{i-1})+h]$$

$$+(x-x_{i-1})^2[2(x_i-x)+h]\}\,dx.$$

Because of

$$\int_{x_{i-1}}^{x_i}(x_i-x)^2(x-x_{i-1})\,dx=\int_{x_{i-1}}^{x_i}(x_i-x)(x-x_{i-1})^2\,dx=\frac{1}{12}h^4$$

we obtain

$$|P|\leqq\frac{A}{6}\|m_i\|h+A\|\hat{S}(x_i)\|=\frac{10A}{3}\|\hat{S}(x_i)\|.$$

Lemma 1 is proved.

Lemma 2. *Let $\hat{U}_i$, $i=1,2,\dots,N-1$, be a solution of* (14) *with U_i satisfying* (19). *Then for small h*

(25) $$0 \leqq \hat{U}_i \leqq C_1,$$

where C_1 *is determined in* (5)

Proof. Denote

$$C_6 \equiv \begin{cases} \max\limits_{0\leqq x\leqq A} d_1(x), & \text{if } d_1'(x) \text{ is bounded on } [0, A], \\ \max\limits_{0\leqq x\leqq A_1} d_1(x), & \text{if } d_1'(x) \text{ is not bounded on } [0, A] \end{cases}$$

$$C_7 \equiv \max_{0\leqq t\leqq B} d_2(t),\ C_8 \equiv \max_{0\leqq P\leqq A_2} d_3(P),$$

$$C_9 \equiv \begin{cases} \max\limits_{0\leqq x\leqq A} |d_1'(x)|, & \text{if } d_1'(x) \text{ is bounded on } [0, A], \\ \max\limits_{0\leqq x\leqq A_1} |d_1'(x)|, & \text{if } d_1'(x) \text{ is not bounded on } [0, A] \end{cases}$$

$$C_{10} \equiv \max_{0\leqq t\leqq B} |d_2'(t)|,$$

(26) $$C_{11} \equiv \max_{0\leqq P\leqq A_2} |d_3'(P)|,\ C_{12} \equiv \max_{0\leqq P\leqq A_2} |d_3''(P)|,$$

(27) $$C_{13} \equiv C_6 + C_7 + C_8,\ C_{14} \equiv C_9 + C_{10} + C_{11} C_1 (1 + C_5 C_{13}).$$

Then $d(x, t, P) \leqq C_{13}$, $|d_1'(x)| + |d_2'(t)| + |P'|\,|d_3'(P)| \leqq C_{14}$.

Firstly we shall prove that $\hat{U}_i \geqq 0$. Let consider the case when $d_1'(x)$ is bounded on $[0, A]$. From (14) we get the following equality

(28) $$\hat{U}_i = \frac{1 - \frac{2h}{3} d(x_{i-1}, t_{j-1}, P) - \frac{h^2}{6} F_{i-1, j-1}, (P, P')}{1 + \frac{h}{3} d(x_i, t_j, \hat{P})} U_{i-1},$$

$$i = 1, 2, \ldots, N-1,$$

We note that

$$1 - \frac{2}{3} y + \frac{1}{6} y^2 \geqq \frac{1}{3}.$$

Then for $0 < h \leqq h_0$ with

(29) $$0 < h_0 \leqq \sqrt{\frac{2}{C_{14}}}$$

from Lemma 1 and (28) we obtain that

$$1 - \frac{2h}{3} d(x_{i-1}, t_{j-1}, P) - \frac{h^2}{6} F_{i-1, j-1}(P, P')$$

$$\geqq 1 - \frac{2h}{3} d(x_{i-1}, t_{j-1}, P) + \frac{h^2}{6} d^2(x_{i-1}, t_{j-1}, P) - \frac{h^2}{6} C_{14}$$

$$\geqq \frac{1}{3}-\frac{h^2}{6}C_{14}\geqq 0.$$

Consider now the case when $d_1'(x)$ is not bounded on $[0, A]$. In this case for $x_{i-1}\leqq A_1$ we can use (28) and (30) to prove that $\hat{U}_i\geqq 0$. Let $x_{i-1}>A_1$. From assumption 2) one has

$$1-\frac{2h}{3}d(x_{i-1},\ t_{j-1},\ P)-\frac{h^2}{6}F_{i-1,j-1}(P,\ P')$$

$$\geqq 1-\frac{2h}{3}d(x_{i-1},\ t_{j-1},\ P)+\frac{h^2}{6}[(1-C_3)\,d^2(x_{i-1},\ t_{j-1},\ P)$$

$$-|d_2'(t_{j-1})+P'\,d_3'(P)|]$$

$$\text{(31)}\ \geqq 1-\frac{2h}{3}d(x_{i-1},\ t_{j-1},\ P)+\frac{h^2}{6}(1-C_3)\,d^2(x_{i-1},\ t_{j-1},\ P)-\frac{h^2}{6}C_{14}.$$

Denote

$$g(y)\equiv 1-\frac{2}{3}y+\frac{1-C_3}{6}y^2.$$

If $C_3<\frac{1}{3}$ then $g(y)\geqq g(y_0)>0$, where $y_0=\frac{2}{1-C_3}$. Let

$$\text{(32)}\qquad 0<h_1\leqq\sqrt{\frac{6g(y_0)}{C_{14}}}.$$

For $0<h\leqq h_1$ we have

$$1-\frac{2h}{3}d(x_{i-1},\ t_{j-1},\ P)-\frac{h^2}{6}F_{i-1,j-1}(P,\ P')\geqq g(y_0)-\frac{h^2}{6}C_{14}\geqq 0.$$

Consider the case $\frac{1}{3}\leqq C_3<1$.

Let

$$0<y_1<\frac{2-\sqrt{6C_3-2}}{1-C_3}$$

and be fixed. Then

$$g(y)\geqq g(y_1)>0,\ 0<y\leqq y_1.$$

Let

$$\text{(33)}\qquad h_2\equiv\min\left(\frac{y_1}{C_{13}},\ \sqrt{\frac{6g(y_1)}{C_{14}}}\right).$$

For $0<h\leqq h_2$ we have

$$1-\frac{2h}{3}d(x_{i-1}, t_{j-1}, P)-\frac{h^2}{6}F_{i-1,j-1}(P, P')\geqq g(hd(x_{i-1}, t_{j-1}, P)$$
$$-\frac{h^2}{6}C_{14}\geqq 0.$$

So we obtain that $\hat{U}_i\geqq 0$.
Now we are going to prove that $\hat{U}_1\leqq U_{i-1}$. Let

$$0<h_3\leqq\frac{6C_4}{C_{13}^2+C_{14}}, \tag{34}$$

where C_4 is given in the assumption 3). Then for $0<h\leqq h_3$ one has

$$1-\frac{2h}{3}d(x_{i-1}, t_{j-1}, P)-\frac{h^2}{6}F_{i-1,j-1}(P, P')\leqq 1-\frac{2h}{3}C_4$$
$$+\frac{h^2}{6}(C_{13}^2+C_{14})\leqq 1+\frac{h}{3}C_4\leqq 1+\frac{h}{3}d(x_i, t_j, \hat{P}).$$

Denote

$$h_4\equiv\min(h_0, h_1, h_2, h_3), \tag{35}$$

h_0, h_1, h_2, h_3 are determined in (29), (32), (33), (34) respectively. Then for $0<h\leqq h_4$ the inequalities $0\leqq\hat{U}_i\leqq U_{i-1}$ hold. Lemma 2 is proved.

Corollary 1. *For* $0<h\leqq h_4$ *if* $\hat{U}_i$ *are solutions of* (14)-(18), *then*

$$0\leqq\hat{P}\leqq\frac{10}{3}C_1A\leqq A_2, \tag{36}$$

i.e. the value $d_3(\hat{P})$ *is correctly determined.*

Lemma 3. *Let* $F=\max(0, \int_0^A f(x)\,dx)$, $\hat{P}=\max(0, \int_0^A \hat{f}(x)\,dx)$. *Then*

$$|F-\hat{F}|\leqq|\int_0^A[f(x)-\hat{f}(x)]\,dx|.$$

Proof. If $\int_0^A f(x)\,dx>0$, $\int_0^A \hat{f}(x)\,dx>0$, then

$$|F-\hat{F}|=|\int_0^A[f(x)-\hat{f}(x)]\,dx|.$$

If $\int_0^A f(x)\,dx\leqq 0$, $\int_0^A \hat{f}(x)\,dx\leqq 0$, then

$$0=|F-\hat{F}|\leqq|\int_0^A[f(x)-\hat{f}(x)]\,dx|.$$

Now consider the case $\int_0^A f(x)\,dx>0$, $\int_0^A \hat{f}(x)\,dx\leqq 0$. Then

$$|F-\hat{F}|=\int_0^A f(x)\,dx\leqq\int_0^A[f(x)-\hat{f}(x)]\,dx.$$

At last, consider the case $\int_0^A f(x)\,dx\leqq 0$, $\int_0^A \hat{f}(x)\,dx>0$. Then

$$|F-\hat{F}|=\int_0^A \hat{f}(x)\,dx\leqq-\int_0^A[f(x)-\hat{f}(x)]\,dx.$$

Lemma 3 is proved.

Now we are going to show that the solutions $\hat{U}_i$, $i=1, 2,\dots, N-1$ exist. For this purpose, we consider the following sequences of k, $k=1, 2,\dots$

$$\hat{U}_{i,k}=U_{i-1}[1-\frac{2h}{3}d(x_{i-1}, t_{j-1}, P)-\frac{h^2}{6}F_{i-1,j-1}(P, P')]$$

$$-\frac{h}{3}d(x_i, t_j, \hat{P}_{k-1})\hat{U}_{i,k}, \tag{37}$$

$$i=1, 2,\dots, N-1$$

$$\hat{U}_{0,k}=\psi(t_j),\ \hat{U}_{N,k}=0,\ \hat{U}_{i,0}=U_i,$$

$$\hat{P}_{k-1}=\max(0, \textstyle\int_0^A \hat{S}_{k-1}(x)\,dx),\ \hat{P}_0=P, \tag{38}$$

$\hat{S}_{k-1}(x)$ is a cubic spline, interpolating $\hat{U}_{i,k-1}$, $i=0, 1,\dots, N$ and $\hat{S}'_{k-1}(x)=0$, $\hat{S}'''_{k-1}(x_1-0)=\hat{S}'''_{k-1}(x_1+0)$.

For $0<h\leqq h_4$ from lemma 2 we obtain

$$0\leqq \hat{U}_{i,k}\leqq C_1 \text{ for all } k=1, 2,\dots$$

Consequently, using Corollary 1 we conclude that $\hat{P}_{k-1}\in[0\,A_2]$.

Denote

$$V_{i,k}=\hat{U}_{i,k}-\hat{U}_{i,k-1},\ k=2, 3,\dots$$

Then

$$V_{0,k}=V_{N,k}=0.$$

$$V_{i,k}=-\frac{h}{3}[d(x_i, t_j, \hat{P}_{k-1})\hat{U}_{i,k}-d(x_i, t_j, \hat{P}_{k-2})\hat{U}_{i,k-1}]$$

$$=-\frac{h}{3}[d(x_i, t_j, \hat{P}_{k-1})\hat{U}_{i,k}-d(x_i, t_j, \hat{P}_{k-1})\hat{U}_{i,k-1}$$

$$+d(x_i, t_j, \hat{P}_{k-1})\hat{U}_{i,k-1}-d(x_i, t_j, \hat{P}_{k-2})\hat{U}_{i,k-1}]$$

$$=-\frac{h}{3}d(x_i, t_j, \hat{P}_{k-1})V_{i,k}+\frac{h}{3}\hat{U}_{i,k-1}d'_3(\xi)(\hat{P}_{k-1}-\hat{P}_{k-2}), \tag{39}$$

$$i=1, 2,\dots, N-1,$$

where ξ is a value between $\hat{P}_{k-1}$ and $\hat{P}_{k-2}$. Now we have to estimate $|\hat{P}_{k-1}-\hat{P}_{k-2}|$. From lemma 3 we obtain

$$|\hat{P}_{k-1}-\hat{P}_{k-2}|\leqq|\textstyle\int_0^A[\hat{S}_{k-1}(x)-\hat{S}_{k-2}(x)]\,dx|.$$

It is clear that $\hat{S}'''_{k-1}(x)-\hat{S}_{k-2}(x)$ is a cubic spline, interpolating $\hat{U}_{i,k-1}-\hat{U}_{i,k-2}=\hat{V}_{i,k-1}$ and $\hat{S}'_{k-1}(x_N)-\hat{S}'_{k-2}(x_N)=0$, $\hat{S}'''_{k-1}(x_1-0)-\hat{S}'''_{k-2}(x_1-0)=\hat{S}'''_{k-1}(x_1+0)-\hat{S}'''_{k-2}(x_1+0)$. Then from lemma 1 we get that

$$|\hat{P}_{k-1}-\hat{P}_{k-2}|\leqq|\textstyle\int_0^A[\hat{S}_{k-1}(x)-\hat{S}_{k-2}(x)]\,dx|\leqq C_5\,\|V_{i,k-1}\|. \tag{40}$$

From (39) and (40) we obtain an inequality

$$\|V_{i,k}\| \leqq \frac{h}{3} C_1 C_5 C_{11} \|V_{i,k-1}\|, \tag{41}$$

where C_1, C_5 and C_{11} are determined in (5), (20) and (26) respectively.
Now we take

$$0 < h_5 < \frac{3}{C_1 C_5 C_{11}}. \tag{42}$$

Then for $0 < h \leqq h_5$ we have

$$\|V_{i,k}\| \leqq q_1 \|V_{i,k-1}\| \leqq \cdots \leqq q_1^{k-1} \|V_{i,1}\|, \quad q_1 \equiv \frac{h}{3} C_1 C_5 C_{11} < 1.$$

So for every i, $i=1, 2, \ldots, N-1$ the sequence $\hat{U}_{i,k}$ converges to some $\hat{U}_i$. Denote $\hat{U}_0 = \psi(t_j)$, $\hat{U}_N = 0$. Let $\hat{S}(x)$ be a cubic spline, interpolating $\hat{U}_i$, $i=0, 1, \ldots, N$ and $\hat{S}'(x)=0$, $\hat{S}'''(x_1-0)=\hat{S}'''(x_1+0)$. Then $\int_0^A \hat{S}_k(x)\,dx$ converges to $\int_0^A \hat{S}(x)\,dx$. From (40) it follows that, if $\int_0^A \hat{S}(x)\,dx < 0$, then $\hat{P}_k$ converges to 0. If $\int_0^A \hat{S}(x)\,dx \geqq 0$, then $\hat{P}_k$ converges to $\int_0^A \hat{S}(x)\,dx$. So $\hat{P}_k$ converges to $\max(0, \int_0^A \hat{S}(x)\,dx)$, i.e. $\hat{U}_i$ and $\hat{P} = \max(0, \int_0^A S(x)\,dx)$ are solutions of (14)-(18).
We resume this result in the following theorem:

Theorem 1. *For* $0 < h \leqq \min(h_4, h_5)$, h_4 *and* h_5 *are determined in* (35) *and* (42) *respectively, the solutions* $\hat{U}_i$, $i=1, 2, \ldots, N-1$ *of the system* (14)-(18) *exist. These solutions can be obtained by sequences* (37), (38).

4. Uniqueness of the numerical solutions

Theorem 2. *For* $0 < h \leqq \min(h_4, h_5)$ *the solutions* $\hat{U}_i$, $i=1, 2, \ldots, N-1$ *of the system* (14)-(18) *are unique.*

Proof. Let $\tilde{U}_i$, also be solutions of (14)-(18), i.e.

$$\tilde{U}_i = U_{i-1}\Big[1 - \frac{2h}{3} d(x_{i-1}, t_{j-1}, P) - \frac{h^2}{6} F_{i-1,j-1}(P, P')\Big]$$

$$-\frac{h}{3} d(x_i, t_j, \tilde{P})\, \tilde{U}_i, \tag{43}$$

$$\tilde{U}_0 = \psi(t_j), \quad \tilde{U}_N = 0.$$

Let

$$\tilde{P} = \max(0, \int_0^A \tilde{S}(x)\,dx),$$

$\tilde{S}(x)$ is a cubic spline, interpolating $\tilde{U}_i$, $i=0, 1, \ldots, N$ and $\tilde{S}'(x_N)=0$, $\tilde{S}'''(x_1-0)=\tilde{S}'''(x_1+0)$. Here $\tilde{U}_i$ may not satisfy (25).
Denote

$$V_i = \hat{U}_i - \tilde{U}_i.$$

Then from (14) and (43) we obtain the equalities

$$V_i = -\frac{h}{3}[d(x_i, t_j, \hat{P})\tilde{U}_i - d(x_i, t_j, \tilde{P})\tilde{U}_i]$$

$$= -\frac{h}{3}[d(x_i, t_j, \hat{P})\hat{U}_i - d(x_i, t_j, \tilde{P})\tilde{U}_i$$

$$+ d(x_i, t_j, \hat{P})\hat{U}_i - d(x_i, t_j, \tilde{P})\tilde{U}_i]$$

$$= \frac{h}{3}d(x_i, t_j, \tilde{P})V_i - \frac{h}{3}d'_3(\tau)\hat{U}_i(\hat{P} - \tilde{P}),$$

where τ is a value between $\hat{P}$ and $\tilde{P}$.

According to Lemmas 1 and 3 we obtain

$$|\hat{P} - \tilde{P}| \leqq |\int_0^A [\hat{S}(x) - \tilde{S}(x)]\, dx| \leqq C_5 \|\hat{U}_i - \tilde{U}_i\| = C_5 \|V_i\|.$$

Then

$$\|V_i\| \leqq \frac{h}{3} C_1 C_5 C_{11} \|V_i\|.$$

For $0 < h \leqq h_5$ it follows that $\hat{U}_i = \tilde{U}_i$. Theorem 2 is proved.

5. Convergence of the numerical solutions

In this part we shall study the convergence of $U_{i,j}$ to $u(x_i, t_j)$. We note that if $G(\tau)$ is a quadratic Hermite spline for $f(\tau) = \bar{d}(\tau)\bar{u}(\tau)$, then for smooth $f(\tau)$ one has (see, for example, [7], p. 304)

$$|f(\tau) - G(\tau)| \leqq C_{15} h^3,$$

where C_{15} is independent of h. From (11) we obtain that

$$u(x_i, t_j) = u(x_{i-1}, t_{j-1}) - \frac{2h}{3} d(x_{i-1}, t_{j-1}, P(t_{j-1}))u(x_{i-1}, t_{j-1})$$

$$-\frac{h^2}{6} F_{i-1,j-1}(P(t_{j-1}), \frac{d}{dt}P(t_{j-1}))$$

$$-d(x_i, t_j, P(t_j))u(x_i, t_j) + \tau_{i,j}, \tag{44}$$

with

$$|\tau_{i,j}| \leqq C_{16} h^4, \tag{45}$$

where C_{16} is independent of h.

Now subtracting (14) from (44) we get

$$u(x_i, t_i) - U_{i,j} = [u(x_{i-1}, t_{j-1}) - U_{i-1,j-1}]$$

$$-\frac{2h}{3}[d(x_{i-1}, t_{j-1}, P(t_{j-1}))u(x_{i-1}, t_{j-1}) - d(x_{i-1}, t_{j-1}, P)U_{i-1,j-1}]$$

$$-\frac{h^2}{6}[F_{i-1,j-1}(P(t_{i-1}), \frac{d}{dt}P(t_{j-1}))u(x_{i-1}, t_{j-1})$$

$$-F_{i-1,j-1}(P, P')U_{i-1,j-1}]$$

$$-\frac{h}{3}[d(x_i, t_j, P(t_j))u(x_i, t_j)-d(x_i, t_j, \hat{P})U_{i,j}]+\tau_{i,j}$$

$$=[u(x_{i-1}, t_{j-1})-U_{i-1,j-1}]$$

$$*[1-\frac{2h}{3}d(x_{i-1}, t_{j-1}, P]-\frac{h^2}{6}F_{i-1,j-1}(P, P')]$$

$$-u(x_{i-1}, t_{j-1})\{\frac{2h}{3}[d(x_{i-1}, t_{j-1}, P(t_{j-1}))-d(x_{i-1}, t_{j-1}, P)]$$

$$-\frac{h^2}{6}[F_{i-1,j-1}(P(t_{i-1}), \frac{d}{dt}P(t_{j-1}))-F_{i-1,j-1}(P, P')]\}$$

$$-\frac{h}{3}[u(x_i, t_j)-U_{i,j}]d(x_i, t_j, \hat{P})$$

$$-\frac{h}{3}u(x_i, t_j)[d(x_i, t_j, P(t_j))-d(x_i, t_j, \hat{P})]+\tau_{i,j}$$

$$=[u(x_{i-1}, t_{j-1})-U_{i-1,j-1}]$$

$$*[1-\frac{2h}{3}d(x_{i-1}, t_{j-1}, P]-\frac{h^2}{6}F_{i-1,j-1}(P, P')]$$

$$-u(x_{i-1}, t_{j-1})\{\frac{2h}{3}[d_3(P(t_{j-1}))-d_3(P)]$$

$$-\frac{h^2}{6}[F_{i-1,j-1}(P(t_{i-1}), \frac{d}{dt}P(t_{j-1}))-F_{i-1,j-1}(P, P')]\}$$

$$-\frac{h}{3}[u(x_i, t_j)-U_{i,j}]d(x_i, t_j, \hat{P})$$

$$-\frac{h}{3}u(x_i, t_j)[d_3(P(t_j))-d_3(\hat{P})]+\tau_{i,j}. \tag{46}$$

Now let $s_j(x)$ be a cubic spline, interpolating $u(x_i, t_j)$ and $s_j'(x_N)=0$, $s_j'''(x_1-0)=s_j'''(x_1+0)$. If $(\partial/\partial x)^3 u(x, t)$ is a continuous function, then

$$\max_{0\leq x\leq A} |(\frac{d}{dx})^k [u(x, t_j)-s_j(x)]| \leq C_{17} h^{3-k}, \; k=0, 1, 2 \tag{47}$$

where C_{17} is independent of h. Denote

$$P_j=\max(0, \textstyle\int_0^A s_j(x)\,dx),$$
$$P'_j=\psi(t_j)-\textstyle\int_0^A d(x;\ t_j,\ P_j)s_j(x)dx.$$

From Lemma 3 and (47) we obtain

(48) $$|P(t_j)-P_j|\leqq|\textstyle\int_0^A [u(x,\ t_j)-s_j(x)]\,dx|\leqq AC_{18}h^3.$$

We can now rewrite (46) in the following form

$$u(x_i,\ t_j)-U_{i,j}=\frac{1-\dfrac{2h}{3}d(x_{i-1},\ t_{j-1},\ P)-\dfrac{h^2}{6}F_{i-1,j-1}(P,P')}{1+\dfrac{h}{3}d(x_i,\ t_j,\ \hat{P})}$$

$$*[u(x_{i-1},\ t_{j-1})-U_{i-1,j-1}]$$

(49) $$+\frac{1}{1+\dfrac{h}{3}d(x_i,\ t_j,\ \hat{P})}(R_1+R_2+R_3+R_4+R_5+\tau_{i,j}),$$

where

$$R_1\equiv-u(x_{i-1},\ t_{j-1})\frac{2h}{3}[d_3(P(t_{j-1}))-d_3(P_{j-1})],$$

$$R_2\equiv-u(x_{i-1},\ t_{j-1})\frac{2h}{3}[d_3(P(t_{j-1}))-d_3(P)],$$

$$R_3\equiv-u(x_{i-1},\ t_{j-1})$$
$$*\frac{h^2}{6}[F_{i-1,j-1}(P(t_{i-1}),\ \frac{d}{dt}P(t_{j-1}))-F_{i-1,j-1}(P,\ P')],$$

$$R_4\equiv-u(x_i,\ t_j)\frac{h}{3}[d_3(P(t_j))-d_3(P_j)],$$

$$R_5\equiv-u(x_i,\ t_j)\frac{h}{3}[d_3(P_j)-d_3(\hat{P})].$$

From Lemma 1, 3 and (48) we get the following inequalities

(50) $$|R_1|\leqq\frac{2}{3}C_1C_{11}AC_{18}h^4,$$

(51) $$|R_4|\leqq\frac{1}{3}C_1C_{11}AC_{18}h^4,$$

(52) $$|R_2|\leqq\frac{2h}{3}C_1C_{11}C_5\|u(x_i,\ t_{j-1})-U_{i,j-1}\|,$$

$$(53)\qquad |R_5| \leqq \frac{h}{3} C_1 C_{11} C_5 \|u(x_i, t_j) - U_{i,j}\|,$$

where C_1, C_5, C_{11}, C_{18} are given in (5), (20), (26) and (48) respectively.

Now we are going to estimate R_3. For this purpose we have to estimate $|P(t_{j-1}) - P|$, $|\frac{\mathrm{d}}{\mathrm{d}t} P(t_{j-1}) - P'|$. From Lemma 1, 3 and (48) we have

$$|P(t_{j-1}) - P| \leqq |P(t_{j-1}) - P_{j-1}| + |P_{j-1} - P|$$

$$(54)\qquad \leqq A C_{18} h^3 + C_5 \|u(x_i, t_{j-1}) - U_i\|.$$

Lemma 4. *The inequality*

$$(55)\qquad |\frac{\mathrm{d}}{\mathrm{d}t} P(t_{j-1}) - P'| \leqq C_{19} h^2 + \frac{1}{h} C_{20} \|u(x_i, t_{j-1}) - U_i\|,$$

holds. Here C_{19}, C_{20} are independent of h.

Proof. Using the representation (21) and the system (22), (23) one can prove that

$$(56)\qquad \|(\frac{\mathrm{d}}{\mathrm{d}x})^k [t_{j-1}(x) - S(x)]\| \leqq \frac{1}{h^k} C_{21} \|u(x_i, t_{j-1}) - U_i\|, \quad k = 0, 1, 2$$

C_{21} is independent of h.

From assumption 2) we obtain the equalities

$$(57)\qquad \frac{\mathrm{d}}{\mathrm{d}t} P(t_{j-1}) - P' = R_6 + R_7,$$

$$R_6 \equiv -\int_0^A d_1(x)[u(x, t_{j-1}) - S(x)]\,\mathrm{d}x,$$

$$R_7 \equiv -[d_2(t_{j-1}) + d_3(P(t_{j-1}))] \int_0^A [u(x, t_{j-1}) - S(x)]\,\mathrm{d}x.$$

From (47) and (56) we get that

$$(58)\qquad |R_7| \leqq (C_7 + C_8) A [C_{18} h^3 + C_{21} \|u(x_i, t_{j-1}) - U_i\|].$$

Consider now R_6. In the case, when $d_1'(x)$ is bounded on $[0, A]$, from (47) and (56) we have

$$(59)\qquad |R_6| \leqq C_6 A [C_{18} h^3 + C_{21} \|u(x_i, t_{j-1}) - U_i\|].$$

In the case, when $d_1'(x)$ is not bounded on $[0, A]$ we can rewrite R_6 in the following form

$$R_6 = R_8 + R_9 + R_{10},$$

$$R_8 \equiv -\int_0^{A_1} d_1(x)[u(x, t_{j-1}) - S(x)]\,\mathrm{d}x,$$

$$R_9 \equiv -\int_{A_1}^{x_{N-1}} d_1(x)[u(x, t_{j-1}) - S(x)]\,\mathrm{d}x,$$

$$R_{10} \equiv -\int_{x_{N-1}}^{A} d_1(x)[u(x, t_{j-1}) - S(x)]\,\mathrm{d}x.$$

Obviously

(60) $$|R_8| \leqq C_6 A[C_{18} h^3 + C_{21} \|u(x_i, t_{j-1}) - U_i\|].$$

Using integration by part and the conditions that $u(A, t_{j-1}) - S(A) = \frac{d}{dt}[u(A, t_{j-1}) - S(A)] = 0$ for R_{10} we obtain that

$$R_{10} = \int_{x_{N-1}}^{A} \int_{x_{N-1}}^{x} d_1(\tau)\, d\tau \frac{d}{dx}[u(x, t_{j-1}) - S(x)]\, dx$$

$$= - \int_{x_{N-1}}^{A} \int_{x_{N-1}}^{x} \int_{x_{N-1}}^{\xi} d_1(\tau)\, d\tau\, d\xi \left(\frac{d}{dx}\right)^2 [u(x, t_{j-1}) - S(x)]\, dx.$$

Consequently, from (6), (48) and (56) we can get

(61) $$|R_{10}| \leqq C_{22} h[C_{18} h + \frac{C_{21}}{h^2} \|u(x_i, t_{j-1}) - U_i\|],$$

$$C_{22} \equiv \max_{A_1 \leqq x \leqq A} \int_{x_{N-1}}^{x} \int_{x_{N-1}}^{\xi} d_1(\tau)\, d\tau\, d\xi.$$

We estimate now R_9. If $\alpha \leqq 1$, where α is given in (6), we have

(62) $$|R_9| \leqq A \frac{C_2}{h^\alpha}[C_{18} h^3 + C_{21} \|u(x_i, t_{j-1}) - U_i\|].$$

In the case $1 < \alpha < 2$ we have

$$|R_9| \leqq \max_{A_1 \leqq x \leqq A} |u(x, t_{j-1}) - S(x)| \Sigma_{i=k}^{N-1} \int_{x_{i-1}}^{x_i} d_1(x)\, dx$$

$$\leqq \frac{C_2}{h^{\alpha-1}}[C_{18} h^3 + C_{21} \|u(x_i, t_{j-1}) - U_i\|] \Sigma_{i=k}^{N-1} \frac{1}{(N-i)^\alpha}$$

(63) $$\leqq C_2 C_{23} [C_{18} h^{4-\alpha} + \frac{C_{21}}{h^{\alpha-1}} \|u(x_i, t_{j-1}) - U_i\|]$$

where $x_{k-1} \leqq A_1 < x_k$,

$$C_{23} \equiv \Sigma_{i=1}^{\infty} \frac{1}{(i)^\alpha}$$

From (57)-(63) we obtain (55). Lemma 4 is proved.

Lemma 5. *The following estimate*

(64) $$|R_3| \leqq C_{24} h^2 + C_{25} \|u(x_i, t_{j-1}) - U_i\|,$$

is true. In (64) C_{24}, C_{25} *are independent of* h.

Proof. We have

$$R_3 = R_{11} + R_{12},$$

$$R_{11} \equiv -u(x_{i-1}, t_{j-1})\frac{h^2}{6}[\frac{d}{dt}P(t_{j-1})d'(P(t_{j-1})) - P'd'_3(P)],$$

$$R_{12} \equiv u(x_{i-1}, t_{j-1})\frac{h^2}{6}[d^2(x_{i-1}, t_{j-1}, P(t_{j-1}) - d^2(x_{i-1}, t_{j-1}, P)],$$

For R_{11} from (54) and Lemma 4 we get that

$$|R_{11}| \leqq \frac{h^2}{6}C_1\{|\frac{d}{dt}P(t_{j-1})[d'_3(P(t_{j-1})) - d'_3(P)]|$$

$$+|d'_3(P)|[\frac{d}{dt}P(t_{j-1}) - P']\}$$

$$\leqq \frac{h^2}{6}C_1\{C_{26}C_{12}[AC_{18}h^3 + C_5\|u(x_i, t_{j-1}) - U_i\|]$$

$$+C_{11}[C_{19}h^2 + \frac{1}{h}C_{20}\|iu(x_i, t_{j-1}) - U_i\|]\}, \tag{65}$$

where

$$C_{26} \equiv \max_{0 \leqq t \leqq B} |\frac{d}{dt}P(t)|.$$

Now we rewrite R_{12} in the following form

$$R_{12} = u(x_{i-1}, t_{j-1})\frac{h^2}{6}[d(x_{i-1}, t_{j-1}, P(t_{j-1})) + d(x_{i-1}, t_{j-1}, P)]$$

$$* [d(x_{i-1}, t_{j-1}, P(t_{j-1})) - d(x_{i-1}, t_{j-1}, P)]$$

$$= \frac{h^2}{6}u(x_{i-1}, t_{j-1})[2d(x_{i-1}, t_{j-1}, P(t_{j-1})) + d_3(P) - d_3(P(t_{j-1}))]$$

$$* [d_3(P(t_{j-1})) - d_3(P)].$$

Consequently

$$|R_{12}| \leqq \frac{h^2}{6}[2C_{27} + 2C_1C_8]C_{11}[AC_{18}h^2 + C_5\|u(x_i, t_{j-1}) - U_i\|], \tag{66}$$

$$C_{27} \equiv \max_{0 \leqq x \leqq a,\ 0 \leqq t \leqq B} (u(x, t)d(x, t, P(t))$$

$$= \max_{0 \leqq x \leqq A,\ 0 \leqq t \leqq B} |\frac{\partial}{\partial t}u(x, t) + \frac{\partial}{\partial x}u(x, t)|.$$

From (65) and (66) we get (64). Lemma 5 is proved.

Now we can prove the convergence of $U_{i,j}$ to $u(x_i, t_j)$.

Theorem 3. *The numerical solutions $U_{i,j}$ converge to the exact solution $u(x_i, t_j)$ of the problem* (1)-(4) *when $h \to 0$ and the rate of this convergence is $O(h^3)$.*

Proof. For every j denote

$$\varepsilon_j = \|u(x_i, t_j) - U_{i,j}\|.$$

We shall show that

$$\varepsilon_j \leqq C_{28} h^3,$$

where C_{28} is independent of h. In fact, using Lemma 2, (45), (49)-(53) and (64) we get that

$$\|u(x_i, t_j) - U_{i,j}\| \leqq \frac{1 + C_{29} h}{1 - C_{30} h} \|u(x_i, t_{j-1}) - U_{i,j-1}\| + C_{31} h^4,$$

where

$$C_{29} \equiv \frac{2}{3} C_1 C_5 C_{11} + C_{25}, \quad C_{30} \equiv \frac{1}{3} C_1 C_5 C_{11},$$

$$C_{31} \equiv (C_1 C_{11} C_{19} A + C_{24} + C_{16})/(1 - h_5 C_{30}).$$

So we get the following estimate

$$\varepsilon_j \leqq q_2 \varepsilon_{j-1} + C_{31} h^4, \quad q_2 \equiv (1 + C_{29} h)/(1 - C_{30} h).$$

If $C_{29} = 0$ and $C_{30} = 0$, then

$$\varepsilon_j \leqq C_{31} j h^3. \tag{67}$$

Now consider the case $C_{29} > 0$ or $C_{30} > 0$, i.e. $q_2 > 1$. Then

$$\begin{aligned} \varepsilon_j &\leqq C_{31} h^4 + q_2 C_{31} h^4 + \ldots + (q_2)^{j-1} C_{31} h^4 \\ &= \frac{C_{31}}{C_{29} + C_{30}} [(q_2)^j - 1](1 - C_{30} h) h^3. \end{aligned} \tag{68}$$

For $j = 1, 2, \ldots, [B/h]$, $[f]$ is the integer part of f, from (67) and (68) we get that

$$\varepsilon_j \leqq C_{28} h^3,$$

where C_{28} is inependent of h. Theorem 3 is proved.

References

1. T. V. Kostova. Numerical solutions for a hyperbolic differential-integral equation. *Comput. Math. Applic.*, **15**, 1988, 427-436.
2. M. E. Gurtin, R. C. McCamy. Nonlinear age-dependent population dynamics. *Arch. Ration. Mech. Analysis*, **54**, 1974, 281-300.
3. G. D. Blasio. Asymptotic behavior of an age-structured fish population. *Comput. Math. Applic.*, **9**, 1983, 377-381.

4. T. Arbogast, F. A. Milner. A finite difference method for a two-sex model of population dynamics. *SIAM J. Num. Anal.*, **26**, 1989, 1474-1486.
5. Jia Li. Persistence and extinction in continuous age-structured population models. *Comput. Math. Applic.*, **15**, 1988, 511-523.
6. I. X. Zavialov, B. I. Kvasov, B. L. Miroshnitčenco. The methods of splines. Moscow, 1980.
7. N. P. Korneichuc. Splines in the approximation theory. Moscow, 1984.
8. Hoang Van Lai. On convergence of numerical solution for a hyperbolic differential-integral equation by splines. Centre for System and Management Research, Hanoi, (to appear).

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