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Absolute Summability Factors of Infinite Series

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In this paper a general theorem on φ-$|C, 1|_k$ summability factors has been proved. Also this theorem includes some known results.

1. Introduction

Let Σa_n be a given infinite series with partial sums (s_n). By u_n we denote the n-th $(C, 1)$ mean of the sequence (s_n). The series Σa_n is said to be summable $|C, 1|_k$, $k \geqq 1$, if (see [5])

$$\sum_{n=1}^{\infty} n^{k-1}|u_n - u_{n-1}|^k < \infty. \tag{1.1}$$

Let (φ_n) be a sequence of complex numbers. The series Σa_n is said to be summable $\varphi - |C, 1|_k$, $k \geqq 1$, if (see [1])

$$\sum_{n=1}^{\infty} |\varphi_n (u_n - u_{n-1})|^k < \infty. \tag{1.2}$$

If we take $\varphi_n = n^{1-(1/k)}$, then $\varphi - |C, 1|_k$ summability is the same as $|C, 1|_k$ summability. A sequence (λ_n) is said to be convex if $\Delta^2 \lambda_n \geqq 0$ for every positive integer n, where $\Delta^2 \lambda_n = \Delta(\Delta \lambda_n)$ and $\Delta \lambda_n = \lambda_n - \lambda_{n+1}$.

2. H. Bor [3] proved the following theorem.

Theorem A. *Let*

$$\Delta^2 \lambda_n \geqq 0, \ \Sigma \frac{\lambda_n}{n} < \infty. \tag{2.1}$$

If there exists an $\varepsilon > 0$ such that the sequence $(n^{\varepsilon - k}|\varphi_n|^k)$ is non-increasing and

$$\sum_{v=1}^{n} v^{-k} |\varphi_v s_v|^k = 0 \ (\log n\gamma_n) \ \textit{as} \ n \to \infty, \tag{2.2}$$

where (γ_n) is a positive non-decreasing sequence such that

(2.3) $$n\gamma_n \log n\Delta(1/\gamma_n)=O(1) \text{ as } n\to\infty,$$

then the series $\Sigma a_n \lambda_n (\gamma_n)^{-1}$ *is summable* $\varphi-|C,1|_k$, $k\geqq 1$.

3. The aim of this paper is to prove the following theorem.

Theorem. *Let*

(3.1) $$\lambda_n=o(1) \text{ and } \sum_{v=1}^{n} v\log v|\Delta^2\lambda_v|=O(1) \text{ as } n\to\infty.$$

If there exists an $\varepsilon>0$ *such that the sequence* $(n^{\varepsilon-k}|\varphi_n|^k)$ *is non-increasing and*

(3.2) $$\sum_{v=1}^{n} v^{-k}|\varphi_v t_v|^k=O(\log n\gamma_n) \text{ as } n\to\infty,$$

where (γ_n) *is as in Theorem A and* (t_n) *is the n-th* $(C,1)$ *mean of the sequence* (na_n), *then the series* $\Sigma a_n \lambda_n(\gamma_n)^{-1}$ *is summable* $\varphi-|C,1|_k$, $k\geqq 1$.

It is well known (see [4], [7], [9]) that (2.1) implies (3.1). It can also be easily shown that (2.2) implies (3.2). However, the converse of these implications need not be true.

4. Proof of the theorem. Let u_n and t_n be n-th Cesàro means of order 1 of the series Σa_n and of the sequence (na_n), respectively. But since $t_n=n(u_n-u_{n-1})$ (see [6]), it is enough to show that

(4.1) $$\sum_{n=1}^{\infty} n^{-k}|\varphi_n T_n|^k<\infty,$$

where

(4.2) $$T_n=(n+1)^{-1}\sum_{v=1}^{n} va_v\lambda_v(\gamma_v)^{-1}.$$

Now, applying Abel's transformation to the sum (4.2), we get

$$T_n=(n+1)^{-1}\sum_{v=1}^{n-1}\Delta\lambda_v(v+1)t_v(\gamma_v)^{-1}+(n+1)^{-1}\sum_{v=1}^{n-1}\lambda_{v+1}\Delta(1/\gamma_v)(v+1)t_v$$
$$+\lambda_n t_n(\gamma_n)^{-1}=T_{n,1}+T_{n,2}+T_{n,3}, \text{ say.}$$

To prove the theorem, by Minkowski's inequality, it is sufficient to show that

(4.3) $$\sum_{n=1}^{\infty} n^{-k}|\varphi_n T_{n,r}|^k<\infty, \text{ for } r=1,2,3.$$

Now, applying Hölder's inequality, we have that

$$\sum_{n=2}^{m+1} n^{-k}|\varphi_n T_{n,1}|^k=\sum_{n=2}^{m+1} n^{-k}|\varphi_n|^k(n+1)^{-k}\left|\sum_{v=1}^{n-1}\Delta\lambda_v(v+1)t_v(1/\gamma_v)\right|^k$$
$$=\sum_{n=2}^{m+1} n^{-k}(n+1)^{-k}|\varphi_n|^k\left|\sum_{v=1}^{n-1}\Delta\lambda_v(1/\gamma_v)t_v\frac{v+1}{v}v\right|^k$$

$$= O(1) \sum_{n=1}^{m+1} n^{-2k} |\varphi_n|^k \left\{ \sum_{v=1}^{n-1} v |\Delta\lambda_v| (1/\gamma_v) |t_v| \right\}^k$$

$$= O(1) \sum_{n=2}^{m+1} \frac{1}{n^{k+1}} |\varphi_n|^k \left\{ \sum_{v=1}^{n-1} v |\Delta\lambda_v| (1/\gamma_v) |t_v|^k \right\} x \left\{ \frac{1}{n} \sum_{v=1}^{n-1} v |\Delta\lambda_v| (1/\gamma_v) \right\}^{k-1}$$

$$= O(1) \sum_{v=1}^{m} v |\Delta\lambda_v| (1/\gamma_v) |t_v|^k \sum_{n=v+1}^{m+1} \frac{|\varphi_n|^k}{n^{k+1}}$$

$$= O(1) \sum_{v=1}^{m} v |\Delta\lambda_v| (1/\gamma_v) |t_v|^k v^{\varepsilon-k} |\varphi_v|^k \int_v^\infty \frac{dx}{x^{1+\varepsilon}} = O(1) \sum_{v=1}^{m} v |\Delta\lambda_v| (1/\gamma_v) v^{-k} |\varphi_v t_v|^k$$

$$= O(1) \sum_{v=1}^{m-1} \Delta \left\{ v |\Delta\lambda_v| \frac{1}{\gamma_v} \right\} \sum_{p=1}^{v} p^{-k} |\varphi_p t_p|^k + O(1) m |\Delta\lambda_m| \frac{1}{\gamma_m} . \sum_{v=1}^{m} v^{-k} |\varphi_v t_v|^k$$

$$= O(1) \sum_{v=1}^{m-1} \Delta \left\{ v |\Delta\lambda_v| \frac{1}{\gamma_v} \right\} \gamma_v \log v + O(1) m |\Delta\lambda_m| \log m$$

$$= O(1) \sum_{v=1}^{m-1} v |\Delta^2\lambda_v| \log v + O(1) \sum_{v=1}^{m-1} v |\Delta\lambda_{v+1}| \gamma_v \Delta(1/\gamma_v) \log v$$

$$+ O(1) \sum_{v=1}^{m-1} |\Delta\lambda_{v+1}| \log v + O(1) m |\Delta\lambda_m| \log m.$$

Since $v\gamma_v \Delta(1/\gamma_v) \log v = O(1)$, by (2.3), we have that

$$\sum_{n=2}^{m+1} n^{-k} |\psi_n T_{n,1}|^k = O(1) \sum_{v=1}^{m-1} v |\Delta^2\lambda_v| \log v + O(1) \sum_{v=1}^{m-1} |\Delta\lambda_{v+1}|$$

$$+ O(1) \sum_{v=1}^{m-1} |\Delta\lambda_{v+1}| \log v + O(1) m |\Delta\lambda_m| \log m = O(1) \text{ as } m \to \infty,$$

by virtue of the hypotheses of the theorem.
Again

$$\sum_{n=2}^{m+1} n^{-k} |\varphi_n T_{n,2}|^k = \sum_{n=2}^{m+1} n^{-k} (n+1)^{-k} |\varphi_n|^k \left| \sum_{v=1}^{n-1} \lambda_{v+1} \Delta(1/\gamma_v) \frac{v+1}{v} v t_v \right|^k$$

$$= O(1) \sum_{n=2}^{m+1} n^{-2k} |\varphi_n|^k \left\{ \sum_{v=1}^{n-1} |\lambda_{v+1}| v |t_v| \Delta(1/\gamma_v) \right\}^k .$$

Since $v\Delta(1/\gamma_v) = O(1) \dfrac{1}{\gamma_v \log v}$, by (2.3), as in $T_{n,1}$ we have that

$$\sum_{n=2}^{m+1} n^{-k} |\varphi_n T_{n,2}|^k = O(1) \sum_{n=2}^{m+1} n^{-2k} |\varphi_n|^k \left\{ \sum_{v=1}^{n-1} \frac{|\lambda_{v+1}| |t_v|}{\gamma_v \log(v+1)} \right\}^k$$

$$= O(1) \sum_{n=2}^{m+1} \frac{1}{n^{k+1}} |\varphi_n|^k \sum_{v=1}^{n-1} \frac{|\lambda_{v+1}|}{\gamma_v} |t_v|^k x \left\{ \frac{1}{n} \sum_{v=1}^{n-1} \frac{|\lambda_{v+1}|}{\gamma_v} \right\}^{k-1}$$

$$=O(1)\sum_{v=1}^{m}\frac{|\lambda_{v+1}|}{\gamma_v}v^{-k}|\varphi_v t_v|^k=O(1)\sum_{v=1}^{m}v^{-k}|\varphi_v t_v|^k\left|\sum_{n=v}^{\infty}\Delta\left(\frac{|\lambda_{n+1}|}{\gamma_n}\right)\right|$$

$$=O(1)\sum_{v=1}^{\infty}\left|\Delta\left(\frac{|\lambda_{v+1}|}{\gamma_v}\right)\right|\sum_{n=1}^{v}n^{-k}|\varphi_n t_n|^k=O(1)\sum_{v=1}^{\infty}\left|\Delta\left(\frac{|\lambda_{v+1}|}{\gamma_v}\right)\right|\gamma_v\log v$$

$$=O(1)\sum_{v=1}^{\infty}|\Delta\lambda_{v+1}|\log v+O(1)\sum_{v=1}^{\infty}|\lambda_{v+2}|\gamma_v\Delta(1/\gamma_v)\log v$$

$$=O(1)\sum_{v=1}^{\infty}|\Delta\lambda_{v+1}|\log v+O(1)\sum_{v=1}^{\infty}\frac{|\lambda_{v+2}|}{v}$$

$$=O(1)\sum_{v=1}^{\infty}|\Delta\lambda_{v+1}|\log v+O(1)\sum_{v=1}^{\infty}\frac{1}{v}\left|\sum_{n=v}^{\infty}\Delta|\lambda_{n+2}|\right|$$

$$=O(1)\sum_{v=1}^{\infty}|\Delta\lambda_{v+1}|\log v+O(1)\sum_{v=1}^{\infty}|\Delta\lambda_{v+2}|\sum_{n=1}^{v}\frac{1}{n}$$

$$=O(1)\sum_{v=1}^{\infty}|\Delta\lambda_{v+1}|\log(v+1)+O(1)\sum_{v=1}^{\infty}|\Delta\lambda_{v+2}|\log(v+2).$$

Since $\sum_{v=1}^{\infty}v|\Delta^2\lambda_v|\log v<\infty$, by hypothesis, we have that

$$\sum_{n=2}^{m+1}n^{-k}|\varphi_n|^k|T_{n,2}|^k=O(1)\quad\text{as } m\to\infty.$$

Finally, as in $T_{n,2}$, we get that

$$\sum_{n=1}^{m}n^{-k}|\varphi_n T_{n,3}|^k=O(1)\sum_{n=1}^{m}\frac{|\lambda_n|}{\gamma_n}n^{-k}|\varphi_n t_n|^k=O(1)\quad\text{as } m\to\infty.$$

Therefore, we get

$$\sum_{n=1}^{m}n^{-k}|\varphi_n T_{n,r}|^k=O(1)\quad\text{as } m\to\infty,\ \text{for } r=1,\ 2,\ 3.$$

This completes the proof of the theorem.

Special cases:

1) If we take $\gamma_n=1$ in our theorem, then we get a theorem of H. Bor [2].
2) If we take $\gamma_n=1$, $\varepsilon=1$ and $\varphi_n=n^{1-k^{-1}}$, then we obtain a theorem due to S. M. Mazhar [8].
3) Finally, if we take $\varepsilon=1$ and $\varphi_n=n^{1-k^{-1}}$, in our theorem, then we get a result due to M. Ali Sarigöl [10].

References

1. M. Balci. Absolute φ-summability factors. *Comm. Fac. Sci. Univ. Ankara, Ser. A_1*, **29**, 1980, 63-68.
2. H. Bor. On the summability factors of infinite series. *Comm. Fac. Sci. Univ. Ankara, Ser. A_1*, **32**, 1983, 7-11.
3. H. Bor. The absolute summability factors of infinite series. *Pure and Applied Math. Sciences*, **XVIII**, No 1-2, 1983, 75-79.
4. H. C. Chow. On the summability factors of Fourier series. *J. Lond. Math. Soc.*, **16**, 1941, 215-220.
5. T. M. Flett. On an extension of absolute summability and some theorems of Littlewood Paley. *Proc. Lond. Math. Soc.*, **7**, 1957, 113-141.
6. E. Kogbetliantz. Sur les séries absolument sommables par la méthode des moyennes arithmétiques. *Bull. Sci. Math.* (*2*), **49**, 1925, 234-256.
7. S. M. Mazhar. On the summability factors of infinite series. *Publicationes Math. Debrecen*, **13**, 1966, 229-236.
8. S. M. Mazhar. On $|C, 1|_k$ summability factors of infinite series. *Indian Jour. Math.*, **14**, 1972, 45-48.
9. T. Pati. Absolute Cesaro summability of infinite series. *Math. Zeith.*, **78**, 1962, 293-297.
10. M. Ali Sarigöl. On $|C, 1|_k$ summability factors. *Pure and Appl. Math. Sciences*, **XXVIII**, No 1-2, 1988, 85-89.

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