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On Conformally Flat Parakählerian Manifolds

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Let M be a conformally flat parakählerian manifold. It is proved that M is locally flat if $\dim M \geqq 6$, and M is locally symmetric and its scalar curvature vanishes identically if $\dim M = 4$. Next it is shown that if $\dim M = 4$ and M is not locally flat, then it is either locally a product of two 2-dimensional parakählerian spaces of nonzero constant opposite Gauss curvatures, or locally some special non-decomposable parakählerian space appeared by Patterson.

1. Preliminaries

Let M be a $2n$-dimensional (connected) differentiable manifold of class C^∞. The all objects involved on M will be also of class C^∞. M is said to be an almost paracomplex manifold (P. Libermann [5]) if it is endowed with a (1,1)-tensor field J(called the almost paracomplex structure of M) satisfying the conditions

(1.1) $$J^2 = I \quad (= \text{the identity operator}),$$

(1.2) $$\begin{cases} \text{for each } p \in M, \text{ the } \pm 1\text{-eigenspaces } M_p^{\pm} \text{ of } J_p \text{ are both} \\ n\text{-dimensional subspaces of the tangent space } M_p. \end{cases}$$

Assume that M is an almost paracomplex manifold. Define the Nijenhuis torsion tensor N of J by putting

$$N(X, Y) = [JX, JY] - J[JX, Y] - J[X, JY] + J^2[X, Y]$$

for any $X, Y \in \mathfrak{X}(M)$, $\mathfrak{X}(M)$ being the Lie algebra of vector fields on M. M is said to be paracomplex if $N = 0$. The almost paracomplex manifold M is paracomplex if and only if the distributions $\mathscr{D}^{\pm} : M \ni p \to M_p^{\pm}$ are both completely integrable (S. Kaneyuki and M. Kozai [2]). Suppose that g is a pseudo-Riemannian metric on M satisfying the condition

(1.3) $$g(JX, Y) + g(X, JY) = 0$$

for $X, Y \in \mathfrak{X}(M)$. Thus, M is an almost parahermitian manifold. Define the 2-form Ω on M by $\Omega(X, Y) = g(X, JY)$ for $X, Y \in \mathfrak{X}(M)$. M is said to be a parakäklerian manifold if it is paracomplex and the 2-form Ω is closed (cf. [5], [2]). In this case the pair (J, g) is called the parakählerian structure of M.

It is already known (cf. [2]) that an almost parahermitian manifold M is parakählerian if and only if $\nabla J=0$, where ∇ is the Levi-Civita connection on M. Another interesting characterization of the parakählerian manifolds gives the following theorem.

Theorem 1. *A pseudo-Riemannian manifold M of dimension $2n$ is a parakählerian manifold if and only if there are two n-dimensional totally isotropic and parallel distributions $\mathscr{H}$ and $\mathscr{V}$ on M such that $\mathscr{H}\cap\mathscr{V}=\{0\}$.*

Proof. If M is parakählerian, then it is sufficient to take $\mathscr{H}=\mathscr{D}^-$ and $\mathscr{V}=\mathscr{D}^+$. The isotropy of $\mathscr{D}^-$ and $\mathscr{D}^+$ follows from (1.3), and the parallelity from $\Delta J=0$. Conversely, assume that on a pseudo-Riemannian manifold M, $\dim M=2n$, we have two n-dimensional totally isotropic and parallel distributions $\mathscr{H}$ and $\mathscr{V}$ such that $\mathscr{H}\cap\mathscr{V}=\{0\}$. Then $TM=\mathscr{H}\oplus\mathscr{V}$. Let H and V be the projections corresponding to the distributions $\mathscr{H}$ and $\mathscr{V}$, respectively. Define $J=-H+V$. It is now a straighforward verification that J satisfies (1.1)-(1.3) and moreover $\nabla J=0$. Thus, M is parakählerian. QED.

2. Locally symmetric parakählerian manifolds

Recently, the global structure of parahermitian symmetric spaces (which are necessarily parakählerian manifolds) was investigated by S. Kaneyuki [1], S. Kaneyuki and M. Kozai [2], [3] from the Lie group-theoretic point of view. Especially, they gave the infinitesimal classification of such manifolds with semi-simple automorphism groups.

On the other hand, some time ago pseudo-Riemannian $2n$-dimensional spaces (in the classical sense) admitting two non-intersecting n-dimensional totally isotropic and parallel distributions were studied by R. Raševski [9], B. A. Rosenfeld [10], and independently by H. S. Ruse [11], E. M. Patterson [6], [7]. Such spaces are of course parakählerian spaces (cf. Th. 1). However, these spaces were called by Patterson Kähler spaces because of the defining condition was formally similar to Kähler's condition in the theory of complex manifolds (see [6], p. 117). E. M. Patterson [6], Th. 4 and 5, has found a necessary and sufficient condition for such a space to be a Riemann extension (of order 2; cf. A. G. Walker [15]), and next he proved that such a space is locally symmetric if and only if it is a Riemann extension with respect to the both distributions. This enabled him to get in [7], Th. 2, canonical forms for the metrics of all non-decomposable locally symmetric parakählerian spaces. One of the Patterson's results, which we need in the sequel, can be formulated as follows (see [7]), p. 289; correct the mistake in his formula).

Proposition 1. *The metric of a non-decomposable locally symmetric 4-dimensional parakählerian space which is not an Einstein space can be transformed into the form*

$$ds^2=(x^3)^2\,dx^1dx^2+\{\theta(x^3)^2+x^3x^4\}(dx^2)^2+2dx^1dx^3+2dx^2dx^4, \tag{2.1}$$

where θ is a constant and $(x^1,\dots,x^4)$ are canonical coordinates. The 2-dimensional totally isotropic and parallel distributions are given here as follows (cf. (4.3) in [6]): $\mathscr{H}$ is spanned by the vector fields

$$E_1=\frac{\partial}{\partial x^1}-\frac{1}{4}(x^3)^2\frac{\partial}{\partial x^4},\ E_2=\frac{\partial}{\partial x^2}-\frac{1}{4}(x^3)^2\frac{\partial}{\partial x^3}-\frac{1}{2}\{\theta(x^3)^2+x^3x^4\}\frac{\partial}{\partial x^4},$$

and $\mathscr{V}$ *by the vector fields* $E_3=\frac{\partial}{\partial x^3},\ E_4=\frac{\partial}{\partial x^4}$.

Remark 1. For the components of the curvature tensor R and the Ricci tensor ρ of the metric (2.1), we have

$$R_{3123}=R_{3224}=\frac{1}{2},\ R_{3223}=\theta,\ \text{otherwise}\ R_{hijk}=0,$$

$$\rho_{22}=\frac{1}{2}(x^3)^2,\ \rho_{23}=1,\ \text{otherwise}\ \rho_{ij}=0.$$

Hence, its scalar curvature τ vanishes identically, and the components of the Weyl's conformal curvature tensor C are

$$C_{3223}=\theta,\ \text{otherwise}\ C_{hijk}=0.$$

Consequently, the rank of the Ricci tensor equals 2, and the metric is conformally flat if and only if $\theta=0$.

3. Curvature identities

It is known that there are analogies between the identities for the curvature tensors and the Ricci tensors of parakählerian and kählerian manifolds (P. Libermann [5], M. Prvanović [8], G. Vranceanu and R. Rosca [14], et al.). We recall here basic curvature identities for parakählerian manifolds since we need them in the next section.

Let M be a parakählerian manifold and denote by $R(X, Y)$ the curvature operator $[\nabla_X, \nabla_Y]-\nabla_{[X,Y]}$. Here and in the sequel X, Y, Z, W always denote arbitrary vector fields on M. By the parallelity of J we have $R(X, Y)J=JR(X, Y)$. Hence it follows for the curvature operator that (cf. [2])

$$(3.1)\qquad R(JX, JY)=-R(X, Y),\ R(JX, Y)+R(X, JY)=0,$$

and for the curvature tensor $R(X, Y, Z, W)=g(Z, R(X, Y)W)$

$$(3.2)\ R(JX,JY,JZ,JW)=R(X,Y,Z,W),\ R(JX,Y,JZ,W)=R(X,JY,Z,JW).$$

Let ρ be the Ricci tensor of M, that is,

$$\rho(Y, Z)=\text{trace}\{X\to R(X, Y)Z\}=\sum_{i=1}^{2n}\varepsilon_i R(E_i, Y, E_i, Z),$$

where $\{E_1,\dots, E_{2n}\}$ is an orthonormal frame of M and $\varepsilon_i=g(E_i, E_i)=\pm 1$. As a consequence of (3.2) we can find

$$(3.3)\qquad \rho(JY, JZ)=-\rho(Y, Z),\ \rho(JY, Z)+\rho(Y, JZ)=0.$$

Hence, for the Ricci operator Q defined by the condition $g(QX, Y)=\rho(X, Y)$, we have

(3.4) $$QJ=JQ.$$

Using (3.1) and the first Bianchi identity, we obtain

(3.5) $$\rho(Y,\ Z)=-\frac{1}{2}\sum_{i=1}^{2n}\varepsilon_i R(E_i,\ JE_i,\ Y,\ JZ),$$

and consequently

(3.6) $$(\nabla_X\rho)(Y,\ Z)=-\frac{1}{2}\sum_{i=1}^{2n}\varepsilon_i(\nabla_X R)(E_i,\ JE_i,\ Y,\ JZ).$$

By the relation (3.6) and the second Bianchi identity, we find

(3.7) $$(\nabla_X\rho)(Y,\ JZ)+(\nabla_Y\rho)(Z,\ JX)+(\nabla_Z\rho)(X,\ JY)=0.$$

The identity (3.7) shows that the 2-form ω defined by $\omega(X,\ Y)=\rho(X,\ JY)$ is closed.

4. Conformal flatness in parakählerian manifolds

Since any parakählerian (and generally, any pseudo-Riemannian as well as Riemannian) manifold of dimension 2 is conformally flat, we are interested in dimensions $2n\geqq 4$. At first, using the identities from the previous section, we prove the following theorem.

Theorem 2. *Let M be a conformally flat parakählerian manifold. Then* (i) *M is locally flat if $\dim M\geqq 6$,* (ii) *M is locally symmetric and its scalar curvature vanishes identically if $\dim M=4$.*

Proof. By the vanishing of the Weyl's conformal curvature, we have

(4.1) $$R(X,\ Y,\ Z,\ W)=\frac{1}{2n-2}\{g(X,\ Z)\rho(Y,\ W)+g(Y,\ W)\rho(X,\ Z)$$
$$-g(X,\ W)\rho(Y,\ Z)-g(Y,\ Z)\rho(X,\ W)\}$$
$$-\frac{\tau}{2(2n-1)(n-1)}\{g(X,\ Z)g(Y,\ W)-g(X,\ W)g(Y,\ Z)\},$$

τ being the scalar curvature of M. With the help of (4.1) one immediately gets

$$\sum_{i=1}^{2n}\varepsilon_i R(E_i JE_i,\ Z,\ JW)=-\frac{2}{n-1}\rho(Z,\ W)+\frac{\tau}{(2n-1)(n-1)}g(Z,\ W),$$

in which we have also used the identities (1.1) and (3.3). The last equality compared with (3.5) leads to

(4.2) $$(2n-4)\rho(Z,\ W)=-\frac{\tau}{2n-1}g(Z,\ W).$$

Since $\tau = \operatorname{trace} Q = \sum_{i=1}^{2n} \varepsilon_i \rho(E_i, E_i)$, from (4.2) it follows that $\tau = 0$. Thus, if $\dim M = 2n \geqq 6$, then from (4.2) we have $\rho = 0$, which together with (4.1) gives the local flatness of M. Assume now that $2n = 4$. By (4.2) we have $\tau = 0$. Moreover, as it is well-known, (4.1) implies the following identity for the Ricci tensor

$$(\nabla_X \rho)(Y, Z) - (\nabla_Y \rho)(X, Z) = \frac{1}{6}\{(X\tau) g(Y, Z) - (Y\tau) g(X, Z)\}.$$

This, by $\tau = 0$, yields

$$(\nabla_X \rho)(Y, Z) - (\nabla_Y \rho)(X, Z) = 0. \tag{4.3}$$

On the other hand, note that from (3.3) it follows that

$$(\nabla_X \rho)(Y, JZ) + (\nabla_X \rho)(Z, JY) = 0.$$

Using the above equality and (4.3), we show that

$$(\nabla_X \rho)(Y, JZ) + (\nabla_Y \rho)(Z, JX) = 0.$$

This together with (3.7) gives $\nabla \rho = 0$, which by virtue of (4.1) implies $\nabla R = 0$. QED.

Now we are going to describe completely the local structure of conformally flat parakählerian manifolds of dimension 4. We need the following result.

Proposition 2. *If M is a conformally flat and locally decomposable pseudo-Riemannian manifold, then M is locally symmetric and we have one of the followings:*

(i) *M is locally flat,*

(ii) *M is locally a product $M_1 \times M_2$ of pseudo-Riemannian or Riemannian spaces M_1 and M_2, where M_1 is of constant curvature $K > 0$ and M_2 is of constant curvature $-K$,*

(iii) *M is locally a product $M_3 \times M_4$ of pseudo-Riemannian or Riemannian spaces M_3 and M_4, where M_3 is of nonzero constant curvature and $\dim M_4 = 1$.*

The assertion of the above proposition seems to be known, and can be obtained by the direct analysis of the products of metrics. For Riemannian manifolds (the metrics are here positive definite), this assertion follows from results of M. Kurita [4].

Corollary 1. *Let M be a 4-dimensional conformally flat parakählerian manifold. Assume that the pseudo-Riemannian metric g of M is locally decomposable. Then M is either*

(a) *locally flat, or*

(b) *locally a product $M_1 \times M_2$ of two 2-dimensional parakählerian spaces M_1 and M_2, where M_1 is constant Gauss curvature $K > 0$ and M_2 is of constant Gauss curvature $-K$.*

Proof. The case (a) is obvious. Assume that M is not locally flat. For the metric g we have two cases (ii) and (iii) as in Proposition 2. With the help of Theorem 2 we easily eliminate the case (iii). Thus, we shall concentrate on the case (ii) only and show that the almost paracomplex structure J of M is locally decomposable according as the metric g decomposes. But it easily follows by using the relation (3.4) and the following two facts: $QX = KX$ (resp., $-KX$) if and only if X is tangent to M_1 (resp., M_2). Finally, from the parallelity of J it can be deduced that both almost parahermitian structures induced on M_1 and M_2 are indeed parakählerian. QED.

Corollary 2. *Let M be a 4-dimensional conformally flat parakählerian manifold, whose the metric is not locally decomposable. Then locally, in appropriate coordinate systems, the parakählerian structure of M can be given as in Proposition* 1 *with* $\theta = 0$.

Proof. From our assumptions and Theorem 2 we see that M is locally symmetric and $\rho \neq \frac{\tau}{4} g$ at each point of M. The assertion follows now from Proposition 1 and Remark 1. It could be added that one has also to take into account the homogenity of the curvature (or the Ricci curvature) of M and Corollary 1. QED.

Remark 2. The assertion of Theorem 2 holds good for Kählerian manifolds. Indeed, the analogy of the case (i) is proved by K. Yano and I. Mogi [16], and the analogy of the case (ii) follows from results of S. Tachibana [12] and S. Tanno [13]. Our Corollary 1 is related to results of S. Tanno [13], and Corollary 2 does not have any analogy in the Kählerian case.

References

[1] S. Kaneyuki. On classification of parahermitian symmetric spaces. *Tokyo J. Math.*, **8**, 1985, 473-482.

[2] S. Kaneyuki, M. Kozai. Paracomplex structures and affine symmetric spaces. *Tokyo J. Math.*, **8**, 1985, 81-98.

[3] S. Kaneyuki, M. Kozai. On the isotropy subgroup of the automorphism group of a parahermitian symmetric space. *Tokyo J. Math.*, **8**, 1985, 483-490.

[4] M. Kurita. On the holonomy group of the conformally flat Riemannian manifold. *Nagoya Math. J.*, **9**, 1955, 161-171.

[5] P. Libermann. Sur le probleme d'equivalence de certaines structures infinitesimales. *Ann. Mat. Pura Appl.*, **36**, 1954, 27-120.

[6] E. M. Patterson. Riemann extensions which have Kähler metrics. *Proc. Roy. Soc. Edinburgh (A)*, **64**, 1954, 113-126.

[7] E. M. Patterson. Symmetric Kähler spaces. *J. London Math., Soc.*, **30**, 1955, 286-291.

[8] M. Prvanović. Holomorphically projective transformations in a locally product space. *Math. Balkanica*, **1**, 1971, 195-213.

[9] R. Raševski. A scalar field in a fibred space (in Russian). *Trudy Sem. Vect. Tens. Anal.*, **6**, 1948, 225-248.

[10] B. A. Rosenfeld. On unitary and fibred spaces (in Russian). *Trudy Sem. Vect. Tens. Anal.*, **7**, 1949, 260-275.

[11] H. S. Ruse. On parallel fields of planes in a Riemannian space. *Quart. J. Math. Oxford (I)*, **20**, 1949, 218-234.

[12] S. Tachibana. On automorphisms of conformally flat K-spaces. *J. Math. Soc. Japan*, **13**, 1961, 183-188.

[13] S. Tanno. 4-dimensional conformally flat Kähler manifolds. *Tôhoku Math. J.*, **24**, 1972, 501-504.

[14] G. Vrănceanu, R. Roşca. Introduction in relativity and pseudo-Riemannian geometry. Bucureşti 1976.

[15] A. G. Walker. Riemann extensions of non-Remannian spaces. Convegno di Geometria Differenziale, pp. 64-70, Roma 1953.

[16] K. Yano, I. Mogi. On real representations of Kählerian manifolds. *Ann. of Math.*, **61**, 1955, 170-189.

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