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Affine and Rigid Motions in Complex Standard Vector Spaces

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Presented by P. Kenderov

No question is ever settled until it is settled right
Rudyard Kipling

The present article is a part of a series of investigations of the authors dedicated to the solution of Hilbert's Sixth Problem concerning the axiomatical consolidation of the foundations of analytical mechanics. It proposes an axiomatical definition of the notions of affine and rigid motions in complex standard vector spaces together with some mathematical corollaries from this definition.

§ 0. Pro domo sua

The rigid body concept has been the object of some previous investigations [1-3] of the authors of this paper. As a matter of fact, however, what was actually studied in these articles were the motions of rigid bodies rather than the rigid bodies themselves.

On the other hand, meanwhile it has been discovered [4] that there exist particular Hermitean spaces H for which it is possible to define a fourth operation (vector multiplication) by the aid of two specific algebraic axioms. More precisely, it has been proved that a necessary and sufficient condition to this end is the 3-dimensionality of H. Such Hermitean spaces supplied with vector multiplication are called complex standard vector spaces and are denoted by V_C by contrast with the real standard vector spaces traditionally denoted by V.

This definition works at considerably more general situations. Namely, H may be replaced by an Hermitean space $H_{C(F)}$ over the complex extension $C(F)$ of any ordered field F, and it is proved that $H_{C(F)}$ is necessarily 3-dimensional. The corresponding standard vector space over $C(F)$ is denoted by $V_{C(F)}$.

Now it is very desirable to establish to what degree the geometric and mechanical notions originally defined and developed in V may be extended in V_C and even in $V_{C(F)}$.

As regards the analytic geometry, for instance, such a parallelism has been established in the articles [5, 6], the first of which proposes an axiomatical consolidation of the real 3-dimensional linear analytic geometry, while the second concerns the complex case.

Another example of real and complex standard vectors analogies is proposed by the a l g e b r a i c t h e o r y o f a r r o w s (vecteurs glissants, gleitende Vektoren, скользящие векторы), the real case being studied in the article [7, 8], and some aspects of the complex case in the papers [9-11].

The main object of the present investigation are the a f f i n e and r i g i d m o t i o n s, the definitions and the main properties of which are given and studied in complex standard vector spaces.

Strange though it may seem, the mechanical literary sources even of recent time are entirely void of a strict mathematical definition of the rigid body concept capable of satisfying the logical standards of twentieth century's mathematics. The authors of mechanical textbooks, treatises, and monographs purely and simply take it for granted that the readers are familiar with the rigid body concept on an intuitive (alias physical) level at least; that this notion is *a priori* self-evident and requires, therefore, no mathematical definition at all; and that all that remains to be done is to describe mathematically the mechanical behaviour of this statical and dynamical entity in various situations under the action of certain systems of forces.

A vivid picture of the present state of affairs in rational mechanics, as regards the rigid body concept, has been drawn by W. N o l l in his article [12]:

"It is a wide spread belief even today that classical mechanics is a dead subject, that its foundations were made clear long ago, and that all that remained to be done is to solve special problems. This is not so. It is true that mechanics of systems of a finite number of mass points has been on a sufficiently rigorous basis since Newton. Many textbooks on theoretical mechanics dismiss continuous bodies with the remark that they can be regarded as the limiting case of a particle system with an increasing number of particles. They can not. The erroneous belief that they can had the unfortunate effect that no serious attempt was made for a long period to put classical continuous mechanics on a rigorous axiomatic basis" (p. 266).

This anachronism is a lamentable outgrowth of the Lagrangean dynamical tradition. It has been conceived in Lagrange's notorious work *Méchanique Analitique* (sic) [13] and has been brought up by his uncritical adepts. In the *Avertisssement* of his book Lagrange has written:

"On ne trouvera point de Figures dans cette Ouvrage. Les méthodes que j'y expose ne demandent ni constructions, ni raisonnements géométriques ou méchaniques, mais seulement des opérations algébriques assujettis à une marche régulière et uniforme."

As it is more often than not the case, before misleading his readers Lagrange misleads himself. The only true something the above excerpt contains is that there are no figures indeed in the whole of *Méchanique Analitique*. No mechanics can be made without geometry – neither mathematical, nor technical. Without geometry analytical mechanics demotes to flesh without a skeleton. Everybody who has some, even though elementary, experience in solving dynamical problems *with understanding* (mathematically, professionally, non-extemporaneously, we mean) is aware of the cold fact that it is absolutely unthinkable to make a single step in analytical dynamics without geometry.

The fundamental ideas lying at the bottom of the basic for analytical mechanics concept of m o t i o n and r i g i d b o d y are purely geometrical. The belief that it is possible to make analytical mechanics by the aid of "méthodes... ne demandent ni constructions, ni raisonnements géométriques ou méchaniques" is not only erroneous, skin-deep and immature: this belief is dangerous at the same time. Its immediate sequels are formalism, superficiality, and incompetence.

It deprives analytical mechanics of its very essence. In the course of two clear centuries (since the publication of the *Méchanique Analitique*, as a matter of fact) this ill-fated belief has played a fatal rôle for analytical mechanics · it tricked this science out of its most profound problems and drove it to a state the most characteristic feature of which consists in the fact that — *horresco referens* — not a single dynamical problem concerning rigid bodies subjected to geometrical constraints has been solved in the proper manner since the appearance of the *Méchanique Analitique* in God's earth.

Such statements might sound heretically for Lagrange's most ardent admirers. For a series of mechanical generations the *Méchanique Analitique* has meant neither more nor less than the Bible for an orthodox Christian. *Voilà* some most instructive samples of almost religious admirations:

F. W. A. Murhard (the German translator of the *Méchanique Analitique*) has written in the Vorrede of his version [14]:

"Herr de la Grange hatte sich dabei vorgestellt, alle Grundlehren der mechanischen Wissenschaften unter einem Gesichtspunkt darzustellen, und auf allgemeine Formeln zu bringen. Hierzu war er nun freylich mehr als irgend jemand anders im Stande, er, der schon so manchen Preis davon getragen, so manches Problem glücklich aufgelöst hatte, der mit dem, was vor ihm gethan war, völlig bekannt war, und der sogar einen Euler an Stärke in Kalkül übertraf... Schon Herr Euler forderte ja in seiner Mechanik, daß der Leser sowohl in der Analysis endlicher als unendlicher Größen genugsam geübt sey, konnte dies Herr de la Grange in unsern aufgeklärten Zeiten nach einem verflossenen Zeitraum von mehr als 50 Jahren nicht auch fordern? — Ja sicher konnte er es fordern, wo nicht in Deutschland, doch in seinem Vaterlande! — Sowenig also dies Werk Anfängern im Calcul irgend eine angenehme Stunde verschaffen wird; ... um so mehr werden Liebhaber tiefsinniger analytischer Untersuchungen mit dem größten Vergnügen alle mechanischen Gesetze hier entwikkelt finden... Mehrmals habe ich dies Meisterwerk mit dem größten Fleiße ganz durchstudiert, und stets nur aus der Hand gelegt, um es mit doppeltem Vergnügen bald wiederum vorzunehmen."

About half a century later, in his article [15], W. R. Hamilton has written:

"The theoretical development of the laws of motion of bodies is of such interest and importance, that it has engaged the attention of all the most eminent mathematicians, since the invention of dynamics as a mathematical science by Galileo... Among the successors of those illustrious men, Lagrange has perhaps done more than any other analyst, to give extent and harmony to such deductive researches, by showing that the most varied consequences respecting the motions of systems of bodies may be derived from one radical formula; the beauty of the method so suiting the dignity of the results, as to make his great work a kind of scientific poem."

In our times, another most glowing Lagrangeanist has written in his treatise [16] on analytical dynamics the following eulogy:

"The whole of analytical dynamics is based upon, and is developed from, the theorem of Lagrange that I call the *fundamental equation*... The beautiful and powerful theorem contained in the equations (6.2.1) and (6.2.2) was established by Lagrange in 1760. It provides a simple and expeditious method of forming the equations of motion for any dynamical system... The equations have a central place in Lagrange's great work, the "Mécanique Analytique" [sic]... published in 1788, one of the epoch-making books in the history of mathematics... The Mécanique Analytique is the primary source of the subject of analytical dynamics, and it is rightly regarded as one of the outstanding intellectual achievements of mankind" (p. VII, 76).

Mais revenons à nos moutons. Referring to the geometry that the analytical mechanics necessitates, we by no means bear in mind the synthetic Euclidean geometry at all costs. It is true that mechanics is compelled to reject entirely, finally, and once for all, the synthetic geometry, in order to become analytical. In the same time it is also true that the Euclidean geometry has its algebraic equivalent in the face of analytic geometry in standard vector spaces.

The standard vector spaces being — in consequence of a *dira necessitas* called by the mechanical praxis in the course of the last century — accepted in the capacity of a constructional material for analytical mechanics, in a quite natural manner a mathematically social errand at once arises: to create an analytical geometry in standard vector spaces. In other words, on the basis of appropriate axioms and definitions involving $V_{C(F)}$, only $V_{C(F)}$, and nothing but $V_{C(F)}$, to construct an analytic geometry not only free from the aegis of the synthetic Euclidean geometry, but also incarnating the categorical repudiation of anything connected with the latter. This problem can be solved, and it has been actually solved in the paper [6] already quoted above.

The theory of arrows, contrary to the wide-spread opinion, is applicable, in analytical mechanics, not only to statics and dynamics, providing it with the genuine mathematical device for a formal representations of the f o r c e s, but also to kinematics, supplying it with the mathematical tool for the s t a t i c a l-k i n e m a t i c a l a n a l o g y. This circumstance, as well as the tendency for a peremptory emancipation of analytical mechanics from synthetic Euclidean geometry substantiated above, bring forth a program toward an algebraic theory of arrows on the basis of corresponding axioms and definitions involving again $V_{C(F)}$, only $V_{C(F)}$, and nothing but $V_{C(F)}$. This program can be realized, and it is actually realized, in the papers [10, 11] cited above.

After these preliminary remarks, we can proceed to our essential task: to propose an analytic theory of affine and rigid motions in complex standard vector spaces. It is true that, for analytical mechanics in the traditional sense of the word, the real case is sufficient. Mathematicians, however, have an enormous appetite: they are striving for swallowing as big a morsel as they can sink their teeth into. If seriously, as regards analytical mechanics, working in $V_{C(F)}$ instead of V only, one responds Hilbert's directive formulated in his famous *Mathematische Probleme* [17] (Problem No 6) envisaging the axiomatical consolidation of the logical foundations of rational mechanics:

"Auch wird der Mathematiker, wie er es in der Geometrie getan hat, nicht bloß die der Wirklichkeit nahe kommenden, sondern überhaupt alle logisch möglichen Theorien zu berücksichtigen haben und stets darauf bedacht sein, einen vollständigen Überblick über die Gesamtheit der Folgerungen zu gewinnen, die das gerade angenommene Axiomensystem nach sich zieht" (S. 307).

For the sake of brevity, the symbols Sgn, sgn :, Ax, Df, Pr, Dm, and Sch replace the words notation, denotes (by definition), axiom, definition, proposition, proof, and scholium respectively. The letters R and C are reserved for the fields of all real and all complex numbers respectively, and F and $C(F)$ denote any ordered field and its complex extension respectively. The real and the complex standard vector spaces are denoted by V and V_C respectively, and $V_{C(F)}$, as mentioned above, stands for the standard vector space over $C(F)$.

Quotations are made in the following manner. Sgn 1, Ax 2, Df 3, Pr 4, Sch 5, and relation (6) of § 7, for instance (the example is a fictitious one) are quoted symply as Sgn 1, Ax 2, Df 3, Pr 4, Sch 5, and (6) in § 7 itself, but as 7 Sgn 1, 7 Ax 2, 7 Df 3, 7 Pr 4, 7 Sch 5, and 7 (6) elsewhere.

§ 1. Affine and rigid repers

Sch 1. All considerations in the first 3 paragraphs of this article are made in a rather general, if not in the most general, mathemathical situation underlying *in rerum natura*: in complex standard vector spaces $V_{C(F)}$ over the complex extensions $C(F)$ of arbitrary ordered fields F; possibilities for further generalizations are proposed by the Hermitean spaces $H_{C(F)}$ over $C(F)$, see for instance the paper [3]. Yet $V_{C(F)}$ has the priority over $H_{C(F)}$ to have at its disposal the most effective in technical respect operation vector multiplication. It is true that this priority is depreciated to some extent by the fact that this operation is rich in content in the 3-dimensional case only. In the same time it is also true that the most reliable patron of these mathematical wares is analytical mechanics and it certainly does not display a particular zest to travel in multidimensional spaces.

As it is well known, however, the more general a mathematical theory is, the more void of content *in re* it is. *La généralisation pour la généralisation* itself is, however, no nonsense in mathematics; abreast of satisfying a purely epistemological interest, it affords mathematicians to determine the exact borders across which a mathematical notion becomes amorphous. The generalizations are of greatest importance especially in those cases when mathematical notions are still lacking in strict definitions: familiar examples in point from mathematical history are legion in order to be adduced here; in any case, affine and rigid motions, as well as affine and rigid kinematical bodies are still in such a deplorable situation, as far as one could pass judgement upon in virtue of the current mechanical literary sources. Anyway, a sound mathematician would scarcely contest the importance of the mathematical policy of *la generalization pour l'axiomatisation*, and many developments of modern mathematics are fortunate offshoots of this everlasting trunk of mathematical creativity.

Sch 2. For the sake of brevity, a convention will be now proposed: a symbolic settlement, convenient though incorrect, the only vindication of which consists in the considerable technical simplifications due to economy of notations and formulations.

As it is well known from the classical analysis, f being a differentiable function over $T \subset \mathsf{R}$, the relations

$$(1) \qquad \frac{df}{dt}=0 \qquad (\forall\, t \in T),$$

and

$$(2) \qquad f=\text{constant}$$

are equivalent provided T is an interval.

In the sequel relations of the type (2) will be permanently used for the aims of definitions and propositions, for which the relations of the type (1) are, simply and purely, meaningless, for the plain reason that the left-hand sides of (1) do downright not exist.

Indeed, no hypotheses will be made in the first 3 paragraphs of this article either as regards the set-theoretical structure of the definitional domains T of the functions f, or concerning the analytical character of the latter. In particular, T may be nowhere dense, or f can be discontinuous at any point of T. It is more than obvious that, under these conditions, no claims regarding relations of the type (1) could be made, in spite of the fact that conditions of the type (2) may take place either by definition or by proof.

Now the convention mentioned in the beginning of this scholium consists in the acceptance that we shall write down relations of the type (1) and we shall consider them as equivalent with relations of the type (2) notwithstanding the fact that the left-hand sides of (1) are by no means granted. In other words, we shall write (1) as a stenographic record of (2) not only when the existence of the left-hand sides of (1) is most problematic, but even when it is certainly lacking.

The convenience of this agreement will become obvious from the following exposition in § 1-3. The cases, when the existence of the left-hand sides of (1) is *conditio sine qua non* for the mathematical consistency of the definitions and propositions involved, will be clear from the context: in such cases the existence in question will be secured by the conditions of the corresponding definitions and propositions.

For brevity's sake, when referring to the agreement so elaborately explained above, we shall speak of it as the constancy-convention.

Df 1. An affine T-reper in $V_{C(F)}$ is called any triad

(3) $$\tilde{a}=\{\boldsymbol{a}_\nu\}_{\nu=1}^3$$

of vector functions

(4) $$\boldsymbol{a}_\nu : T \to V_{C(F)} \qquad (\nu=1,\ 2,\ 3)$$

with

(5) $$\boldsymbol{a}_1 \times \boldsymbol{a}_2 \cdot \boldsymbol{a}_3 \neq 0 \qquad (\forall t \in T)$$

provided

(6) $$T \subset C(F).$$

Sgn 1. R_T sgn: the set of all affine T-reper in $V_{C(F)}$.

Pr 1. *If*

(7) $$\tilde{a}=\{\boldsymbol{a}_\nu\}_{\nu=1}^3 \in R_T,$$

(8) $$\boldsymbol{a}_\nu^{-1} = \frac{\boldsymbol{a}_{\nu+1} \times \boldsymbol{a}_{\nu+2}}{\boldsymbol{a}_1 \times \boldsymbol{a}_2 \cdot \boldsymbol{a}_3} \qquad (\nu=1,\ 2,\ 3;\ \forall t \in T)$$

provided

(9) $$\boldsymbol{a}_{\nu+3} = \boldsymbol{a}_\nu \qquad (\nu=1,\ 2;\ \forall t \in T),$$

then

(10) $$\{\boldsymbol{a}_\nu^{-1}\}_{\nu=1}^3 \in R_T.$$

Dm. (7), Sgn 1, Df 1 imply (5), hence the vectors (8) with (9) exist. Now [4, Pr 87, Pr 29] imply

(11) $$\boldsymbol{a}_1^{-1} \times \boldsymbol{a}_2^{-1} \cdot \boldsymbol{a}_3^{-1} \neq 0 \qquad (\forall t \in T)$$

and (11), Df 1, Sgn 1 imply (10).

Sgn 2. $\tilde{a}^{-1}$ sgn: $\{\boldsymbol{a}_\nu^{-1}\}_{\nu=1}^3$ iff (7).

Df 2. The affine T-reper $\tilde{a}^{-1}$ is called reciprocal to the affine T-reper $\tilde{a}$.

Df 3. The affine T-reper (7) is called orthonormal iff

(12) $$\boldsymbol{a}_\mu \boldsymbol{a}_\nu = \begin{cases} 1 & (\mu=\nu) \\ 0 & (\mu \neq \nu) \end{cases} \qquad (\mu,\ \nu=1,\ 2,\ 3;\ \forall t \in T).$$

Pr 2. *If*

(13) $$\tilde{a} \in R_T,$$

then

(14) $$\tilde{a}^{-1} = \tilde{a}$$

iff $\tilde{a}$ *is orthonormal.*

Dm. [4, Pr 94] implies

(15) $$\boldsymbol{a}_\nu^{-1} = \boldsymbol{a}_\nu \qquad (\nu=1,\ 2,\ 3;\ \forall t \in T)$$

iff (12). Now Df 3, Sgn 2.

Sgn 3. $\tilde{a} \sim \tilde{b}$ sgn:

(16) $$\frac{d}{dt}(\boldsymbol{a}_\mu \boldsymbol{b}_\nu^{-1}) = 0 \qquad (\mu,\ \nu=1,\ 2,\ 3;\ \forall t \in T)$$

provided (7) and

(17) $$\tilde{b} = \{\boldsymbol{b}_\nu\}_{\nu=1}^{3} \in R_T.$$

Df 4. The affine T-reper $\tilde{a}$ is called e q u i v a l e n t with the affine T-reper $\tilde{b}$ iff

(18) $$\tilde{a} \sim \tilde{b}.$$

Sgn 4. $\tilde{a} \not\sim \tilde{b}$ sgn: one at least of the conditions (16) is violated provided (7) and (17).

Df 5. The affine T-reper $\tilde{a}$ is called n o n - e q u i v a l e n t with the affine T-reper $\tilde{b}$ iff

(19) $$\tilde{a} \not\sim \tilde{b}.$$

Pr 3. (13) *implies*

(20) $$\tilde{a} \sim \tilde{a}.$$

Dm. If (7), then [4, Pr 85] implies

(21) $$\boldsymbol{a}_\mu \boldsymbol{a}_\nu^{-1} = \begin{cases} 1 & (\mu=\nu) \\ 0 & (\mu \neq \nu) \end{cases} \qquad (\mu,\ \nu=1,\ 2,\ 3;\ \forall t \in T).$$

Now Sgn 3.

Pr 4. (13)

(22) $$\tilde{b} \in R_T,$$

(18) *imply*

(23) $$\tilde{b} \sim \tilde{a}.$$

Dm. If (7) and (17), then (18) implies (16) (Sgn 3). On the other hand

(24) $$b_\mu = \sum_{\nu=1}^{3} (b_\mu a_\nu^{-1}) a_\nu \qquad (\mu=1, 2, 3; \forall t \in T)$$

[4, Pr 88] imply

(25) $$\sum_{\nu=1}^{3} (b_\mu a_\nu^{-1})(a_\nu b_\lambda^{-1}) = b_\mu b_\lambda^{-1} = \begin{cases} 1 & (\lambda=\mu) \\ 0 & (\lambda \neq \mu) \end{cases}$$

(λ, $\mu=1, 2, 3; \forall t \in T$) [4, Ax 10, Ax 9, Pr 85], and (25), (16) imply

(26) $$\sum_{\nu=1}^{3} \left(\frac{d}{dt}(b_\mu a_\nu^{-1})\right)(a_\nu b_\lambda^{-1}) = 0$$

(λ, $\mu=1, 2, 3; \forall t \in T$). Let μ be fixed ($1 \leqq \mu \leqq 3$) and let λ take successively the values 1, 2, 3; then (26) represents a system of 3 linear homogeneous algebraic equations with respect to the unknown quantities

(27) $$\frac{d}{dt}(b_\mu a_\nu^{-1}) \qquad (1 \leqq \mu \leqq 3, \ \nu=1, 2, 3; \forall t \in T).$$

The determinant D of the system

(28) $$\sum_{\nu=1}^{3} \left(\frac{d}{dt}(b_\mu a_\nu^{-1})\right)(a_\nu b_\lambda^{-1}) = 0$$

($1 \leqq \mu \leqq 3$; $\lambda=1, 2, 3; \forall t \in T$) is

(29) $$D = \begin{vmatrix} a_1 b_1^{-1} & a_2 b_1^{-1} & a_3 b_1^{-1} \\ a_1 b_2^{-1} & a_2 b_2^{-1} & a_3 b_2^{-1} \\ a_1 b_3^{-1} & a_2 b_3^{-1} & a_3 b_3^{-1} \end{vmatrix} \qquad (\forall t \in T).$$

Now (29) and [4, Pr 27] imply

(30) $$D = (a_3 . a_1 \times a_2)(b_1^{-1} \times b_2^{-1} . b_3^{-1}) \qquad (\forall t \in T).$$

On the other hand,

(31) $$a_3 . a_1 \times a_2 = \overline{a_1 \times a_2 . a_3} \qquad (\forall t \in T)$$

[4, Ax 8], and (31), (5) imply

(32) $$a_3 . a_1 \times a_2 \neq 0 \qquad (\forall t \in T).$$

Now (30), (32), and

(33) $$b_1^{-1} \times b_2^{-1} . b_3^{-1} \neq 0 \qquad (\forall t \in T)$$

[4, Pr 87, Pr 29] imply

(34) $$D \neq 0 \qquad (\forall t \in T).$$

Hence the system (28) admits only the zero-solution for the unknown quantities (27), i.e.

(35) $$\frac{d}{dt}(b_\mu a_\nu^{-1}) = 0 \qquad (\mu, \ \nu=1, 2, 3; \forall t \in T),$$

and (35), Sgn 3 imply (23).

Pr 5. (13), (22), (18),

(36) $$\tilde{c} \in R_T,$$

(37) $$\tilde{b} \sim \tilde{c}$$

imply

(38) $$\tilde{a} \sim \tilde{c}.$$

Dm. If (7), (17), and

(39) $$\tilde{c} = \{c_\nu\}_{\nu=1}^{3} \in R_T,$$

then (18) and

(40) $$\frac{d}{dt}(b_\mu c_\nu^{-1}) = 0 \qquad (\mu,\ \nu = 1,\ 2,\ 3;\ \forall t \in T)$$

(Sgn 3). On the other hand,

(41) $$a_\mu = \sum_{\lambda=1}^{3} (a_\mu b_\lambda^{-1})\, b_\mu \qquad (\mu = 1,\ 2\ 3;\ \forall t \in T)$$

[4, Pr 88], whence

(42) $$a_\mu c_\nu^{-1} = \sum_{\lambda=1}^{3} (a_\mu b_\lambda^{-1})(b_\lambda c_\nu^{-1})$$

(μ, $\nu = 1, 2, 3$; $\forall t \in T$) [4, Ax 10, Ax 9]. Now (42), (18), (40) imply

(43) $$\frac{d}{dt}(a_\mu c_\nu^{-1}) = 0 \qquad (\mu,\ \nu = 1,\ 2,\ 3;\ \forall t \in T)$$

whence (38) (Sgn 3).

Pr 6. *The relation* $\sim$ *in* R_T *defined by Sgn* 3 *is an equivalence relation in* R_T.

Dm. Pr 3-Pr 5.

Sch 3. In the above definitions the adjective *affine* has been repeatedly used. This has been done in order to turn the reader's attention to the fact that in all former considerations the vector functions (4) are *not obliged* either to have constant (with respect to T) lengths, or to make constant angles between themselves (provided these notions "length" and "angle" are defined). Extremely important for the applications, however, are those namely repers, whose vector functions (4) have constant length and make constant angles between themselves. This circumstance gives good ground for the following definitions.

Df 6. An affine T-reper is called rigid (solid, Euclidean) iff

(44) $$\frac{d}{dt}(a_\mu a_\nu) = 0 \qquad (\mu,\ \nu = 1,\ 2,\ 3;\ \forall t \in T).$$

Df 7. An affine T-reper is called non-rigid (non-solid, *non-Euclidean, properly affine*) iff it is not rigid.

Sgn 5. E_T sgn: the set of all rigid T-repers.

Pr 7. *Any orthonormal T-reper is rigid.*

Dm. (7), (12) (Df 3) imply (44).

Pr 8. *If*

(45) $$\tilde{a} \in E_T,$$

then

(46) $$\tilde{a}^{-1} \in E_T.$$

Dm. (7), (45) imply (44) (Df 6). On the other hand, (7) and Sgn 2, Df 6 imply that the proposition will be proved if it is established that

(47) $$\frac{d}{dt}(a_\mu^{-1} a_\nu^{-1})=0 \qquad (\mu,\ \nu=1,\ 2,\ 3;\ \forall t \in T).$$

To this end let us note that (8), (9) imply

(48) $$a_\mu^{-1} a_\nu^{-1} = \frac{a_{\mu+1} \times a_{\mu+2}}{a_1 \times a_2 . a_3} \cdot \frac{a_{\nu+1} \times a_{\nu+2}}{a_1 \times a_2 . a_3} = \frac{\Delta_{\mu\nu}}{\Delta}$$

$(\mu,\ \nu=1,\ 2,\ 3;\ \forall t \in T)$ where

(49) $$\Delta_{\mu\nu} = \begin{vmatrix} a_{\nu+1} a_{\mu+1} & a_{\nu+1} a_{\mu+2} \\ a_{\nu+2} a_{\mu+1} & a_{\nu+2} a_{\mu+2} \end{vmatrix}$$

$(\mu,\ \nu=1,\ 2,\ 3;\ \forall t \in T)$ and

(50) $$\Delta = (a_1 \times a_2 . a_3)\overline{a_1 \times a_2 . a_3} = \begin{vmatrix} a_1^2 & a_1 a_2 & a_1 a_3 \\ a_2 a_1 & a_2^2 & a_2 a_3 \\ a_3 a_1 & a_3 a_2 & a_3^2 \end{vmatrix}.$$

$(\forall t \in T)$ [4, Pr 1, Pr 4, Pr 28]. Now (48)-(50), (44) imply (47).

Pr 9. (45), (22), (18) *imply*

(51) $$\tilde{b} \in E_T.$$

Dm. (45), (3) imply (44) (Sgn 5, Df 6). On the other hand, (18) implies (23) (Pr 4), and (3), (17), (23) imply (35) (Sgn 3). Now (24) imply

(52) $$b_\mu b_\nu = \sum_{\sigma=1}^{3} \sum_{\tau=1}^{3} (b_\mu a_\sigma^{-1})(a_\tau^{-1} b_\nu)(a_\sigma a_\tau)$$

$(\mu,\ \nu=1,\ 2,\ 3;\ \forall t \in T)$ [4, Ax 8-Ax 10] and (52), (35), (44) imply

(53) $$\frac{d}{dt}(b_\mu b_\nu)=0 \qquad (\mu,\ \nu=1,\ 2,\ 3;\ \forall t \in T)$$

whence (51) (Sgn 5, Df 6).

§ 2. Affine and rigid systems of reference

Sch 1. The notion of a m o v i n g s y s t e m o f r e f e r e n c e is traditionally introduced in analytical mechanics as an *a priori-notion*, i.e. on an intuitive physical-geometrical level. It turns out, however, that this notion is susceptible of a strict mathematical formalization in situations much more general than those that have given rise to it, conserving at that its basic attributes: as a matter of fact this notion is definable in arbitrary standard vector spaces over the complex extensions of any ordered fields.

There are two ways, not essentially different from one another, to define moving systems of reference in $V_{C(F)}$. The first of them is more symmetrical, while the second one is technically more convenient. We shall adduce them both consecutively, bringing forward their mutual connection.

Pr 1. 1(4)-1(6),

(1) $$A_\nu : T \to V_{C(F)} \qquad (\nu = 1, 2, 3),$$

(2) $$\boldsymbol{a}_\mu A_\nu + \boldsymbol{a}_\nu A_\mu = 0 \qquad (\mu, \nu = 1, 2, 3; \forall t \in T)$$

imply:

(3) $$\vec{a}_\nu = (\boldsymbol{a}_\nu, A_\nu) \qquad (\nu = 1, 2, 3; \forall t \in T)$$

are non-zero arrows in $V_{C(F)}$.

Dm. The conditions 1(5) imply

(4) $$\boldsymbol{a}_\nu \neq \boldsymbol{o} \qquad (\nu = 1, 2, 3; \forall t \in T)$$

and (2) with $\mu = \nu$ imply

(5) $$\boldsymbol{a}_\nu A_\nu = 0 \qquad (\nu = 1, 2, 3; \forall t \in T).$$

Now [10, 2 Df 5].

Df 1. The set

(6) $$\tilde{\alpha} = \{\vec{a}_\nu\}_{\nu=1}^{3}$$

provided 1(4)-1(6), (1)-(3) is called a (moving) affine T-system of reference in $V_{C(F)}$ (of the first kind).

Sgn 1. $\tilde{A}_T$ sgn: the set of all (6).

Sch 2. The conditions 1 (5), (2) and [4, Pr 103] imply that the system of vector equations

(7) $$\boldsymbol{a} \times \boldsymbol{a}_\nu = A_\nu \qquad (\nu = 1, 2, 3; \forall t \in T)$$

has axactly one solution

(8) $$\boldsymbol{a} : T \to V_{C(F)}$$

namely

(9) $$\boldsymbol{a} = \frac{1}{2} \sum_{\nu=1}^{3} \boldsymbol{a}_\nu^{-1} \times A_\nu \qquad (\forall t \in T).$$

Df 2. The arrows (3) are called the axes of the system of reference (6).

Df 3. The functions 1 (2) are called the axis vectors of the system of reference (6).

Df 4. The function (8) defined by (9) is called the o r i g i n of the system of reference (6).

Pr 2. 1(4)-1(6), (1), (2), (9) *imply*

(10) $$\boldsymbol{a}\,z\,\mathrm{dir}\,\vec{a}_\nu \qquad (\nu=1,\ 2,\ 3;\ \forall t\in T).$$

Dm. (7), [6, 4 Sgn 1].

Df 1 bis. The ordered pair

(11) $$\alpha=(\boldsymbol{a},\ \tilde{a})$$

of a function (8) and a T-reper 1 (13) is called a (m o v i n g) a f f i n e T-s y s t e m o f r e f e r e n c e i n $V_{C(F)}$ (o f t h e s e c o n d k i n d).

Sgn 2. A_T sgn: the set of all (11).

Sch 3. The equivalence of Df 1 and Df 1 bis is obvious. Indeed, let first

(12) $$\tilde{\alpha}=\{\vec{a}_\nu\}_{\nu=1}^{3}\in\tilde{A}_T$$

be given. Then the functions 1 (4), (1) with the properties 1 (5), (2) are known, and the function (8) is defined by (9). In other words,

(13) $$\alpha=(\boldsymbol{a},\ \{\boldsymbol{a}_\nu\}_{\nu=1}^{3})\in A_T$$

is defined.

Inversely, let (13) be given. Then the functions 1 (4) with 1 (5), as well as (8) with (9), are known, and the functions (1) are defined by (7); it is immediately seen that they satisfy the conditions (2). In other words, (12) is defined.

Sch 4. Due to the correspondence between $\tilde{A}_T$ and A_T (see Sch 3), the terms introduced by Df 2-Df 4 for the systems of reference of the first kind may be adapted for the systems of reference of the second kind too. Since the corresponding definitions are obvious, we shall not adduce them explicitly here.

Sch 5. In the following exposition a preference is given to Df 1 bis.

Pr 3. *If*

(14) $$(\boldsymbol{a},\ \tilde{a})\in A_T$$

then

(15) $$(\boldsymbol{a},\ \tilde{a}^{-1})\in A_T.$$

Dm. (14) implies 1 (13), whence

(16) $$\tilde{a}^{-1}\in R_T.$$

Now (14), 6) imply (15) (Df 1 bis, Sgn 2).

Sgn 3. α^{-1} sgn: $(\boldsymbol{a},\ \tilde{a}^{-1})$ iff (14).

Df 5. α^{-1} is called *reciprocal to* α.

Sgn 4. r z α sgn:

(17) $$\frac{d}{dt}((\boldsymbol{r}-\boldsymbol{a})\,\boldsymbol{a}_\nu^{-1})=0 \qquad (\nu=1,\ 2,\ 3;\ \forall t\in T)$$

provided (13), 1 (8), 1 (9),

(18) $$r : T \to \dot{V}_{C(F)}.$$

Df 6. r is called invariable with respect to α iff

(19) $$r \, z \, \alpha$$

provided (13), 1 (8), 1 (9), (18).

Sgn 5. $r \,\bar{z}\, \alpha$ sgn: one at least of the conditions (17) is violated provided (13), 1 (8), 1 (9), (18).

Df 7. r is called non-invariable with respect to α iff

(20) $$r \,\bar{z}\, \alpha$$

provided (13), 1 (8), 1 (9), (18).

Sch 6. In the light of Sch 3 the meaning of the symbols $r z \tilde{\alpha}$ and $r \bar{z} \tilde{\alpha}$ provided (12) and (18) is obvious.

Sch 7. (13), Sgn 2, Df 1 bis, 1 Df 1, (18), [4, Pr 88] imply

(21) $$r - a = \sum_{\nu=1}^{3} ((r-a) a_\nu^{-1}) a_\nu \qquad (\forall t \in T).$$

Now (21), (17) manifest that the meaning of the relation (19) reduces to the requirement that the components $(r-a)a_\nu^{-1}$ $(\nu = 1, 2, 3)$ of the vector function $r-a$ with respect to the reper a_ν $(\nu = 1, 2, 3)$ must remain constant over T.

Pr 4. (13) *implies*

(22) $$a \, z \, \alpha.$$

Dm. Trivial in the light of Sgn 4.

Pr 5. (13) *implies*

(23) $$a + a_\nu \, z \, \alpha \qquad (\nu = 1, 2, 3).$$

Dm. (23) is equivalent with

(24) $$\frac{d}{dt}(a_\mu a_\nu^{-1}) = 0 \qquad (\mu, \nu = 1, 2, 3; \forall t \in T)$$

(Sgn 4). Now 1 (21) implies (24).

Sch. 8. As it is well known from analytic geometry, a necessary and sufficient condition for the colinearity of 3 vectors $r_\nu \in V_{C(F)}$ $(\nu = 1, 2, 3)$ is

(25) $$r_1 \times r_2 + r_2 \times r_3 + r_3 \times r_1 = o$$

[6, 4 Pr 17]. The following proposition displays that the property (25) remains invariant with regard to the relation (19) defined by means of Sgn 4.

Pr 6. *If*

(26) $$\alpha \in A_T$$

(27) $$r_\nu : T \to V_{C(F)} \qquad (\nu = 1, 2, 3),$$

(28) $$r_\nu \, z \, \alpha \qquad (\nu = 1, 2, 3),$$

(29) $\boldsymbol{r}_1 \times \boldsymbol{r}_2 + \boldsymbol{r}_2 \times \boldsymbol{r}_3 + \boldsymbol{r}_3 \times \boldsymbol{r}_1 = \boldsymbol{o}\ (t=\tau \in T)$,

then

(30) $$\boldsymbol{r}_1 \times \boldsymbol{r}_2 + \boldsymbol{r}_2 \times \boldsymbol{r}_3 + \boldsymbol{r}_3 \times \boldsymbol{r}_1 = \boldsymbol{o} \qquad (\forall\, t \in T).$$

Dm. (29) is equivalent with

(31) $$(\boldsymbol{r}_1 - \boldsymbol{r}_3) \times (\boldsymbol{r}_2 - \boldsymbol{r}_3) = \boldsymbol{o} \qquad (t=\tau \in T)$$

and (28) is equivalent with

(32) $$\frac{d}{dt}((\boldsymbol{r}_\mu - \boldsymbol{a})\,\boldsymbol{a}_\nu^{-1}) = 0 \qquad (\mu,\ \nu = 1,\ 2,\ 3;\ \forall\, t \in T)$$

provided (13), whence

(33) $$\frac{d}{dt}((\boldsymbol{r}_\lambda - \boldsymbol{r}_\mu)\,\boldsymbol{a}_\nu^{-1}) = 0 \qquad (\lambda,\ \mu,\ \nu = 1,\ 2,\ 3;\ \forall\, t \in T).$$

The identities

(34) $$\boldsymbol{r}_\lambda - \boldsymbol{r}_\mu = \sum_{\nu=1}^{3} ((\boldsymbol{r}_\lambda - \boldsymbol{r}_\mu)\,\bar{\boldsymbol{a}}_\nu^{-1})\,\boldsymbol{a}_\nu \qquad (\lambda,\ \mu = 1,\ 2\ 3;\ \forall\, t \in T)$$

[4, Pr 88], together with (33), imply

(35) $$\boldsymbol{r}_1 - \boldsymbol{r}_3 = \alpha_1 \boldsymbol{a}_1 + \alpha_2 \boldsymbol{a}_2 + \alpha_3 \boldsymbol{a}_3 \qquad (\forall\, t \in T),$$

(36) $$\boldsymbol{r}_2 - \boldsymbol{r}_3 = \beta_1 \boldsymbol{a}_1 + \beta_2 \boldsymbol{a}_2 + \beta_3 \boldsymbol{a}_3 \qquad (\forall\, t \in T),$$

where

(37) $$\frac{d\alpha_\nu}{dt} = \frac{d\beta_\nu}{dt} = 0 \qquad (\nu = 1,\ 2,\ 3;\ \forall\, t \in T).$$

Now (35), (36) imply

(38) $$(\boldsymbol{r}_1 - \boldsymbol{r}_3) \times (\boldsymbol{r}_2 - \boldsymbol{r}_3) = (\bar{\alpha}_1 \bar{\beta}_2 - \bar{\alpha}_2 \bar{\beta}_1)\,\boldsymbol{a}_1 \times \boldsymbol{a}_2 + (\bar{\alpha}_2 \bar{\beta}_3 - \bar{\alpha}_3 \bar{\beta}_2)\,\boldsymbol{a}_2 \times \boldsymbol{a}_3 + (\bar{\alpha}_3 \bar{\beta}_1 - \bar{\alpha}_1 \bar{\beta}_3)\,\boldsymbol{a}_3 \times \boldsymbol{a}_1$$

$(\forall\, t \in T)$ [4, Pr 37], $\bar{\alpha}_\nu$ and $\bar{\beta}_\nu$ denoting the conjugate values of α_ν and β_ν $(\nu = 1, 2, 3)$ respectively, and (38), 1 (8), 1 (9),

(39) $$\alpha_{23} = \bar{\alpha}_2 \bar{\beta}_3 - \bar{\alpha}_3 \bar{\beta}_2,\quad \alpha_{31} = \bar{\alpha}_3 \bar{\beta}_1 - \bar{\alpha}_1 \bar{\beta}_3,\quad \alpha_{12} = \bar{\alpha}_1 \bar{\beta}_2 - \bar{\alpha}_2 \bar{\beta}_1$$

imply

(40) $$(\boldsymbol{r}_1 - \boldsymbol{r}_3) \times (\boldsymbol{r}_2 - \boldsymbol{r}_3) = (\boldsymbol{a}_1 \times \boldsymbol{a}_2 \,.\, \boldsymbol{a}_3)(\alpha_{23}\, \boldsymbol{a}_1^{-1} + \alpha_{31}\, \boldsymbol{a}_2^{-1} + \alpha_{12}\, \boldsymbol{a}_3^{-1})$$

$(\forall\, t \in T)$, where

(41) $$\frac{d\alpha_{12}}{dt} = \frac{d\alpha_{23}}{dt} = \frac{d\alpha_{31}}{dt} = 0 \qquad (\forall\, t \in T),$$

according to (39), (37). On the other hand, (40), 1 (5), (31) imply

(42) $$\alpha_{23}\,a_1^{-1}+\alpha_{31}\,a_2^{-1}+\alpha_{12}\,a_3^{-1}=o \qquad (t=\tau).$$

By virtue of 1(11) the relation (42) implies

(43) $$\alpha_{12}=\alpha_{23}=\alpha_{31}=0 \qquad (t=\tau),$$

and (43), (41) imply

(44) $$\alpha_{12}=\alpha_{23}=\alpha_{31}=0 \qquad (\forall\, t\in T).$$

Now (44), (40) imply

(45) $$(r_1-r_3)\times(r_2-r_3)=o \qquad (\forall\, t\in T).$$

Since (45) is equivalent with (30), the proposition is proved.

Sch 9. As it is well known from analytic geometry, a necessary and sufficient condition for the coplanarity of 4 vectors $\bar{r}_\nu \in V_{C(F)}$ ($\nu=1,\ 2\ 3,\ 4$) is

(46) $$(r_1-r_4)\times(r_2-r_4).(r_3-r_4)=0$$

[6, Pr 19]. The following proposition displays that the property (46) is invariant with regard to the relation (19) defined by Sgn 4.

Pr 7. (26),

(47) $$r_\nu \in V_{C(F)} \qquad (\nu=1,\ 2,\ 3,\ 4),$$

(48) $$r_\nu \quad \alpha \qquad (\nu=1,\ 2,\ 3,\ 4),$$

(49) $$(r_1-r_4)\times(r_2-r_4).(r_3-r_4)=0 \qquad (t=\tau\in T)$$

imply

(50) $$(r_1-r_4)\times(r_2-r_4).(r_3-r_4)=0 \qquad (\forall\, t\in T).$$

Dm. If (13), then in the same manner, as in the demonstration of Pr 6 it is proved that

(51) $$(r_1-r_4)\times(r_2-r_4)=(a_1\times a_2.a_3)(\alpha_{23}\,a_1^{-1}+\alpha_{31}\,a_2^{-1}+\alpha_{12}\,a_3^{-1})$$

($\forall\, t\in T$) with appropriate α_{12}, α_{23}, α_{31} satisfying (41), whence

(52) $$(r_1-r_4)\times(r_2-r_4).(r_3-r_4)=(a_1\times a_2.a_3)(\alpha_{23}(a_1^{-1}(r_3-r_4)) +\alpha_{31}(a_2^{-1}(r_3-r_4))+\alpha_{12}(a_3^{-1}(r_3-r_4)) \qquad (\forall\, t\in T).$$

Now (52), (49), 1(5) imply

(53) $$\alpha_{23}(a_1^{-1}(r_3-r_4))+\alpha_{31}(a_2^{-1}(r_3-r_4))+\alpha_{12}(a_3^{-1}(r_3-r_4))=0$$

($t=\tau$). On the other hand, as in the demonstration of Pr 6 it is proved that (47), (48) imply

(54) $$\frac{d}{dt}((r_3-r_4)\,a_\nu^{-1}=0 \qquad (\nu=1,\ 2,\ 3;\ \forall\, t\in T).$$

Then (54), (41), (53) imply

(55) $$\alpha_{23}(a_1^{-1}(r_3-r_4))+\alpha_{31}(a_2^{-1}(r_3-r_4))+\alpha_{12}(a_3^{-1}(r_3-r_4))=0$$

($\forall\, t\in T$), and (52), (55) imply (50).

Sch 10. Pr 6 and Pr 7 explain the denomination affine system of reference given to the elements of A_T. Physically spoken, the space of the functions (18) satisfying (19) is deformed in the course of the "time" $t \in T$ as an elastic body with a linear deformability, conserving lines and planes.

Sch 11. Let (13),

$$\beta = (b, \{b_\nu\}_{\nu=1}^{3}) \in A_T, \tag{56}$$

$$a \, z \, \beta, \tag{57}$$

$$a + a_\nu \, z \, \beta \qquad (\nu = 1, 2, 3) \tag{58}$$

hold. The definition (17) of the relation (19) implies that the relations (57) and (58) are equivalent with

$$\frac{d}{dt}((a-b)\, b_\nu^{-1}) = 0 \qquad (\nu = 1, 2, 3; \forall t \in T) \tag{59}$$

and

$$\frac{d}{dt}((a+a_\mu-b)\, b_\nu^{-1}) = 0 \qquad (\mu, \nu = 1, 2, 3; \forall t \in T) \tag{60}$$

respectively. On the other hand, (59) and (60) imply

$$\frac{d}{dt}(a_\mu b_\nu^{-1}) = 0 \qquad (\mu, \nu = 1, 2, 3; \forall t \in T) \tag{61}$$

and, inversely, (59) and (61) imply (60). In other words, the systems of conditions (59) and (60), on the one hand, and (59) and (61), on the other hand, are equivalent.

Sch 12. We shall now implant this observation into the definition of a mathematical notion which is basically important for both kinematics and dynamics of rigid bodies: the notion of equivalence of moving affine systems of reference.

Sgn 6. $\alpha \sim \beta$ sgn: (59), (61) provided (13), (56).

Df 8. α is called equivalent with β iff

$$\alpha \sim \beta \tag{62}$$

provided (13), (56).

Sgn 7. $\alpha \not\approx \beta$ sgn: one at least of the conditions (59), (61) is violated provided (13), (56).

Df 9. α is called non-equivalent with β iff

$$\alpha \not\approx \beta \tag{63}$$

provided (13), (56).

Pr 8. (13) *and*

$$\beta \in A_T \tag{64}$$

imply: (62) *iff* (57), (58).

Dm. Sch 11, Sgn 6.

Pr 9. (13), (56) *imply*: (62) *iff* (57) *and*

(65) $$\{a_\nu\}_{\nu=1}^{3} \sim \{b_\nu\}_{\nu=1}^{3}$$

Dm. Pr 8, Sch 11, 1 Sgn 3.

Pr 10. *If*

(66) $$\alpha=(a,\ \tilde{a}) \in A_T,$$

(67) $$\beta=(b,\ \tilde{b}) \in A_T,$$

then (62) *iff* (57) *and* 1 (18).

Dm. Pr 9, Df 1 bis.

Pr 11. (26) *implies*

(68) $$\alpha \sim \alpha$$

Dm. (22), (23), Pr 8.

Pr 12. (26), (64), (62) *imply*

(69) $$\beta \sim \alpha.$$

Dm. If (13), (56), then (69) is equivalent with

(70) $$b\, z\, \alpha,$$

(71) $$\{b_\nu\}_{\nu=1}^{3} \sim \{a_\nu\}_{\nu=1}^{3}$$

(Pr 9). On the other hand, (62) implies (65) (Pr 9), whence (71) by virtue of 1 Pr 4. In other words, the proposition will be proved, if (70) is proved, i. e.

(72) $$\frac{d}{dt}((b-a)\, a_\nu^{-1})=0 \qquad (\nu=1,\ 2,\ 3;\ \forall t \in T)$$

(Sgn 4). To this end let us note that

(73) $$a_\nu^{-1}= \sum_{\mu=1}^{3} (a_\nu^{-1}\, b_\mu)\, b_\mu^{-1} \qquad (\nu=1,\ 2\ 3;\ \forall t \in T)$$

[4, Pr 89] imply

(74) $$(b-a)\, a_\nu^{-1}= \sum_{\mu=1}^{3} (b_\mu\, a_\nu^{-1})((b-a)\, b_\mu^{-1})$$

($\nu=1, 2, 3; \forall t \in T$) [4, Ax 8-Ax 10]. On the other hand, (71) is equivalent with 1 (35) (1 Sgn 3), and (69) implies (59) (Sgn 6). Now (74), 1 (35), (59) imply (72).

Pr 13. (26), (64), (62),

(75) $$\gamma \in A_T,$$

(76) $$\beta \sim \gamma$$

imply

(77) $$\alpha \sim \gamma.$$

Dm. If (13), (56),

(78) $$\gamma=(c, \{c_\nu\}_{\nu=1}^3)\in A_T,$$

then (62) is equivalent with (57) and (65); (76) is equivalent with

(79) $$b \mathrel{z} \gamma,$$

(80) $$\{b_\nu\}_{\nu=1}^3 \sim \{a_\nu\}_{\nu=1}^3;$$

and (77) is equivalent with

(81) $$a \mathrel{z} \gamma,$$

(82) $$\{a_\nu\}_{\nu=1}^3 \sim \{c_\nu\}_{\nu=1}^3$$

(Pr 9). On the other hand, (65) and (80) imply (82) (1 Pr 5). In other words, the proposition will be proved, if (81) is proved, i.e.

(83) $$\frac{d}{dt}((a-c)c_\nu^{-1}=0 \qquad (\nu=1, 2, 3; \forall t\in T)$$

(Sgn 4). To this end let us note that

(84) $$c_\nu^{-1}=\sum_{\lambda=1}^{3}(c_\nu^{-1}b_\lambda)b_\lambda^{-1} \qquad (\nu=1, 2\ 3; \forall t\in T)$$

[4, Pr 89] imply

(85) $$(a-b)c_\nu=\sum_{\lambda=1}^{3}(b_\nu c_\nu^{-1})((a-b)b_\lambda^{-1})$$

($\nu=1, 2, 3; \forall t\in T$) [4, Ax 8-Ax 10]. On the other hand, (80) implies 1 (40) (1 Sgn 3) and (57) implies (59). Now (85), 1 (40), (59) imply

(86) $$\frac{d}{dt}((a-b)c_\nu^{-1})=0 \qquad (\nu=1, 2, 3; \forall t\in T).$$

Besides, (79) implies

(87) $$\frac{d}{dt}((b-c)c_\nu^{-1})=0 \qquad (\nu=1, 2, 3; \forall t\in T)$$

(Sgn 4). Now

(88) $$(a-c)c_\nu^{-1}=(a-b)c_\nu^{-1}+(b-c)c_\nu^{-1}$$

($\nu=1, 2, 3; \forall t\in T$), (86), (87) imply (83).

Pr 14. *The relation* $\sim$ *in* A_T *defined by Sgn 6 is an equivalence relation in* A_T.

Dm. Pr 11-Pr 13.

Pr 15. (26), (64), (62), (18), (19) *imply*

(89) $$r \mathrel{z} \beta.$$

Dm. If (13) and (56), then (62) implies (59), (61) (Sgn 6) and (19) implies (17) (Sgn 4). The relations

(90) $$b_\nu^{-1} = \sum_{\mu=1}^{3} (b_\nu^{-1} a_\mu) a_\mu^{-1} \qquad (\nu=1, 2, 3; \forall t \in T)$$

[4, Pr 88] imply

(91) $$(r-b) b_\nu^{-1} = \sum_{\mu=1}^{3} (a_\mu b_\nu^{-1})((r-b) a_\mu^{-1})$$
$$= \sum_{\mu=1}^{3} (a_\mu b_\nu^{-1})((r-a) a_\mu^{-1}) + \sum_{\mu=1}^{3} (a_\mu b_\nu^{-1})((a-b) a_\mu^{-1})$$

$(\nu=1, 2, 3; \forall t \in T)$ [4, Ax 8-Ax 10], i.e.

(92) $$(r-b) b_\nu^{-1} = \sum_{\mu=1}^{3} (a_\mu b_\nu^{-1})((r-a) a_\mu^{-1}) + (a-b) b_\nu^{-1}$$

$(\nu=1, 2, 3; \forall t \in T)$ [4, Ax 8-Ax 10, Pr 89]. Now (92), (59), (61), (17) imply

(93) $$\frac{d}{dt}((r-b) b_\nu^{-1})=0 \qquad (\nu=1, 2, 3; \ \forall t \in T),$$

whence (89) (Sgn 4).

Pr 16. (26), (64), (18),

(94) $$r \, z \, \alpha \text{ implies } r \, z \, \beta$$

imply (62).

Dm. (22), (23) provided (13) and (94) imply (57), (58), whence (62) (Pr 8).

Pr 17. (26), (64)' *imply: the set of all* (18) *with* (19) *coincides with the set of all* (18) *with* (89) *iff* (62) *holds.*

Dm. Pr 15, Pr 16.

Sch 13. All former considerations in this paragraph concern the general case when no restrictions on the axis vectors 1 (4) are imposed with the only exception of 1 (5). Extremely important for analytical mechanics, however, are those systems of reference (and hence the motions defined by such systems, see § 3) the axis vectors of which have constant lengths and make constant angles between themselves. This circumstance justifies the following definitions.

Df 10. A moving affine T-system of reference is called rigid (solid, Euclidean) iff its axis vectors form a rigid T-reper.

Df 11. A moving affine T-system of reference is called non-rigid (non-solid, non-Euclidean, properly affine) iff its axis vectors form a properly affine T-reper.

Sch 14. Let us note that the definitions of rigidity and non-rigidity of a moving system of reference impose restrictions on its axis vectors only, while its origin may be entirely arbitrary.

Sgn 8. Σ_T sgn: the set of all solid T-systems of reference.

Pr 18. (64),

(95) $$\alpha \in \Sigma_T,$$

(62) *imply*

(96) $$\beta \in \Sigma_T.$$

Dm. Sgn 8, Df 10, 1 Pr 9.

Sch 15. The meaning of Pr 18 is that the property rigidity of a system of reference in $V_{C(F)}$ is invariant with respect to the equivalence relation $\sim$ in A_T defined by Sgn 6. In other words, a rigid system of reference cannot be equivalent with a properly affine system of reference. In the same time, an observant eye has certainly established that the condition (62) has not been used in the proof of Pr 18 in its full range: no condition for the origins of the systems of reference involved have been taken into consideration. This is easily explained in the light of Sch 14.

The following proposition reveals a fundamental property of the rigid systems of reference in

Pr 19. (95), (27), (28) *imply*

(97) $$\frac{d}{dt}((r_1-r_3)(r_2-r_3))=0 \qquad (\forall t \in T).$$

Dm. (95), (13) imply 1 (44) (Sgn 8, Df 10, 1 Df 6). On the other hand, in the same way as in the demonstration of Pr 6 it is proved that the relations (33), (34) hold. The latter imply

(98) $$(r_1-r_3)(r_2-r_3)=\sum_{\mu=1}^{3}\sum_{\nu=1}^{3}((r_1-r_3)a_\mu^{-1})(a_\nu^{-1}(r_2-r_3))(a_\mu a_\nu)$$

$(\forall t \in T)$ [4, Ax 8-Ax 10] and (98), (33), 1 (44) imply (97).

Sch 16. The geometrical interpretation of Pr 19 reads: if 3 points are invariable with respect to a rigid system of reference in $V_{C(F)}$, then their mutual distances (if any), as well as the angles (if any) between the segments these points determine, remain constant in the course of the "time" t.

Let, *exempli gratia*,

(99) $$r_\nu = a + a_\nu \qquad (\nu = 1,\ 2;\ \forall t \in T),$$

(100) $$r_3 = a \qquad (\forall t \in T).$$

Then the conditions (28) are satisfied (Pr 4, Pr 5) and (99), (100), (97) imply

(101) $$\frac{d}{dt}(a_1 a_2)=0 \qquad (\forall t \in T),$$

i.e. one of the relations 1 (44). The other ones can be obtained in the same manner. In this way it is seen that the definitional conditions 1 (44) of the notion of a rigid system of reference are a particular case of the basic property (97) of rigidity.

Pr 20. *If*

(102) $$\alpha=(a, \{a_\nu\}_{\nu=1}^{3})\in\Sigma_T$$

then

(103) $$a+a_\nu^{-1}\, z\,\alpha \qquad (\nu=1, 2, 3).$$

Dm. The relation (103) is equivalent with 1 (47) (Sgn 4), proved in 1 Pr 8 for rigid *T*-repers (3). Now Sgn 8, Df 10.

Pr 21. (95) *implies*

(104) $$\alpha^{-1}\sim\alpha.$$

Dm. Sgn 3, Pr 4, Pr 20, Pr 8.

Df 12. An affine *T*-system of reference is called o r t h o n o r m a l iff its axis vectors form an orthonormal *T*-reper.

Df 13. An affine *T*-system of reference is called n o n-o r t h o n o r m a l iff it is not orthonormal.

Sgn 17. It should be noted that the orthonormality and the non-orthonormality of a system of reference do not depend on the properies of its origin.

Pr 22. *If a system of reference is orthonormal, then it is rigid.*

Dm. Df 12, 1 Pr 7, Df 10.

Pr 23. (95) *imply*

(105) $$\alpha^{-1}=\alpha$$

iff α *is orthonormal.*

Dm. Sgn 3, 1 Pr 2.

§ 3. Affine and rigid motions

Sch 1. We now proceed to the definitions of two most important for the rational mechanics notions: those of affine and rigid motions.

The motion concept in rational mechanics aspires to formalize mathematically as adequately as it is possible phenomena going off in the real world. There is a great variety of physical motions and any of them calls for its specific mathematical description.

The special features of any particular motion are predeterminated by the characteristics of the object that moves. The analytical mechanics concerns itself mainly with r i g i d b o d i e s. Their movements are, therefore, of greatest interest for this science. They are called rigid motions and they are the chief object of present studies.

Physically speaking, in order to describe the movement of a rigid body one ataches to it an apropriate s y s t e m o f r e f e r e n c e ("invariably connected with the rigid body") and, instead of studying the movement of the rigid body itself, one studies the m o t i o n of this system of reference. Now the choice of the system in question can be made in an infinity variety of manners. And yet, there is an amalgamating feature among all of them: they are i n v a r i a n t with respect to each other.

All notions used above, namely rigid body, motion, system of reference, and invariance of such systems, are by no means a priori notions. They are purely mathematical notions — mathematically as pure as the number concept in arithmetic — and the fact that epistemologically their genesis has physical origins does not a whit impair this circumstance. In very deed, the physical genesis of the motion concept does not one jot differ from the physical genesis of the natural number concept or that of the geometrical point-line-plane concepts.

For rational mechanics this origin-problem is a matter of principle, and it is high time, in this connection, to dot the i's and to cross the t's once for ever. First of all, this is a problem the philosophers gladly chew the cud over but mathematician despise: it is a philosophical rather than a mathematical problem. In the second place, contrary to many prejudies, the only difference in the physical genesis of a geometrical ball, for instance, and of a mechanical one, consists in the degree of abstraction and — owing to it — in the amount of mathematical attributes ascribed to both of these notions: in contrast to the geometrical ball, the mechanical one is supplied with density and, as a result, with mass and mass-center; it may be subjected to geometrical constraints, restricting its possible positions in space; it may be acted upon by active forces (given a priori in the condition of any dynamical problem) and by passive forces (generated by the geometrical constraints); the constraints themselves may have a vast variety of dynamical properties resulting in the different characters of the forces they bring forth; at last, the mechanical ball may move in space and its motion originates such dynamical entities, as for instance momentum of motion, kinetical moment, kinetic energy, acceleration energy. And so on, and so forth, etcetera.

Physical origin? For mathematics the physical world is no criterion. The only arguments for a sound mathematician are the irreproachable mathematical definitions and the flawless mathematical demonstrations. In this respect the difference between a mathematical ball and a mechanical ball vanishes into thin air. As a matter of fact, the mechanical ball is susceptible to an abstract axiomatical definition satisfying all modern criteria of mathematical rigour. As regards the rigid motion concept, this definition is one of the main aims of the present work. As already hinted, it is a complicated one. If the following exposition does not seem spacious, the reason is that the preparatory work has been accomplished in the two preceding paragraphs.

Df 1. An affine T-motion in $V_{C(F)}$ is called any element of a set M_T, for which a set N_T exists satisfying the following conditions:

Ax 1. $N_T \subset A_T \times M_T$.

Ax 2. $\alpha \in A_T$ implies: there exists $m \in M_T$ with $(\alpha, m) \in N_T$.

Ax 3. $(\alpha_\nu, m_\nu) \in N_T$ $(\nu = 1, 2)$, $\alpha_1 \sim \alpha_2$ imply $m_1 = m_2$.

Ax 4. $m \in M_T$ implies: there exists $\alpha \in A_T$ with $(\alpha, m) \in N_T$.

Ax 5. $(\alpha_\nu, m) \in N_T$ $(\nu = 1, 2)$ implies $\alpha_1 \sim \alpha_2$.

Pr 1. *The system of axioms Ax 1-Ax 5 is consistent.*

Dm. [18].

Pr 2. *The system of axioms Ax 1-Ax 5 is categorical.*

Dm. [18].

Sgn 1. $\alpha \& m$ sgn: $(\alpha, m) \in N_T$ provided $\alpha \in A_T$, $m \in M_T$.

Df 2. α is called associated with m iff

$$\alpha \& m \tag{1}$$

provided $\alpha \in A_T$, $m \in M_T$.

Sgn 2. $\alpha \overline{\&} m$ sgn: $(\alpha, m) \bar{\in} N_T$ provided $\alpha \in A_T$, $m \in M_T$.

Df 3. α is called non-associated with m iff

$$\alpha \& m \tag{2}$$

provided $\alpha \in A_T$, $m \in M_T$.

Sch 2. Owing to the notation (1), the axioms Ax 2-Ax 5 may be rewritten in the form:

Ax 2&. $\alpha \in A_T$ implies: there exists $m \in M_T$ with $\alpha \& m$.

Ax 3&. $\alpha_\nu \& m_\nu$ $(\nu = 1, 2)$, $\alpha_1 \sim \alpha_2$ imply $m_1 = m_2$.

Ax 4&. $m \in M_T$ implies: there exists $\alpha \in A_T$ with $\alpha \& m$.

Ax 5&. $\alpha_\nu \& m$ $(\nu = 1, 2)$ imply $\alpha_1 \sim \alpha_2$.

Sch 3. The development of the theory of the affine T-motions in $V_{C(F)}$ (rigid body kinematics in the case of V instead of $V_{C(F)}$) consists in the revealing of those properties of the affine T-systems of reference in $V_{C(F)}$ which remain invariant with respect to the equivalence relation $\sim$ in A_T defined by 2 Sgn 6.

Sch 4. 2 Pr 17 manifest that the relation 2(19) defined by 2 Sgn 4 is invariant with respect to the equivalence relation $\sim$ in A_T defined by 2 Sgn 6. In other words, this relation is transferable from the affine T-systems of reference in $V_{C(F)}$ into the affine T-motions in $V_{C(F)}$. This circumstance justifies the following definitions.

Sgn 3. $\boldsymbol{r}\, z\, m$ sgn: $\boldsymbol{r}\, z\, \alpha$ provided 2(18), 2(26), $m \in M_T$, $\alpha \& m$.

Df 4. $\boldsymbol{r}$ is called invariable with respect to m iff

$$\boldsymbol{r}\, z\, m \tag{3}$$

provided 2(18), 2(26), $m \in M_T$, $\alpha \& m$.

Sgn 4. $\boldsymbol{r} \not{z}\, m$ sgn: $\boldsymbol{r} \not{z}\, \alpha$ provided 2(18), 2(26), $m \in M_T$, $\alpha \& m$.

Df 5. $\boldsymbol{r}$ is called non-invariable with respect to m iff

$$\boldsymbol{r} \not{z}\, m \tag{4}$$

provided 2(18), 2(26), $m \in M_T$, $\alpha \& m$.

The M_T-versions of 2 Pr 6 and 2 Pr 7 read as follows.

Pr 3. 2(27), 2(29),

$$m \in M_T, \tag{5}$$

(6) $$r_\nu \, z \, m \qquad (\nu = 1, 2, 3)$$

imply 2(30).

Dm. (5), Ax 4& imply: there exists $\alpha \in A_T$ with (1). Then (6), Sgn 3 imply 2(28). Now 2 Pr 6.

Pr 4. 2(47), 2(49), (5),

(7) $$r_\nu \, z \, m \qquad (\nu = 1, 2, 3, 4)$$

imply 2(50).

Dm. (5), Ax 4& imply: there exists $\alpha \in A_T$ with (1). Then (7), Sgn 3 imply 2(48). Now 2 Pr 7.

Sch 5. 1 Pr 18 manifests that two systems of references in $V_{C(F)}$ cannot be associated with the same motion in $V_{C(F)}$ unless they are simultaneously either rigid or properly affine: supposing the contrary one arrives at a contradiction with Ax 5& and 2 Pr 18. In such a manner, the notions of rigidity and non-rigidity are transferred from the systems of reference in $V_{C(F)}$ onto the motions in $V_{C(F)}$ these systems are associated with. This circumstance justifies the following definitions.

Df 6. A T-motion m in $V_{C(F)}$ is called rigid (solid, Euclidean) iff $\alpha \in A_T$, $\alpha \,\&\, m$ imply $\alpha \in \Sigma_T$.

Df 7. A T-motion m in $V_{C(F)}$ is called non-rigid (non-solid, non-Euclidean, properly affine) if it is not rigid.

Sgn 5. S_T sgn: the set of all rigid T-motions in $V_{C(F)}$.

The S_T-versions of 2 Pr 19 reads:

Pr 5. 2(27),

(8) $$m \in S_T,$$

(6) *imply* 2(97).

Dm. (8), Ax 4&, Sgn 5, Df 6 imply: there exists $\alpha \in \Sigma_T$ with (1). Then (6), Sgn 3 imply 2(28). Now 2 Pr 19.

Pr 6. 2(95), (8), (1) *imply*

(9) $$\alpha^{-1} \,\&\, m.$$

Dm. Sgn 5, Df 6, 2 Pr 21, Ax 3&.

Pr 7. *If* $\alpha \in A_T$ *is orthogonal and* (5), (1), *then* (8).

Dm. 2 Pr 22, Sgn 5, Df 6.

§ 4. Instantaneous angular velocity

Sch 1. In 1 Sch 1 some words have been said about the generality of the formulations of the first 3 paragraphs of the present paper. As regards motions, however, we have reached the *Ultima Thule* in our exposition: any further progress is *eo ipso* foreordained by mathematical specializations. These specifications concern in the utmost degree three mathematical factors: the standard vector space as the stage of events; the set-theoretical nature of the definitional domains of the functions involved as *locus standi*; and, last but not least, the analytical character of the functions themselves as *dramatis personae*.

To begin with, we shall impose the most severe restrictions on these factors: the standard vector space will be, by hypothesis, the real one V; $T \in \mathsf{R}$ will be an interval; and the functions involved will be supposed at least two times differentiable in any point of T (sometimes without an explicit reference to this condition). Right from the start we shall prove under these conditions a most remarkable kinematical theorem, due to Euler (although in a considerably different form) and we shall derive from it important mathematical corollaries; afterwards we shall examine the possibilities of its generalization.

Pr 1. *If*

$$T \subset \mathsf{R} \tag{1}$$

is an interval, the functions

$$\boldsymbol{a}_\nu : T \to V \qquad (\nu = 1, 2, 3) \tag{2}$$

are differentiable in T, *and the function*

$$\bar{\omega} : T \to V \tag{3}$$

satisfies

$$\frac{d\boldsymbol{a}_\nu}{dt} = \bar{\omega} \times \boldsymbol{a}_\nu \qquad (\nu = 1, 2, 3;\ \forall t \in T), \tag{4}$$

then 1 (44).

Dm. A necessary condition for the consistency of the system of vector equations (4) is

$$\boldsymbol{a}_\mu \frac{d\boldsymbol{a}_\nu}{dt} + \boldsymbol{a}_\nu \frac{d\boldsymbol{a}_\mu}{dt} = 0 \qquad (\mu, \nu = 1, 2\ 3;\ \forall t \in T) \tag{5}$$

[4, Pr 100]. On the other hand, (2) and

$$\frac{d\boldsymbol{a}_\nu}{dt} : T \to V \qquad (\nu = 1, 2, 3) \tag{6}$$

imply

$$\boldsymbol{a}_\mu \frac{d\boldsymbol{a}_\mu}{dt} = \frac{d\boldsymbol{a}_\mu}{dt} \boldsymbol{a}_\nu \qquad (\mu, \nu = 1, 2, 3;\ \forall t \in T) \tag{7}$$

and (5), (7) imply

$$\frac{d\boldsymbol{a}_\mu}{dt} \boldsymbol{a}_\nu + \boldsymbol{a}_\mu \frac{d\boldsymbol{a}_\nu}{dt} = 0 \qquad (\mu, \nu = 1, 2\ 3;\ \forall t \in T) \tag{8}$$

whence 1 (44).

Pr 2. (1), (2), 1 (5), 1 (44) *imply: the system of vector equations* (4) *has exactly one solution* (3), *namely*

(9) $$\bar{\omega}=\frac{1}{2}\sum_{\nu=1}^{3} a_\nu^{-1}\times\frac{da_\nu}{dt} \qquad (\forall t\in T).$$

Dm. [4, Pr 103].

Sch 2. 1 Sgn 5, 1 Df 6 imply that Pr 1 and Pr 2 hold iff

(10) $$\{a_\nu\}_{\nu=1}^{3}\in E_T.$$

Df 1. The function (9) is called the instant angular velocity (or briefly angular velocity) of the reper (10).

Pr 3. (1), (2), 1 (5) *imply*

(11) $$\sum_{\nu=1}^{3} a_\nu^{-1}\times\frac{da_\nu}{dt}=\sum_{\nu=1}^{3} a_\nu\times\frac{da_\nu^{-1}}{dt} \qquad (\forall t\in T).$$

Dm. The identity

(12) $$\sum_{\nu=1}^{3} a_\nu\times a_\nu^{-1}=o \qquad (\forall t\in T)$$

[4, Pr 122] implies

(13) $$\sum_{\nu=1}^{3}\frac{da_\nu}{dt}\times a_\nu^{-1}+\sum_{\nu=1}^{3} a_\nu\times\frac{da_\nu^{-1}}{dt}=o \qquad (\forall t\in T),$$

whence (11) [4, Pr 19].

Pr 4. (1), (2), 1 (5), (9) *imply*

(14) $$\bar{\omega}=\frac{1}{2}\sum_{\nu=1}^{3} a_\nu\times\frac{da_\nu^{-1}}{dt} \qquad (\forall t\in T).$$

Dm. Pr 3.

Pr 5. (1), (2), 1 (5), 1 (44), (14) *imply*

(15) $$\frac{da_\nu^{-1}}{dt}=\bar{\omega}\times a_\nu^{-1} \qquad (\nu=1,\ 2,\ 3;\ \forall t\in T).$$

Dm. 1 (11), 1 (47), Pr 2, and

(16) $$(a_\nu^{-1})^{-1}=a_\nu \qquad (\nu=1,\ 2,\ 3;\ \forall t\in T)$$

[4, Pr 90].

Pr 6. (9) *implies*

(17) $$\bar{\omega}=\frac{1}{2}\sum_{\nu=1}^{3} a_\nu\times\frac{da_\nu}{dt} \qquad (\forall t\in T)$$

iff the reper (10) *is orthonormal.*

Dm. 1 Df 3, [4, Pr 94].

Pr 7. 1 (7), 1 (17), 1 (18) *imply*

(18) $$\sum_{\nu=1}^{3} a_\nu^{-1}\times\frac{da_\nu}{dt}=\sum_{\nu=1}^{3} b_\nu^{-1}\times\frac{db_\nu}{dt} \qquad (\forall t\in T).$$

Dm. 1 (18) implies 1 (16) (1 Sgn 3). Hence the identity 1 (41) imply

$$\frac{da_\mu}{dt} = \sum_{\lambda=1}^{3} (a_\mu b_\lambda^{-1}) \frac{db_\lambda}{dt} \qquad (\mu=1, 2, 3; \forall t \in T), \tag{19}$$

whence

$$\sum_{\mu=1}^{3} a_\mu^{-1} \times \frac{da_\mu}{dt} = \sum_{\lambda=1}^{3} \sum_{\mu=1}^{3} (b_\lambda^{-1} a_\mu) a_\mu^{-1} \times \frac{db_\lambda}{dt} \tag{20}$$

($\forall t \in T$). Now (20), 2 (90) imply

$$\sum_{\mu=1}^{3} a_\mu^{-1} \times \frac{da_\mu}{dt} = \sum_{\lambda=1}^{3} b_\lambda^{-1} \times \frac{db_\lambda}{dt} \qquad (\forall t \in T), \tag{21}$$

i.e. (18).

Pr 8. 1 (7), 1 (17), 1 (18) *imply*

$$\sum_{\nu=1}^{3} a_\nu \times \frac{da_\nu^{-1}}{dt} = \sum_{\nu=1}^{3} b_\nu \times \frac{db_\nu^{-1}}{dt} \qquad (\forall t \in T). \tag{22}$$

Dm. Pr 7, Pr 3.

Pr 9. *If* $\bar{\omega}_a$ *and* $\bar{\omega}_b$ *denote the instantaneous angular velocities of the repers* 1 (45) *and* 1 (51) *respectively and if* 1 (18), *then*

$$\bar{\omega}_a = \bar{\omega}_b \qquad (\forall t \in T). \tag{23}$$

Dm. Df 1, Pr 7.

Pr 10. *If* $\bar{\omega}_a$ *and* $\bar{\omega}_b$ *denote the instantaneous angular velocities of the repers* 1 (45) *and* 1 (51) *respectively and if* (23), *then* 1 (18) *holds.*

Dm. If 1 (3) and 1 (17), then Df 1, (23) imply (18). On the other hand, (4) and (15) imply

$$\frac{da_\nu}{dt} = \bar{\omega}_a \times a_\nu \qquad (\nu=1, 2, 3; \forall t \in T) \tag{24}$$

and

$$\frac{db_\nu^{-1}}{dt} = \bar{\omega}_b \times b_\nu^{-1} \qquad (\nu=1, 2, 3; \forall t \in T) \tag{25}$$

respectively, and (24), (25) imply

$$\frac{d}{dt}(a_\mu b_\nu^{-1}) = \frac{da_\mu}{dt} b_\nu^{-1} + a_\mu \frac{db_\nu^{-1}}{dt} \tag{26}$$

$$= \bar{\omega}_a \times a_\mu . b_\nu^{-1} + a_\mu . \bar{\omega}_b \times b_\nu^{-1} \quad (\mu, \nu=1, 2\ 3; \forall t \in T).$$

Now (26), (23) imply 1 (16), i.e. 1 (18) (1 Sgn 3).

Pr 11. *If* $\bar{\omega}_a$ *and* $\bar{\omega}_b$ *denote the instantaneous angular velocities of the repers* 1 (45) *and* 1 (51) *respectively, then* (23) *iff* 1 (18) *holds.*

Dm. Pr 9, Pr 10.

Sch 3. Pr 11 displays that the equality of the instantaneous angular velocities of two rigid repers is a necesarry and sufficient condition for their equivalence. In other words, the notion of instantaneous angular velocity may be transferred from the particular rigid systems of reference onto the classes of equivalence these systems determine by virtue of the equivalence relation $\sim$ in R_T, defined by 1 Sgn 3. In other words, the instantaneous angular velocities are mathematical characteristics for these classes of equivalence which – being dependent on the rigid repers only – may be transferred onto the rigid motions these repers determine. This is soon done. For the time being we shall derive some other corollaries from the definition (9).

Pr 12. *If the functions* (2) *are infinitely many times differentiable, then* 1 (5) *and* 1 (44) *imply*

$$\sum_{\nu=1}^{3} \frac{d^m \boldsymbol{a}_\nu}{dt^m} \times \frac{d^n \boldsymbol{a}_\nu^{-1}}{dt^n} = \sum_{\nu=1}^{3} \frac{d^m \boldsymbol{a}_\nu^{-1}}{dt^m} \times \frac{d^n \boldsymbol{a}_\nu}{dt^n} \tag{27}$$

$(m,\ n=0,\ 1,\ 2, \ldots;\ \forall t \in T)$.

Dm. [4, Pr 89] implies

$$\boldsymbol{a}_\nu = \sum_{\mu=1}^{3} (\boldsymbol{a}_\nu \boldsymbol{a}_\mu) \boldsymbol{a}_\mu^{-1} \qquad (\nu=1,\ 2,\ 3;\ \ \forall t \in T) \tag{28}$$

and (28), 1 (44) imply

$$\frac{d^m \boldsymbol{a}_\nu}{dt^m} = \sum_{\mu=1}^{3} (\boldsymbol{a}_\nu \boldsymbol{a}_\mu) \frac{d^m \boldsymbol{a}_\mu^{-1}}{dt^m} \qquad (\nu=1,\ 2,\ 3;\ \forall t \in T), \tag{29}$$

whence

$$\sum_{\nu=1}^{3} \frac{d^m \boldsymbol{a}_\nu}{dt^m} \times \frac{d^n \boldsymbol{a}_\nu^{-1}}{dt^n} = \sum_{\nu=1}^{3} \sum_{\mu=1}^{3} (\boldsymbol{a}_\nu \boldsymbol{a}_\mu) \frac{d^m \boldsymbol{a}_\mu^{-1}}{dt^m} \times \frac{d^n \boldsymbol{a}_\nu^{-1}}{dt^n} \tag{30}$$

$(\forall t \in T)$. On the other hand, (28), 1 (44) imply

$$\frac{d^n \boldsymbol{a}_\nu}{dt^n} = \sum_{\mu=1}^{3} (\boldsymbol{a}_\nu \boldsymbol{a}_\mu) \frac{d^n \boldsymbol{a}_\mu^{-1}}{dt^n} \qquad (\nu=1,\ 2,\ 3;\ \forall t \in T) \tag{31}$$

whence

$$\sum_{\nu=1}^{3} \frac{d^m \boldsymbol{a}_\nu^{-1}}{dt^m} \times \frac{d^n \boldsymbol{a}_\nu}{dt^n} = \sum_{\nu=1}^{3} \sum_{\mu=1}^{3} (\boldsymbol{a}_\nu \boldsymbol{a}_\mu) \frac{d^m \boldsymbol{a}_\nu^{-1}}{dt^m} \times \frac{d^n \boldsymbol{a}_\mu^{-1}}{dt^n} \tag{32}$$

$(\forall t \in T)$. Now (32) may obviously be written in the form

$$\sum_{\nu=1}^{3} \frac{d^m \boldsymbol{a}_\nu^{-1}}{dt^m} \times \frac{d^n \boldsymbol{a}_\nu}{dt^n} = \sum_{\mu=1}^{3} \sum_{\nu=1}^{3} (\boldsymbol{a}_\mu \boldsymbol{a}_\nu) \frac{d^m \boldsymbol{a}_\nu^{-1}}{dt^m} \times \frac{d^n \boldsymbol{a}_\mu^{-1}}{dt^n} \tag{33}$$

$(\forall t \in T)$ and a formal change of μ and ν with ν and μ respectively in the right-hand side of (33) implies

$$\sum_{\nu=1}^{3} \frac{d^m \boldsymbol{a}_\nu^{-1}}{dt^m} \times \frac{d^n \boldsymbol{a}_\nu}{dt^n} = \sum_{\nu=1}^{3} \sum_{\mu=1}^{3} (\boldsymbol{a}_\nu \boldsymbol{a}_\mu) \frac{d^m \boldsymbol{a}_\mu^{-1}}{dt^m} \times \frac{d^n \boldsymbol{a}_\nu^{-1}}{dt^n} \tag{34}$$

$(\forall t \in T)$. Now (30), (34).

Pr 13. *If the functions* (2) *are infinitely many times differentiable, then* 1 (5) *and* 1 (44) *imply*

$$\sum_{\nu=1}^{3} \frac{d^n \boldsymbol{a}_\nu}{dt^n} \times \frac{d^n \boldsymbol{a}_\nu^{-1}}{dt^n} = \boldsymbol{o} \tag{35}$$

($n=0,\ 1,\ 2, \ldots;\ \forall t \in T$).

Sch 4. The cases $m=n=0$, $n=0$, $n=1$, and $m=1$, $n=0$ in Pr 12 and Pr 13 coincide with (12) and (13) respectively; hence in these cases the condition 1 (44) is superfluous.

Df 2. The derivative

$$\bar{\varepsilon} = \frac{d\bar{\omega}}{dt} \qquad (\forall t \in T) \tag{36}$$

of the instantaneous angular velocity $\bar{\omega}$ of the reper (10) is called the instantaneous angular acceleration (or briefly angular acceleration) of (10).

Pr 14. (10), (36) *imply*

$$\bar{\varepsilon} = \frac{1}{2} \sum_{\nu=1}^{3} \boldsymbol{a}_\nu^{-1} \times \frac{d^2 \boldsymbol{a}_\nu}{dt^2} \qquad (\forall t \in T). \tag{37}$$

Dm. (9), Pr 13 with $n=1$.

Pr 15. (10), (36) *imply*

$$\bar{\varepsilon} = \frac{1}{2} \sum_{\nu=1}^{3} \boldsymbol{a}_\nu \times \frac{d^2 \boldsymbol{a}_\nu^{-1}}{dt^2} \qquad (\forall t \in T). \tag{38}$$

Dm. (14), Pr 13 with $n=1$.

Pr 16. (37) *implies*

$$\bar{\varepsilon} = \frac{1}{2} \sum_{\nu=1}^{3} \boldsymbol{a}_\nu \times \frac{d^2 \boldsymbol{a}_\nu}{dt^2} \qquad (\forall t \in T). \tag{39}$$

iff the reper (10) *is orthonormal.*

Dm. 1 Df 3, [4, Pr 94].

Pr 17. (10), (9), (36) *imply*

$$\frac{d^2 \boldsymbol{a}_\nu}{dt^2} = \bar{\varepsilon} \times \boldsymbol{a}_\nu + \bar{\omega} \times (\bar{\omega} \times \boldsymbol{a}_\nu) \qquad (\nu=1,\ 2,\ 3;\ \forall t \in T). \tag{40}$$

Dm. (4).

Pr 18. (10), (9), (36) *imply*

$$\frac{d^2 \boldsymbol{a}_\nu^{-1}}{dt^2} = \bar{\varepsilon} \times \boldsymbol{a}_\nu^{-1} + \bar{\omega} \times (\bar{\omega} \times \boldsymbol{a}_\nu^{-1}) \qquad (\nu=1,\ 2\ 3;\ \forall t \in T). \tag{41}$$

Dm. (15).

Pr 19. *If* $\bar{\varepsilon}_a$ *and* $\bar{\varepsilon}_b$ *denote the instantaneous angular accelerations of the repers* 1 (45) *and* 1 (51) *respectively, then*

$$\bar{\varepsilon}_a = \bar{\varepsilon}_b \qquad (\forall t \in T) \tag{42}$$

iff 1 (18).

Dm. Pr 11, Df 2.

Sch 5. Similar to Sch 3 with $\bar{\varepsilon}$ instead of $\bar{\omega}$.

Pr 20. (1), (2),

$$a : T \to V, \tag{43}$$

$$r : T \to V, \tag{44}$$

2 (102), 2 (19), (9) *imply*

$$\frac{d}{dt}(r-a) = \bar{\omega} \times (r-a) \qquad (\forall t \in T). \tag{45}$$

Dm. 2 (102) implies: the function (3) defined by (9) exists and satisfies (4) (2 Sgn 8, 2 Df 10, 2 Df 1 bis, Pr 2). On the other hand, 2 (19) implies 2 (17) (2 Sgn 4). Besides, 2 (21) holds [4, Pr 88]. Now 2 (21), 2 (17), (4) imply

$$\frac{d}{dt}(r-a) = \sum_{\nu=1}^{3} ((r-a)\, a_\nu^{-1}) \frac{da_\nu}{dt} \tag{46}$$

$$= \sum_{\nu=1}^{3} ((r-a)\, a_\nu^{-1}) \bar{\omega} \times a_\nu = \bar{\omega} \times \sum_{\nu=1}^{3} ((r-a)\, a_\nu^{-1}) a_\nu$$

($\forall t \in T$), i.e. (45).

Pr 21. 2 (102), (44), (9), (45) *imply* 2 (19).

Dm. 2 (102) implies: the function (3) defined by (9) exists and satisfies (15) (2 Sgn 8, 2 Df 10, 2 Df 1 bis, Pr 5). On the other hand, (45) and (15) imply

$$\frac{d}{dt}((r-a)\, a_\nu^{-1}) = \left(\frac{d}{dt}(r-a)\right) a_\nu^{-1} + (r-a) \frac{da_\nu^{-1}}{dt} \tag{47}$$

$$= \bar{\omega} \times (r-a) \,.\, a_\nu^{-1} + (r-a) \,.\, \bar{\omega} \times a_\nu^{-1} = 0$$

($\nu = 1, 2, 3$; $\forall t \in T$), i.e. 2 (19), (2 Sgn 4).

Pr 22. 2 (102), (44), (9) *imply*: (45) *is a necessary and sufficient condition for* 2 (19).

Dm. Pr 20, Pr 21.

Sch 6. The following proposition manifests that there exists exactly one function (3) with the property described by Pr 22.

Pr 23. 2 (102) *implies*: *if*

$$\Omega : T \to V \tag{48}$$

and (44), 2 (19) *imply*

(49) $$\frac{d}{dt}(\boldsymbol{r}-\boldsymbol{a})=\Omega\times(\boldsymbol{r}-\boldsymbol{a}) \qquad (\forall t\in T),$$

then

(50) $$\Omega=\bar{\omega} \qquad (\forall t\in T).$$

Dm. The definitional property (49) of (48) implies

(51) $$\frac{d\boldsymbol{a}_\nu}{dt}=\Omega\times\boldsymbol{a}_\nu \qquad (\nu=1, 2, 3;\ \forall t\in T)$$

(2 Pr 5). Now 2 (102), 2 Df 1 bis, 2 Sgn 8, 2 Df 10 imply 1 (5), 1 (44), and the latter imply that the system of vector equations (51) has exactly one solution Ω [4, Pr 103]. Then Pr 2 implies (50).

Df 3. The function (3) defined by (9) is called the instantaneous angular velocity (or briefly angular velocity) of the system of reference 2 (102).

Sch 7. As already mentioned in Sch 1, the existence of the instantaneous angular velocity has been discovered by Euler. That is why Pr 22 is usually called Euler's theorem of instantaneous angular velocity. This denomination is, in a sense, disputable. In the first place, the root of the matter lies maybe deeper: in Pr 1 and Pr 2 rather than in Pr 22 itself. In the second place, Euler's way to introduce the instantaneous angular velocity in rigid body kinematics is essentially different from the manner it has been defined here. In Euler's version intuitive geometric-kinematical considerations have been used on a large scale instead of strict mathematical definitions as those introduced here, and one is at a loss where namely did Euler's center of gravity lie. So, for instance, Euler has never used properly affine systems of reference, whence he has not had the oportunity to establish the importance (necessity!) of the property rigidity for the existence of instantaneous angular velocities (Pr 1). At any rate, the whole of the exposition in this paragraph is organically connected with Euler's discovery, and any of the above propositions may be justly attached to his name.

Sch 8. Similarly to the remark of 2 Sch 15 it should be noted that — as an observant eye has certainly established — the definition of rigid system of reference 2 (102) to which the instantaneous angular velocity (9) is attached by virtue of Df 3 has not been used in the latter in its full range: as in the definition 2 Df 10 of the notion rigidity of a moving T-system of reference, the origin $\boldsymbol{a}$ of α is neglected in the definition of the angular velocity $\bar{\omega}$ of α. In other words, both definitions are concerned with the axis vectors $\boldsymbol{a}_\nu$ ($\nu=1$, 2 3) of α only and are entirely indifferent with respect to $\boldsymbol{a}$. Naturally, the same remark holds for the instantaneous angular acceleration $\bar{\varepsilon}$ of α as well, defined immediately below, because of its relation (36) with $\bar{\omega}$.

Df 4. The function defined by (36) is called the instantaneous angular acceleration (or briefly angular acceleration) of the system of reference 1 (102).

Pr 24. (1), (2), (43), (44), 2 (102), 2 (19), (9), (36) *imply*

(52) $$\frac{d^2}{dt^2}(\boldsymbol{r}-\boldsymbol{a})=\bar{\varepsilon}\times(\boldsymbol{r}-\boldsymbol{a})+\bar{\omega}\times(\bar{\omega}\times(\boldsymbol{r}-\boldsymbol{a})) \qquad (\forall t\in T).$$

Dm. Pr 20.

Sch 9. By virtue of Pr 11, the notion of instantaneous angular velocity may be transferred from the rigid systems of reference onto the equivalence classes in A_T (generated by the equivalence relation $\sim$ in A_T defined by 2 Sgn 6) these systems define, i. e. on the rigid bodies. This remark gives rise to the following definitions and propositions.

Df 5. If 3 (8), 3 (1), 2 (102) with (1), (2), (43), and (9), then $\bar{\omega}$ is called the instantaneous angular velocity (or briefly the angular velocity) of m.

Df 6. If 3 (8), 3 (1), 2 (102) with (1), (2), (43), and (9), (37), then $\bar{\varepsilon}$ is called the instantaneous angular acceleration (or briefly the angular acceleration) of m.

Pr 25. 3 (8), 2 (102), 3 (1), (9), 2 (18) *imply: a necessary and sufficient condition for* 3 (3) *is* (45).

Dm. 3 Sgn 3, Pr 22.

Sch 10. All considerations in this paragraph are carried out under the hypothesis that the stage of events is the real standard vector space V. Now the question quite naturally arises whether they admit a complex extension, i. e. whether all, or a part of them, remain valid if the complex standard vector space V_C over the field C of all complex numbers is substituted for V. The answer is a categorical *no*. The reasons for this answer are very simple and they are analyzed here.

Pr 26. *If*

(53) $$T \subset \mathrm{R},$$

(54) $$\boldsymbol{a}_\nu : T \to V_C \qquad (\nu = 1, 2, 3),$$

(10), *then in the general case*

(55) $$\boldsymbol{a}_\mu \frac{d\boldsymbol{a}_\nu}{dt} + \boldsymbol{a}_\nu \frac{d\boldsymbol{a}_\mu}{dt} \neq 0 \qquad (\mu, \nu = 1, 2, 3; \forall t \in T).$$

Dm. (10), 1 Sgn 5, 1 Df 6 imply 1 (44), i. e. (8). On the other hand

(56) $$\frac{d\boldsymbol{a}_\mu}{dt} \boldsymbol{a}_\nu = \overline{\boldsymbol{a}_\nu \frac{d\boldsymbol{a}_\mu}{dt}} \qquad (\mu, \nu = 1, 2 3; \forall t \in T)$$

[4, Ax 8] where the vinculum in the right-hand side of (56) denotes the conjugate value of the scalar product. Now, in the general case

(57) $$\overline{\boldsymbol{a}_\nu \frac{d\boldsymbol{a}_\mu}{dt}} \neq \boldsymbol{a}_\nu \frac{d\boldsymbol{a}_\mu}{dt} \qquad (\mu, \nu = 1, 2, 3; \forall t \in T)$$

and (8), (56), (57) imply (55).

Pr 27. (53), (54), (10),

(58) $$\bar{\omega} : T \to V_C$$

imply: the system of vector equations (4) *is inconsistent.*

Dm. By virtue of [4, Pr 100] the relations (5) are necessary conditions for the consistency of (4). Now Pr 26.

Sch 11. Pr 27 explains the negative answer of the question put in Sch 10: if 1 (3) is a rigid reper in V_C, it does not possess an instantaneous angular velocity (58) in the general case. Indeed, the definitional condition of (58) are the relations (4); according to Pr 27, however, there does not exist a function (58) satisfying (4).

Sch 12. An immediate corollary from the conclusion of Sch 11 is the inference that any of the V_C-analogues of the definitions and propositions in the present paragraph are, simply and purely, meaningless. This does not mean that there does not exist rigid body kinematics in V_C. As in the case of the C-analysis in comparison with the R-analysis, however, denuded of angular velocity with its fundamental property (45), the C-rigid body kinematics is out and away scantier than the R-one. *Dura lex, sed lex*: worthy of regret though this conclusion is, it is a mathematical *nuda veritas* one is compelled to become reconciled with.

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