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On the General Mountain Pass Principle of Ghoussoub-Preiss

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Presented by P. Kenderov

We present here a version of the general mountain pass principle of Ghoussoub-Preiss for locally Lipschitz functions. Our theorem is a generalization of a result of Chang for the case when "the separating mountain range has zero altitude". The proof is simplified by means of a deformation lemma.

1. Introduction

Since A. Ambrosetti and P. H. Rabinowitz published in 1973 their famous variational principle known as "mountain pass lemma" (cf. [AR]), it has been successfully applied in different mathematical fields. For the time being numerous generalizations of this result are known. In 1981 K. C. Chang [Ch] extended the classical theorem (concerning C^1 functions) to locally Lipschitz functions. In the present paper we generalize his result.

The usual way of proving "the mountain pass lemma" is based on a "deformation lemma" and some kind of compactness condition imposed on the considered function. To prove existence of deformations one needs an equivalent of $\|f'(x)\|_{X^*}$ (where X is a Banach space, X^* is its conjugate and f is C^1) for locally Lipschitz functions:

Definition 1. Let X be a Banach space, $S \subseteq X$ be a neighbourhood of $x \in X$, $f: S \to \mathrm{R}$ be Lipschitz continuous and $f^0(x, h)$ be the Clarke derivative of f at x in direction $h \in X$. The number

$$\inf\{f^0(x;\ h)\ (1 \in X,\ \|h\|_X = 1\}$$

is called steepness of f at x and is denoted by $stf(x)$.

Remark 1. In [Ch] the respective notion is

$$\lambda(x) := \min\{\|x^*\|_{X^*} \mid x^* \in \partial f(x)\}\ \ (\text{p. } 113),$$

where $\partial f(x)$ is the Clarke gradient of f at x. In fact $\lambda(x) = 0$ iff $stf(x) \geqq 0$ and $\lambda(x) = -stf(x)$ otherwise.

In contrast to Chang we prove a version of the deformation lemma for locally Lipschitz functions which follows exactly the one proved by Willem in [*W*] for C^1 functions.

We shall denote
$$S_\alpha=\{x\in X \mid \operatorname{dist}(x, S)\leqq\alpha\},$$
$$f^c=\{x\in X \mid f(x)\leqq c\}$$
where f is a function on a Banach space X, S is a subset of X and α, c are real numbers.

Lemma 1. (deformation lemma). *Let X be a Banach space, $f:X\to \mathrm{R}$ be locally Lipschitz, $S\subseteq X$ and $c\in\mathrm{R}$. Let ε and δ be positive reals such that for every y in an open neighbourhood Q of $f^{-1}([c-\varepsilon, c+\varepsilon])\cap S_\delta$ we have*
$$st f(y)<-2\varepsilon/\delta.$$
Then there exists $\eta\in C([0, 1]\times X, X)$ satisfying the following properties:
(i) $\eta(0, x)=x$ *for every* $x\in X$;
(ii) $\eta(t, x)=x$ *for every* $x\in X\backslash Q$ *and for every* $t\in[0, 1]$;
(iii) $\eta(1, f^{c+\varepsilon}\cap S)\subseteq f^{c-\varepsilon}\cap S_\delta$;
(iv) $\eta(t,.)$ *is a homeomorphism of X for every* $t\in[0, 1]$;
(v) $\operatorname{dist}(x, \eta(1, x))\leqq\delta$ *for every* $x\in X$.

Next we introduce the respective Palais-Smale conditions for locally Lipschitz functions.

Definition 2. Let X be a Banach space and $f:X\to\mathrm{R}$ be locally Lipschitz. The real number c is said to be a critical value of f iff there exists $x\in X$ (called critical point of f) such that $c=f(x)$ and $0\in\partial f(x)$.

Definition 3. (compare with [GP], [W]) Let X be a Banach space, $c\in\mathrm{R}$ and $f:X\to\mathrm{R}$ be locally Lipschitz. We say that f satisfies
(i) the condition $(\mathrm{PS})_c$ if whenever a sequence $\{x_n\}_{n=1}^\infty$ is such that $f(x_n)_{n\to\infty}\to c$ and $\liminf st f(x_n)\geqq 0$, than c is a critical value of f;
(ii) the Palais-Smale condition (PS) if each sequence $\{x_n\}_{n=1}^\infty$ such that $\{|f(x_n)|\}_{n=1}^\infty$ is bounded and $\liminf st f(x_n)\geqq 0$ has a convergent subsequence;
(iii) the Weak Palais-Smale condition (WPS) if each bounded sequence $\{x_n\}_{n=1}^\infty$ such that $\{|f(x_n)|\}_{n=1}^\infty$ is bounded and $\liminf st f(x_n)\geqq 0$ has a convergent subsequence;

Remark 2. Let X be a Banach space, $f:X\to\mathrm{R}$ be locally Lipschitz and $\{x_n\}_{n=1}^\infty$ has a cluster point x. Let $c=f(x)$ and $\liminf st f(x_n)\geqq 0$. Then c is a critical value of f by the upper semicontinuity of the Clarke derivative (see proposition 2.1.1 on p. 32 in [C]). This means that (PS) implies $(\mathrm{PS})_c$. The implication (PS) $\to$ (WPS) is obvious.

In the classical mountain pass theorem the critical value occurs because there are two "low points" on either side of a mountain range which has positive altitute", so that between these two points there must be a "mountain pass". The fact that "the separating mountain range has positive altitude" is crucial for its proof. In [GP] N. Ghoussoub and D. Preiss established a general mountain pass principle for smooth functions (Gâteaux – differentiable with strong to weak* continuous derivative) which includes the case when "the altitude" in question is equal to zero.

Following [GP] we introduce some necessary notations: Let X be a Banach space, F be a closed subset of X; u, v be two distinct points in X and $f: X \to \mathbb{R}$ be locally Lipschitz. We set

$$\Gamma := \{g \in C([0,\ 1], X) \mid g(0) = u,\ g(1) = v\},$$

$$c(F, f) := \inf\{\max\{f(g(t)) \mid g(t) \in F,\ t \in [0,\ 1]\} \mid g \in \Gamma\}.$$

Definition 4. It is said that F separates u and v iff u, v belong to disjoint components of $X \setminus F$.

Theorem 1. *Let X be a Banach space and $f: X \to \mathbb{R}$ be locally Lipschitz function. Let F be a closed subset of X and u, v be two points from X separated by F. Assume $c(F, f) = c(X, f) = c$. Then:*
(i) if f verifies $(PS)_c$ then c is a critical value for f;
(ii) if f verifies (PS) or if F is bounded and f verifies (WPS) then there exists a critical point in F with critical value c.*

The following is now immediate.

Corollary 1. *Let X be a Banach space, $f: X \to \mathbb{R}$ be locally Lipschitz, u, v be two distinct points in X and $r \in (0,\ \|u - v\|_X)$ be a real such that*

$$b := \inf\{f(x) \mid \|x - u\|_X = r\} \geqq \max\{f(u), f(v)\} := a.$$

Let $c = c(X, f)$.
(i) if f verifies $(PS)_c$ then c is a critical value of f;
(ii) if f verifies (WPS) and $c = a$ then there exists a critical point x_0 satisfying $\|x_0 - u\|_X = r$.

Corollary 1 differs from the respective result of K. C. Chang (Theorem 3 .4 on p.118 in [Ch]) by lack of reflexivity assumption on X, a weaker Palais-Smale condition and by considering the case when "the separating mountain range may have zero altitude". It also gives some information about the location of the critical points (in (ii)).

The proof of Theorem 1 uses the idea of Ghoussoub and Preiss "for perturbation of the considered function along F", but it is simplified by using the deformation lemma and because "the perturbation" is locally Lipschitz.

2. Proofs

2.1. Proof of deformation lemma

Let x be a point of Q. Then from the definition of the Clarke derivative we obtain an open neighbourhood U_x of x, a vector $h_x \in X$ of norm one and a positive number $\alpha_x > 0$ such that

$$\frac{f(y + th_x) - f(y)}{t} < -\frac{2\varepsilon}{\delta}$$

whenever $y \in U_x$ and $t \in (0, \alpha_x)$. Now $\{U_x\}_{x \in Q}$ is an open cover of the paracompact space Q (as Q is a subspace of the complete metric space $(X, \|.\|)$). Therefore there exists an open cover $\{V_\beta\}_{\beta \in B}$ of Q with V_β of diameter less than one, which is a locally finite refinement of $\{U_x\}_{x \in Q}$. If $V_\beta \subseteq U_x$, let h_β be h_x, x_β be x, α_β be α_x and $\rho_\beta(z) = \operatorname{dist}(z,\ X \setminus V_\beta)$. We define a locally Lipschitz function $h: Q \to X$ by

* In private communication R. Deville informed us that Theorem 1 generalized the main result of Ghoussoub and Preiss [GP].

$$h(x)=\sum_{\beta}\frac{\rho_\beta(x)}{\sum_{\gamma}\rho_\gamma(x)}h_\beta.$$

The so constructed mapping h has the following "decreasing property":

Claim 1. *For every $x\in Q$ there exists $t_x>0$ such that*

$$\frac{f(x+th(x))-f(x)}{t}<-\frac{2\varepsilon}{\delta}$$

whenever $t\in(0,\ t_x)$.

Proof of claim 1.

Let $x\in Q$ be arbitrary. Then $x\in V_{\beta_i}$, $i=1,\ 2,\dots,\ k$. We denote $p_i=\frac{\rho_{\beta_i}(x)}{\sum_{\beta\in B}\rho_\beta(x)}\in(0,1]$ and $h_i=h_{\beta_i}$, $i=1,\ 2,\dots,\ k$. Thus $h(x)=\sum_{i=1}^{k}p_ih_i$ is a convex combination of h_i, $i=1,\ 2,\dots,\ k$. If

$$t_x=\sup\{t_0>0\,|\,x+t\sum_{i=j+1}^{k}p_ih_i\in V_{\beta_j}\text{ for each }t\leqq t_0,\ j=1\div k-1,\ t_0<\alpha_{\beta_i},\ i=1\div k\},$$

then for every $t\in(0,\ t_x)$ we have

$$\begin{aligned}f(x+th(x))-f(x)&=f(x+t\sum_{i=1}^{k}p_ih_i)-f(x)\\&=\sum_{j=1}^{k}(f(x+t\sum_{i=j}^{k}p_ih_i)-f(x+t\sum_{i=j+1}^{k}p_ih_i))\\&=\sum_{j=1}^{k}(f(x+t\sum_{i=j+1}^{k}p_ih_i+tp_jh_j)-f(x+t\sum_{i=j+1}^{k}p_ih_i)).\end{aligned}$$

Since $t<t_x$, the vector $x+t\sum_{i=j+1}^{k}p_ih_i$ belongs to V_{β_j} and from $0<p_jt\leqq t<t_x\leqq\alpha_{\beta_j}$ we obtain

$$f(x+t\sum_{i=j+1}^{k}p_ih_i+tp_jh_j)-f(x+t\sum_{i=j+1}^{k}p_ih_i)<-2(\varepsilon/\delta)tp_j$$

for every $j=1\div k$. Hence

$$f(x+th(x))-f(x)<-2(\varepsilon/\delta)t$$

for every $t\in(0,\ t_x)$ (t_x is positive because V_{β_j} are finitely many open sets). ∎

Let U_1, U_2 be two open subsets of X with

$$\bar U_1\cap\bar U_2=\emptyset,\ U_1\supset f^{-1}([c-\varepsilon,\ c+\varepsilon])\cap S_\delta,\ U_2\supseteq X\backslash Q$$

and $\Psi:X\to\mathsf{R}$ be an arbitrary locally Lipschitz function for which $\Psi(x)=1$ whenever $x\in U_1$, $\Psi(x)=0$ if $x\in U_2$ and $0\leqq\Psi(x)\leqq1$ for every $x\in X$. We define a locally Lipschitz mapping $g:X\to X$ by

$$g(x)=\begin{cases}\dfrac{\Psi(x)}{\|h(x)\|}h(x) & \text{if } x\in Q;\\ 0 & \text{if } x\in X\setminus Q.\end{cases}$$

We can do it because $h(x)=0$ for some $x\in Q$ contradicts the "decreasing property" of h.

Let us consider the Cauchy problem

$$\dot z=g(z),$$
$$z(0)=x.$$

There exists a solution $z(.,x)$ of this problem which is defined on $[0,\infty)$. Let $\eta:[0,1]\times X\to X$ be the mapping $\eta(t,x)=z(\delta t,x)$. It is easy to verify the conditions (i), (ii), (iv) stated in the deformation lemma.

If $t\geqq 0$ we have

$$\|z(t,x)-x\|=\|\int_0^t g(z(\tau,x))\,d\tau\|\leqq\int_0^t\|g(z(\tau,x))\|\,d\tau\leqq t.$$

Therefore (v) holds true and $z(t,x)\in S_\delta$ for every $t\in[0,\delta]$. To conclude the proof of the deformation lemma we need the following assertion:

Claim 2. *If $f^0(z(t,x),g(z(t,x)))\leqq a$ for every $t\in[t_1,t_2]$, then $f(z(t_2,x))-f(z(t_1,x))\leqq a(t_2-t_1)$ (here $t_1\geqq 0$).*

Proof of claim 2.

Let $\varepsilon>0$ be an arbitrary positive number. It is clear that it will be done if we prove that

$$f(z(t_2,x))-f(z(t_1,x))\leqq(a+\varepsilon)(t_2-t_1).$$

We will proceed by transfinite induction on $t_2\geqq t_1$. If $t_2=t_1$, then

$$f(z(t_2,x))-f(z(t_1,x))=0=(a+\varepsilon)(t_2-t_1).$$

Let

$$f(z(t,x))-f(z(t_1,x))\leqq(a+\varepsilon)(t-t_1)$$

for every $t\in[t_2,t_1)$. A continuity argument shows that

$$f(z(t_2,x))-f(z(t_1,x))\leqq(a+\varepsilon)(t_2-t_1).$$

Now $f^0(z(t_2,x),g(z(t_2,x)))<a+\varepsilon/2$ yields the existence of $\alpha_0>0$ such that for every $t\in(t_2,t_2+\alpha_0)$ the following inequality holds:

$$f(z(t_2,x)+(t-t_2)g(z(t_2,x)))-f(z(t_2,x))\leqq(a+\varepsilon/2)(t-t_2).$$

Since f is locally Lipschitz continuous, there exists $\alpha_1>0$ and $K\geqq 0$ such that

$$|f(z(t,x))-f(z(t_2,x)+(t-t_2)g(z(t_2,x)))|$$
$$\leqq K\,\|z(t,x)-z(t_2,x)-(t-t_2)g(z(t_2,x))\|$$

for every $t\in(t_2,t_2+\alpha_1)$. Therefore

$$\left|\frac{f(z(t, x))-f(z(t_2, x)+(t-t_2)g(z(t_2, x)))}{t-t_2}\right|$$
$$\leqq K\left\|\frac{z(t, x)-z(t_2, x)}{t-t_2}-g(z(t_2, x))\right\|$$

and the last quantity tends to zero whenever t tends to t_2. But for every $t\in(t_2, t_2+\alpha_0)$ we have

$$\frac{f(z(t, x))-f(z(t_2, x))}{t-t_2}<a+\frac{\varepsilon}{2}+\frac{f(z(t, x))-f(z(t_2, x)+(t-t_2)g(z(t_2, x)))}{t-t_2}.$$

Hence there exists $\alpha_2>0$ with

$$\frac{f(z(t, x))-f(z(t_2, x))}{t-t_2}<a+\varepsilon$$

for every $t\in(t_2, t_2+\alpha_2)$. If we add the inequality for $z(t_2, x)$, we obtain

$$f(z(t, x))-f(z(t_1, x))\leqq(a+\varepsilon)(t-t_1)$$

for every $t\in[t_1, t_2+\alpha_2)$. ■

Let us turn back to the proof of the deformation lemma. The "decreasing property" of h yields

$$\frac{f(z+th(z))-f(z)}{t}<-\frac{2\varepsilon}{\delta}$$

for every $t\in(0, t_z)$ or

$$\frac{f(z+tg(z))-f(z)}{t}<-\frac{2\varepsilon}{\delta}\frac{\Psi(z)}{\|h(z)\|}\leqq 0$$

for every $t\in(0, t_z\frac{\|h(z)\|}{\Psi(z)})$. Now

$$\frac{f(z+tg(y))-f(z)}{t}$$
$$\leqq\frac{f(z+tg(z))-f(z)}{t}+\left|\frac{f(z+tg(y))-f(z+tg(z))}{t}\right|$$
$$\leqq K_1K_2.\|y-z\|$$

for every z in a neighbourhood of $y\in Q$ because f and g are locally Lipschitz. Hence $f^0(y, g(y))\leqq 0$ whenever $y\in Q$. It is clear that $f^0(z, g(z))=0$ whenever $z\in X\setminus Q$. So $f^0(z, g(z))\leqq 0$ for every $z\in X$ and from claim 2 we have $f(z(t_2, x))-f(z(t_1, x))\leqq 0$ if $0\leqq t_1\leqq t_2$. If there exists a real $t_0\in[0, \delta]$ with $f(z(t_0, x))\leqq c-\varepsilon$ for some $x\in f^{c+\varepsilon}\cap S$, then $f(z(\delta, x))\leqq c-\varepsilon$. This completes the proof.

Let us suppose that $f(z(t, x))>c-\varepsilon$ for some $x\in f^{c+\varepsilon}\cap S$ and for every t from $[0, \delta]$. Then (according to the decreasing property) $f(z(t, x))\leqq c+\varepsilon$ for every t from $[0, \delta]$ and therefore $z(t, x)\in f^{-1}([c-\varepsilon, c+\varepsilon])\cap S_\delta$ for every $t\in[0, \delta]$. Then according to the definition of g we have $\Psi(z(t, x))=1$ and $g(z(t, x))=h(z(t, x))/\|h(z(t, x))\|$ for every $t\in[0, \delta]$. Therefore we have

$$\frac{f(z(t, x)+s.h(z(t, x)))-f(z(t, x))}{s}<-\frac{2\varepsilon}{\delta}$$

for every $s\in(0, t_{z(t, x)})$ (because $z(t, x)\in Q$) and

$$\frac{f(z(t, x)+s.g(z(t, x)))-f(z(t, x))}{s}<\frac{-2\varepsilon}{\delta\,\|h(z(t, x))\|}$$

for every $s\in(0, \|h(z(t, x))\|.t_{z(t, x)})$. Since $\|h(z(t, x))\|\leqq 1$ as a convex combination of normed vectors we have

$$\frac{-2\varepsilon}{\delta.\|h(z(t, x))\|}\leqq-\frac{2\varepsilon}{\delta}.$$

Thus we obtain

$$\frac{f(z(t, x)+s.g(z(t, x)))-f(z(t, x))}{s}<-\frac{2\varepsilon}{\delta}$$

for every $s\in(0, \|h(z(t, x))\|.t_{z(t, x)})$ and the same reasoning as above shows

$$f^0(z(t, x), g(z(t, x)))\leqq-2\varepsilon/\delta \text{ for each } t\in[0, \delta].$$

Then claim 2 implies

$$f(z(\delta, x))-f(x)\leqq(-2\varepsilon/\delta).\delta=-2\varepsilon$$

and therefore

$$f(z(\delta, x))\leqq f(x)-2\varepsilon\leqq c+\varepsilon-2\varepsilon=c-\varepsilon$$

which contradicts the assumption. ■

2.2. Proof of theorem 1

Without loss of generality we can suppose that $f\geqq c$ on F. Let $\varepsilon>0$ be a fixed positive number. We set

$$b:=\inf\{f(x)\,|\,x\in F\},\quad a:=\max\{f(u), f(v)\},$$
$$\psi_\varepsilon(x):=\max\{0, \varepsilon^2/4-\varepsilon/2.\operatorname{dist}(F, x)\},\quad x\in X,$$
$$f_\varepsilon(x):=f(x)+\psi_\varepsilon(x),\quad x\in X.$$

It is easy to check that

$$c(X, f_\varepsilon)=c(F, f_\varepsilon)=c+\varepsilon^2/4.$$

We denote this quantity by c_ε.

We claim that for every $\varepsilon>0$ satisfying

$$\varepsilon<(1/2)\min\{\operatorname{dist}(u, F), \operatorname{dist}(v, F)\}$$

there exists a point

$$x_\varepsilon\in f_\varepsilon^{-1}([c_\varepsilon-\varepsilon^2/6, c_\varepsilon+\varepsilon^2/6])\cap F_\varepsilon$$

such that $stf_\varepsilon(x_\varepsilon) \geqq -\varepsilon/2$. Supposing the contrary, we apply the deformation lemma for the following choice of $S, f, c, \varepsilon, \delta$ and Q respectively: $F_{\varepsilon/3}, f_\varepsilon, c_\varepsilon, \varepsilon^2/12$, $\varepsilon/3$ and

$$Q_\varepsilon := \operatorname{int}(f_\varepsilon^{-1}([c_\varepsilon - \varepsilon^2/6,\ c_\varepsilon + \varepsilon^2/6]) \cap F_\varepsilon).$$

So we obtain that there exists $\eta \in C([0, 1] \times X, X)$ satisfying the following properties:
(i) $\eta(0, x) = x$ for every $x \in X$;
(ii) $\eta(t, x) = x$ for every $x \in X \setminus Q_\varepsilon$ and for every $t \in [0, 1]$;
(iii) $\eta(1, f_\varepsilon^{c_\varepsilon + \varepsilon^2/12} \cap F_{\varepsilon/3}) \subseteq f_\varepsilon^{c_\varepsilon - \varepsilon^2/12} \cap F_{2\varepsilon/3}$;
(iv) $\eta(t, .)$ is a homeomorphism of X for every $t \in [0, 1]$;
(v) $\operatorname{dist}(x, \eta(1, x)) \leqq \varepsilon/3$ for every $x \in X$.

Let us fix an element g_ε from the set Γ such that $\max\{f_\varepsilon(g_\varepsilon(t)) | g_\varepsilon(t) \in F_{\varepsilon/3}, t \in [0, 1]\} < c_\varepsilon + \varepsilon^2/12$, which is possible because

$$c(F, f_\varepsilon) \leqq c(F_{\varepsilon/3}, f_\varepsilon) \leqq c(X, f_\varepsilon) = c(F, f_\varepsilon) = c_\varepsilon.$$

We set

$$h_\varepsilon(t) := \eta(1, g_\varepsilon(t)),\ t \in [0, 1].$$

By the choice of ε we have $u \notin Q_\varepsilon$, $v \notin Q_\varepsilon$ and therefore $h_\varepsilon \in \Gamma$ follows from (ii). Now

$$\{g_\varepsilon(t) \mid g_\varepsilon(t) \in F_{\varepsilon/3},\ t \in [0, 1]\} \subseteq f_\varepsilon^{c_\varepsilon + \varepsilon^2/12} \cap F_{\varepsilon/3}$$

and (iii) imply

$$\{h_\varepsilon(t) \mid g_\varepsilon(t) \in F_{\varepsilon/3},\ t \in [0, 1]\} \subseteq f_\varepsilon^{c_\varepsilon - \varepsilon^2/12} \cap F_{2\varepsilon/3}.$$

The property (v) gives us

$$\begin{aligned} &\{h_\varepsilon(t) \mid h_\varepsilon(t) \in F,\ t \in [0, 1]\} \\ &\subseteq \{h_\varepsilon(t) | g_\varepsilon(t) \in F_{\varepsilon/3},\ t \in [0, 1]\} \end{aligned}$$

and hence

$$\begin{aligned} c_\varepsilon = c(F, f_\varepsilon) &\leqq \max\{f_\varepsilon(h_\varepsilon(t)) : h_\varepsilon(t) \in F,\ t \in [0, 1]\} \\ &\leqq c_\varepsilon - \varepsilon^2/12 \end{aligned}$$

which is a contradiction.

Let us estimate $f(x_\varepsilon)$:

$$\begin{aligned} f(x_\varepsilon) = f_\varepsilon(x_\varepsilon) - \psi_\varepsilon(x_\varepsilon) &\leqq c_\varepsilon + \varepsilon^2/6 = c + \varepsilon^2/4 + \varepsilon^2/6 \\ &= c + 5\varepsilon^2/12. \end{aligned}$$

$$f(x_\varepsilon) = f_\varepsilon(x_\varepsilon) - \psi_\varepsilon(x_\varepsilon) \geqq c_\varepsilon - \varepsilon^2/6 - \varepsilon^2/4 = c - \varepsilon^2/6.$$

The both inequalities imply that

$$(*) \qquad f(x_\varepsilon) \xrightarrow[\varepsilon \to 0]{} c.$$

Since $f_\varepsilon = f + \psi_\varepsilon$, using Proposition 2.3.3, [C, p. 43], we get $f_\varepsilon^0(x, h) \leqq f^0(x, h) + \psi_\varepsilon^0(x, h)$, for every $h \in X$, $\|h\| = 1$. Hence $stf(x) \geqq stf_\varepsilon(x) + \inf\{-\psi_\varepsilon^0(x, h) \mid \|h\| = 1\}$.

But ψ_ε is a globally Lipschitz function with Lipschitz constant $\varepsilon/2$. Therefore $|-\psi_\varepsilon^0(x, h)| \leqq \varepsilon/2$, for every $h \in X$, $\|h\|=1$, and so

$$st f(x_\varepsilon) \geqq st f_\varepsilon(x_\varepsilon) - \varepsilon/2 \geqq -\varepsilon/2 - \varepsilon/2 = -\varepsilon.$$

In this way we obtain that

$$(**) \qquad \liminf_{\varepsilon > 0} st f(x_\varepsilon) \geqq 0$$

and the theorem follows from the various Palais-Smale conditions, (*) and (**). The localization of the critical point in part (b) of the theorem is due to $x_\varepsilon \in F_\varepsilon$, for every $\varepsilon > 0$.

2.3. Proof of corollary 1

If $c > a$ there exist open balls B_1, B_2 centered at u and v respectively so that

$$\sup\{f(x) \mid x \in B_1 \cup B_2\} < c.$$

Denoting $F = X \setminus (B_1 \cup B_2)$, we have $c(F, f) = c$ and Theorem 1 applies.

If $c = a$ we denote $F = \{x \in X \mid \|x-u\| = r\}$. Then we have

$$c = a \leqq b \leqq c(F, f) \leqq c(X, f) = c$$

and Theorem 1 applies again.

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