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Conditions Related to Selectibility

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Conditions known to be necessary for the contractibility of curves are investigated to see if they are also necessary for the selectibility of these and other classes of curves.

0. Introduction

When investigating selectible continua and contractible curves one can observe that these two classes of spaces are related to each other. And although no internal characterization is known even now of continua to belong to either of these classes, various conditions are considered in the literature which either imply or are implied by the fact that a particular continuum belongs to one of these two classes. Three of these conditions, (1.13), (1.14) and (1.5) in Theorem 1.9, are of special importance; they are used by L. G. Oversteegen in the characterization of contractible fans [32]. The aim of this paper is to verify if some of them have an influence on the existence of a continuous selection for the hyperspace of all subcontinua of a given fan or dendroid. In other words we try to find conditions which are responsible not only for noncontractibility, but also for nonselectibility of dendroids or of fans.

I am much obliged to T. J. Lee for calling Example 1.7 to my attention, and for many fruitful suggestions during the preparation of this paper.

All spaces considered in the paper are metric continua, and all mappings are assumed to be continuous. A curve means a one-dimensional continuum. Given a continuum X, we denote by $C(X)$ the hyperspace of all nonempty subcontinua of X equipped with the Hausdorff metric. A continuous selection on $C(X)$ means a mapping $\sigma: C(X) \to X$ such that $\sigma(A) \in A$ for each $A \in C(X)$. If $C(X)$ admits a continuous selection, then X is said to be selectible.

A continuum X is said to be contractible provided there exists a mapping $H: X \times [0, 1] \to X$ (called a homotopy) such that for some point $p \in X$ and for each point $x \in X$ we have $H(x, 0) = x$ and $H(x, 1) = p$.

A property of a continuum X is said to be hereditary provided each subcontinuum of X has this property. A continuum is said to be hereditarily unicoherent if the intersection of each two its subcontinua is connected. A hereditarily unicoherent and arcwise connected continuum is called a dendroid. It is known that each dendroid is a curve. If a dendroid is locally connected, it is called a dendrite. The unique arc joining points a and b in a dendroid is denoted by ab. A point of a dendroid X is called an end point of X if it is an end point of each arc contained in X and containing it. If a dendroid has countably many end points, it is said to be **countable**. By a **ramification point** of a dendroid X we understand a point which is the centre of a simple triod contained in X, i.e., a point $p \in X$ such that there are three arcs pa, pb and pc in X, with the intersection of each two of them being just the singleton $\{p\}$. A dendroid having exactly one ramification point is called a **fan**, and this point is then called its **top**.

We use the symbols Li, Ls and Lim to denote the lower limit, the upper limit and the topological limit of a sequence of sets. If two points p and q of either the Euclidean plane or three-space are given, then $\overline{pq}$ denotes the straight line segment with ends p and q. $\mathbb{N}$ means the set of all positive integers.

Recall that a continuum X is said to be **uniformly arcwise connected** provided it is arcwise connected and for every positive number ε there is a number $k \in \mathbb{N}$ such that every arc contained in X can be divided into at most k subarcs whose diameters do not exceed ε (see [4], p. 193; compare also [23]).

The following known result illustrates that the two classes of continua considered, i.e. selectible continua and contractible curves, have some important properties in common.

0.1. Theorem. *If a continuum is either selectible or one-dimensional and contractible, then*
(0.2) *it is a dendroid;*
(0.3) *it is a continuous image of the Cantor fan and thus it is uniformly arcwise connected;*
(0.4) *it is locally connected if and only if it is a dendrite.*

Proof. Assume first that the continuum X under consideration is selectible. S. B. Nadler and L. E. Ward showed in Lemma 3 of [30], p. 370, that X is a dendroid. According to Proposition 2 of [7], p. 110, each selectible dendroid satisfies condition (0.3). And (0.4) is stated as Corollary in [30], p. 371.

Assume next that X is one-dimensional and contractible. Then (0.2) is just Proposition 1 of [5], p. 73. Theorem 3 of [10], p. 94, states that every contractible dendroid X is uniformly arcwise connected, which is known to be equivalent to the existence of a mapping from the Cantor fan onto X (W. Kuperberg [23], Theorem 3.5, p. 322); thus (0.3) is true. Finally note that (0.4) follows easily from (0.2) and the definitions. Thus the proof is complete.

On the other hand, the author's Propositions 3 and 4 of [7], p. 110 and 111, contain examples of noncontractible and selectible dendroids (even with some extra properties). S. B. Nadler asked in his book ([29], (5.11), p. 259) if every contractible dendroid is selectible. The question has been solved in the negative by T. Maćkowiak ([28], Example, p. 321) who constructed a contractible and nonselectible dendroid D by combining properties of two examples, viz. his own Example 1 of [27], p. 548 and B. G. Graham's example A of the Appendix in [20], p. 89. See [8] for a discussion concerning some further properties of the dendroid D and problems related to this topic. Nevertheless, the question is still open if we require that the contractible dendroid has some additional properties, e.g. is a fan. And though we know an internal characterization of contractible fans (cited here as Theorem 1.9 below), interrelations between contractibility and selectibility for these continua are not clear enough, and the results on this topic seem to be rather far from being the final ones. Thus, the following question is open ([8], Problem 7, p. 28).

0.5. Question. Does there exists a contractible and nonselectible fan?

To see other differences between contractibility and selectibility for dendroids, recall that selectibility is a hereditary property (see [7], p. 113), while contractibility is not, even for countable plane fans, as it was indicated by F. B. Jones in [22] (compare also S. T. Czuba's [17]).

1. The conditions

Now we recall the three conditions which were used by L. G. Oversteegen in [32] to characterize contractible fans.

A dendroid X is said to be of type N (between points p and q) provided there exist in X: two sequences of arcs $p_n p'_n$ and $q_n q'_n$, and points $p''_n \in q_n q'_n \setminus \{q_n, q'_n\}$ and $q''_n \in p_n p'_n \setminus \{p_n, p'_n\}$, such that the following conditions are satisfied:

(1.1) $$pq = \operatorname{Lim} p_n p'_n = \operatorname{Lim} q_n q'_n ;$$

(1.2) $$p = \lim p_n = \lim p'_n = \lim p''_n ;$$

(1.3) $$q = \lim q_n = \lim q'_n = \lim q''_n .$$

The above concept is due to L. G. Oversteegen ([31], p. 837) and is related to B. G. Graham's condition "contains a zigzag" (see [20], p. 78). Namely we say that a dendroid X contains a zigzag provided there exist in X: an arc pq, a sequence of arcs $p_n q_n$ and two sequences of points p'_n and q'_n situated in these arcs in such a manner that $p_n < q'_n < p'_n < q_n$ (where $<$ denotes the natural order on $p_n q_n$ from p_n to q_n), for which the following conditions hold:

(1.4) $$pq = \operatorname{Lim} p_n q_n ;$$

(1.5) $$p = \lim p_n = \lim p'_n ;$$

(1.6) $$q = \lim q_n = \lim q'_n .$$

It is evident that if a dendroid contains a zigzag, then it is of type N ([32], p. 393) but not inversely, even for fans, as it can be seen from Example 2.7 in [9], which is repeated here for the reader's convenience.

1.7. Example. *There is a countable plane fan which is of type N and contains no zigzag.*

Proof. Let v be the pole (i.e. the origin) of a polar coordinate system in the Euclidean plane. For each $n \in \mathbb{N}$ define the following points in polar coordinates (ρ, φ):

$$a = (1, 0),\ a_n = (1, 2^{1-n}),\ p_n = (1/3, (3/4) \cdot 2^{1-n}),$$
$$q_n = (2/3, (3/4) \cdot 2^{1-n}),\ r_n = (2/3, 2^{1-n}).$$

Let

$$X = \overline{va} \cup \bigcup \{\overline{va_{2n-1}} \cup \overline{a_{2n-1} p_{2n-1}} \cup \overline{p_{2n-1} q_{2n-1}} : n \in \mathbb{N}\}$$
$$\cup \bigcup \{\overline{vr_{2n}} \cup \overline{r_{2n} p_{2n}} : n \in \mathbb{N}\}.$$

Then X is a fan with top v. Putting $p = \lim p_n = (1/3, 0)$ and $q = \lim q_n = (2/3, 0)$ we see that X is of type N between p and q, while it contains no zigzag.

A point p of a dendroid X is called a Q-point of X provided there exists a sequence of points p_n of X converging to p such that $\operatorname{Ls} pp_n \neq \{p\}$ and, if for each $n \in \mathbb{N}$ the arc $p_n q_n$ is irreducible between p_n and the continuum $\operatorname{Ls} pp_n$, then the sequence of points q_n converges also to p. This concept is due to R. B. Bennett

[3] and it was intensively exploited in investigations of the contractibility of fans, e. g. in [20] and [32]. The following result is related to selectibility and the concept of a Q-point.

1.8. Proposition. *Let a dendroid X contain a Q-point p and let $K = \mathrm{Ls}\, pp_n$, where $\{p_n\}$ is the sequence of points of X mentioned in the definition of the Q-point. Then for every continuous selection $\sigma\colon C(X) \to X$ we have $\sigma(K) = p$.*

Proof. Indeed, the above proposition is a stronger version of the Corollary on p. 118 of [7], which has the same conclusion under an additional assumption that $\lim \operatorname{diam} pq_n = 0$, where q_n has the same meaning as in the definition of the Q-point p. Note that $pq_n \subset K$. If K is locally connected, then it is a dendrite, whence the assumption is satisfied, and so the conclusion holds. If not, the continuum $K \cup \bigcup \{pp_n;\ n \in \mathbb{N}\}$ is not uniformly arcwise connected, whence it follows that the whole of X also is not uniformly arcwise connected, contrary to condition (0.3) of Theorem 0.1.

The third concept we recall here is pairwise smoothness; it was formulated by B. G. Graham in [20], p. 78, and was shown to be an important tool in studies of the contractibility of fans in [20] and L. G. Oversteegen's [32]. Let two sequences of points r_n^1 and r_n^2 of a dendroid X be given, both converging to a common limit point r. We say that the former sequence *dominates* the latter one provided that whenever there is a point s in X and a sequence of points s_n^1 of X converging to s with the property that the arcs $r_n^1 s_n^1$ converge to the arc rs, then it follows that there also exists a sequence of points s_n^2 of X converging to s such that the arcs $r_n^2 s_n^2$ converge to rs. A dendroid X is said to be *pairwise smooth* provided that whenever a pair of sequences converge to a common limit point, then one of the pairs dominates the other.

The following internal characterization of contractibility of fans is due to L. G. Oversteegen (see [32], Theorem 3.4, p. 393; cf. B. G. Graham's [20], Theorems 2.1, 2.3, 2.4 and 3.10, p. 81, 82 and 88; compare also Theorem (2.7) below).

1.9. Theorem (L. G. Oversteegen). *For every fan X the following conditions are equivalent:*

(1.10) *X is contractible;*

(1.11) *X is not of type N, contains no Q-point and is pairwise smooth;*

(1.12) *X contains no zigzag, contains no Q-point and is pairwise smooth.*

The above characterization describes three possible reasons for the non-contractibility of a fan:

(1.13) being of type N (in particular containing a zigzag),

(1.14) containing a Q-point, and

(1.15) being not pairwise smooth.

Let us note that no one of the above three conditions (1.13), (1.14) and (1.15) implies any of the other two. Namely a fan of type N without any Q-point that is pairwise smooth is shown by B. G. Graham in [20], Fig. 5 (also Fig. 6), p. 92. A fan which is not pairwise smooth but contains no Q-point and is not of type N is pictured also by B. G. Graham in [20], Fig. 3 (and Fig. 4), p. 91. And finally the third needed example of a fan which contains a Q-point, is not of type N and is pairwise smooth is constructed as Example 2.16 in [9]. Since this example is used for further purposes in this paper we recall it below.

1.16. Example. *There is a countable plane fan which contains a Q-point, is not of type N and is pairwise smooth.*

Proof. Take in the plane $\mathbb{R}^2$ a dendrite K such that the closure of the set of all ramification points of K is an arc A. The simplest example of such a dendrite is pictured in Fig. 6 of [24], §49, VI, Remark, p. 247. Let p and q denote the end points of A. Take, again in $\mathbb{R}^2$, a sequence of arcs pp_n such that (i) $q=\lim p_n$, (ii) for each distinct $m, n \in \mathbb{N}$ we have $pp_n \cap pp_m = \{p\} = pp_n \cap K$, and (iii) the arcs pp_n approximate K without folding back so that $K = \operatorname{Lim} pp_n$. Next consider the union $K \cup \bigcup \{pp_n : n \in \mathbb{N}\}$ and shrink the arc $A \subset K$ to the point p. The resulting space is a fan with its top p being a Q-point. In a routine way we may verify the two other properties of this fan.

2. Bend intersection property

The following concept was introduced by T. Maćkowiak in [27], p. 548. Let a continuum X and its subcontinuum A be given. A continuum $B \subset A$ is called a bend set of A provided there are two sequences $\{A_n\}$ and $\{A'_n\}$ of subcontinua of X satisfying the following conditions:

(2.1) $A_n \cap A'_n \neq 0$ for each $n \in \mathbb{N}$;

(2.2) $A = \operatorname{Lim} A_n = \operatorname{Lim} A'_n$;

(2.3) $B = \operatorname{Lim} (A_n \cap A'_n)$.

We say that a continuum X has the bend intersection property provided for each continuum $A \subset X$ the intersection of all bend sets of A is nonempty. It is shown in [27], Corollary, p. 548 that

(2.4) *each selectible dendroid has the bend intersection property.*

T. Maćkowiak's Example 1 in [27], p. 548, shows that the converse is not true, and he observes (on the same page) that

(2.5) *if a dendroid is of type N, then it has not the bend intersection property, and hence it is not selectible.*

In [25] T. J. Lee studies some further relations for dendroids between the bend intersection property and the concept of having a Q-point and of being of type N. In particular, he showed in Statement 1 of [25] that if a dendroid X contains a Q-point p, then there is a subcontinuum of X that has $\{p\}$ as its bend set. The main result of [25] says that if a fan contains no Q-point and is not of type N, then it has the bend intersection property. Thus by Theorem 1.9 one gets that

(2.6) *every contractible fan has the bend intersection property.*

Moreover, the following characterization of contractible fans is a consequence of L. G. Oversteegen's Theorem 1.9 and of Statements (2.5) and (2.6).

(2.7) *A fan is contractible if and only if it contains no Q-point, has the bend intersection property and is pairwise smooth.*

Another result which is related to the properties discussed was obtained by T. J. Lee in [26]. He proved that a dendroid X is not of type N if and only if for each arc A contained in X the intersection of all bend sets of A is nonempty. In particular, this condition is satisfied for all contractible dendroids (see [26], Corollary). However, we would like to attain a stronger result, which corresponds to (2.4) and (2.6).

2.8. Question (T. J. Lee). Does every contractible dendroid have the bend intersection property?

3. Interrelations

As we already know from (2.5), one of the three conditions (1.13), (1.14) and (1.15), viz. (1.13), implies not only noncontractibility of dendroids ([31], Theorem 2.1, p. 838), but also their nonselectibility. Now we are going to show that the other two, (1.14) and (1.15), do not have similar consequences. We recall some examples first.

In T. Maćkowiak's Example 3 of [27], p. 549 a selectible countable dendroid D is constructed which has exactly three ramification points, contains no Q-point, and is not pairwise smooth (these properties can easilly be observed just from the definition of D). Therefore we see that property (1.15) of being not pairwise smooth does not suffice for nonselectibility of a dendroid. The same conclusion can be drawn for fans from an example described in Proposition 4 of [7], Fig. 2, p. 111 and 112, because the fan constructed there is selectible, contains no Q-point and is not pairwise smooth. But even if we join the existence of a Q-point, i. e. (1.14) to (1.15), the two conditions together still are not strong enough to imply nonselectibility. This can be seen from an example below.

3.1. Example. *There exists a countable plane dendroid which has exactly two ramification points, contains a Q-point, is not of type N, is not pairwise smooth, and which is selectible.*

Proof. Let the plane $\mathbb{R}^2$ be equipped with the polar coordinate system (ρ, φ), and let $|z|$ denote the Euclidean norm of a point $z \in \mathbb{R}^2$ (i. e., if $z=(\rho, \varphi)$, then $|z|=\rho$). Further, given a nonnegative real number k and a point $z=(\rho, \varphi)$, we let kz denote the point $(k\rho, \varphi)$. With this notation let $p=(0, 0)$ be the origin, and let $a=(1, 0)$, $b=(1, \pi/2)$, and $c=(1, \pi)$. Further, for each $n \in \mathbb{N}$ we put

$$a_n=(1, -\pi/2n),\ b_n=(1+1/n, \pi/2),\ c_n=(1+1/n, \pi),$$
$$d_n=(1/n, \pi/4), \text{ and } e_n=(1/n, 3\pi/4).$$

Define

$$X_1=\overline{ac}\cup\overline{pb}\cup\bigcup\{(\overline{ad_n}\cup\overline{d_n b_n}\cup\overline{b_n e_n}\cup\overline{e_n c_n}\cup\overline{c_n a_n}) : n\in\mathbb{N}\}$$

and note that X_1 is a dendroid having the points a and p as the only ramification points. It is apparent that the point a is a Q-point of X_1, and that X_1 is not of type N. To see that X_1 is not pairwise smooth note that the sequences of points d_n and e_n have the point p as their common limit, and neither of them dominates the other. In fact, the sequence of straight line segments $\overline{d_n a}$ tends to the segment $\overline{pa}$, while there is no sequence of points x_n in the dendroid X_1 with the point a as its limit and such that the arcs $e_n x_n$ have $\overline{pa}$ as the limit. This shows that the sequence $\{d_n\}$ does not dominate $\{e_n\}$. Similarly, note that $c=\lim c_n$ and that $\overline{pc}=\operatorname{Lim} \overline{e_n c_n}$, while there is no sequence of points y_n in X_1 with the property that $c=\lim y_n$ and that $\overline{pc}$ is the limit of the sequence of arcs $d_n y_n$. So $\{e_n\}$ has not dominate $\{d_n\}$.

Now we prove that the dendroid X_1 is selectible. The idea of this proof is taken from T. Maćkowiak's description of a selection of the above mentioned dendroid D in Example 3 of [27], p. 550. Let $T=\overline{ac}\cup\overline{pb}$ stand for the limit triod of X_1. Define a mapping $s: T\times T\to T$ as follows. Given two points x and y in T, we let $s(x, y)$ to be a point of T such that

$$|s(x, y)|=||x|-|y||/2$$

and such that $s(x, y)$ belongs either to the segment $\overline{px}$ if $|x|\geqq|y|$, or to the segment $\overline{py}$ otherwise. In other words, $s(x, y)$ is the mid point of the arc from x to y contained in T. Further, put $C(T, p)=\{K\in C(T): p\in K\}$. We shall use a function $\alpha: C(T, p)\times \mathsf{R}^2\to T$ that assigns to each subcontinuum K of T with $p\in K$ and to each point $z\in \mathsf{R}^2$ a point $\alpha(K, z)$ of $\overline{pz}\cap K$ having the greatest norm.

Now define an auxiliary mapping $\sigma': C(T)\to T$ as follows. Let $K\in C(T)$.

1) If $p\in K$, we put

$$a'=\alpha(K, a),\ b'=\alpha(K, b),\ \text{and}\ c'=\alpha(K, c),$$

and define

$$\sigma'(K)=\alpha(K, 4s(s(3a', b'), s(b', c'))). \tag{3.2}$$

2) If $p\in T\setminus K$, observe that K is an arc, and denote by x and y its end points, with $|x|\geqq|y|$. Then put $\sigma'(K)=x$.

It is easy to show that σ' is a continuous selection for $C(T)$ having the following properties:

$$\text{if } a\in K\subset \overline{pa}\cup\overline{pb}\subset T, \text{ then } \sigma'(K)=a; \tag{3.3}$$

$$\text{if } b\in K\subset \overline{pb}\cup\overline{pc}\subset T, \text{ then } \sigma'(K)=b; \tag{3.4}$$

$$\text{if } c\in K\subset \overline{pc}\subset T, \text{ then } \sigma'(K)=c; \tag{3.5}$$

$$\text{if } a, c\in K\subset T, \text{ then } \sigma'(K)\in\overline{ac}; \tag{3.6}$$

$$\sigma'(T)=a. \tag{3.7}$$

Now let a mapping $\beta: X_1\to T$ be a retraction from X_1 onto T such that for each $n\in\mathsf{N}$ we have

$$\beta(a_n)=a,\ \beta(b_n)=b,\ \beta(c_n)=c,\ \beta(d_n)=\beta(e_n)=p$$

and that β is linear on each straight line segment which is contained in X_1 but not in T. Further, we put

$$X_0=\{(\rho, \varphi)\in X_1: \varphi\in[0, \pi]\},$$

(see Fig 4 of [15], p. 78, for a picture of a homeomorphic copy of X_0) and we denote by $\leqq_p$ the partial order on X_1 with respect to the point p, i.e., $x\leqq_p y$ if and only if $px\subset py$. We define a selection $\sigma: C(X_1)\to X_1$ as follows. Let $K\in C(X_1)$ be given. Consider three cases.

(a) If $K\cap T\neq\emptyset$, then we define $\sigma(K)=\sigma'(K\cap T)$.

(b) If $K\cap T=\emptyset\neq K\cap X_0$, then note that the set

$$\beta^{-1}(\sigma'(\beta(K\cap X_0)))\cap K\cap X_0=(\beta|(K\cap X_0))^{-1}(\sigma'(\beta(K\cap X_0)))$$

consists of at most two points; we define $\sigma(K)$ as the first one of them in the ordering $\leqq_p$; i.e. we put

$$\sigma(K) = \min_{\leq p} \{(\beta | (K \cap X_0))^{-1} (\sigma' (\beta (K \cap X_0)))\}.$$

(c) If $K \cap T = \emptyset = K \cap X_0$, then $(\beta | K)^{-1} (\sigma' (\beta (K)))$ is composed of just one point, which we accept as $\sigma(K)$.

Using properties (3.3)-(3.7) of σ' one can verify that σ is continuous. Thus the proof is complete.

3.8. Remarks. According to case (a) of the definition of σ above, we see that conditions (3.3)-(3.7) hold true if σ' is replaced by σ. In connection with this we have the following observations.

1) Since the subcontinuum $ab \cup \bigcup \{ab_n : n \in \mathbb{N}\}$ of X_1 is homeomorphic to the harmonic fan, condition (3.3) holds for each selection σ for $C(X_1)$ by the S. B. Nadler and L. E. Ward Lemma 4 of [30], p. 371, which says that if σ is a continuous selection on the hyperspace of subcontinua of the harmonic fan with top v and with limit segment L, then $\sigma(L) = v$.

2) As consequences of (3.4) and (3.5) we have $\sigma(pb) = b$ and $\sigma(pc) = c$. Both of these equalities have to be satisfied for an arbitrary selection σ for $C(X_1)$ according to T. Maćkowiak's main theorem of [27], p. 547, which runs as follows. Let the hyperspace $C(X)$ of a dendroid X admit a continuous selection σ, and let two sequences $\{A_n\}$ and $\{A'_n\}$ of subcontinua of X satisfy the following conditions:

(2.1) $$A_n \cap A'_n \neq \emptyset \quad \text{for each } n \in \mathbb{N};$$

(2.2') $$\operatorname{Lim} A'_n \subset A = \operatorname{Lim} A_n;$$

(2.3) $$B = \operatorname{Lim} (A_n \cap A'_n).$$

If $\sigma(A_n \cup A'_n) \in A'_n$ for each $n \in \mathbb{N}$, then $\sigma(A) \in B$.

3) Also condition (3.7) must be satisfied for every continuous selection σ for $C(X_1)$, which can be seen from Theorem 1.8 above.

4) The author does not know whether conditions (3.4)-(3.6) must hold if σ is an arbitrary continuous selection for $C(X_1)$.

5) The results quoted above in 1), 2) and 3) imply the choice of the coefficients 3 and 4 in the definition (3.2) of the auxiliary selection σ'. The reader can verify that if we put for example the value 2 in place of 3 in formula (3.2), then continuity of σ' is violated: we then get $\sigma(ab) = p$ instead of $\sigma(ab) = a$ in (3.3), which contradicts to the S. B. Nadler and L. E. Ward lemma stated in 1) above.

3.9. Remark. It can be observed that if we add to the dendroid X_1 a straight line segment $\overline{pd}$ which is situated in three-space in such a way that it is perpendicular to the plane $\mathbb{R}^2$ the dendroid X_1 is located in, then the union $X_2 = \overline{pd} \cup X_1$ is a nonplanar dendroid having all the other properties of X_1. The fact that X_2 is nonplanar follows from an observation that the point p is a strongly inaccessible point of X_1 (i.e., there is no embedding $h: X_1 \to h(X_1) \subset \mathbb{R}^2$ of X_1 into the plane $\mathbb{R}^2$ such that $h(p)$ is accessible from the complement $\mathbb{R}^2 \setminus X_1$), and from Proposition 2 of [12], p. 206, saying that if a space X contains a planar subset S and an arc pq such that $pq \cap S = \{p\}$, where p is a strongly inaccessible point of S, then X is nonplanar.

Now we give an example of a fan with similar properties. The existence of such an example was known to T. Maćkowiak, as was announced in the last paragraph of [7], p. 118, but no proof of its properties has been published by him. The proof presented below is again patterned after T. Maćkowiak's ideas taken from his paper [27].

3.10. Eaxample. *There exists a countable plane fan which contains a Q-point, is not of type N, is not pairwise smooth, and which is selectible.*

Proof. We apply the same notation as in Example 3.1. Let again $p=(0,0)$ be the origin, and for each n, $m\in\mathrm{N}$ let

$$a_n=(1/2^{n-1},\ \pi/2^n),\ a_{n,m}=((1/2^{n-1})(1+1/m),\ \pi/2^n),$$
$$b_m=(1/m,\ \pi),\ \text{and}\ p_{n,m}=(1/(m\cdot 2^n),\ 3\pi/2^{n+2}).$$

Put

$$T=\bigcup\{\overline{pa_n}:n\in\mathrm{N}\}, \tag{3.11}$$

and for a fixed $m\in\mathrm{N}$ let

$$pb_m=\overline{b_m a_{1,m}}\cup\bigcup\{\overline{a_{n,m}p_{n,m}}\cup\overline{p_{n,m}a_{n+1,m}}:n\in\mathrm{N}\}\cup\{p\}.$$

It is evident that T is a countable fan with the top p and that for each natural m the union pb_m is an arc from p to b_m. Note further that the sequence $\{pb_m:m\in\mathrm{N}\}$ limits on the fan T, and consequently the union

$$X_3=T\cup\bigcup\{pb_m:m\in\mathrm{N}\}$$

is a countable fan (see p. 301) of [13] for a picture of this fan). It can easily be seen that p is a Q-point of X_3 and that X_3 is not of type N. To see that X_3 is not pairwise smooth note that the sequences of points $p_{1,m}$ and $p_{2,m}$ have the point p as their common limit, and neither of them dominates the other. This last statement can easily be shown in the same way is it was done before, in Example 3.1. So we omit the details.

To describe a selection for $C(X_3)$ we employ the same functions $s:T\times T\to T$ and $\alpha:C(T,p)\times\mathrm{R}^2\to T$ as before in the proof of Example 3.1, and so we need not repeat them here. We define a selection $\sigma':C(T)\to T$ first. Let $K\in C(T)$.

1) If $p\in K$, put $x_n=\alpha(K,\ a_n)$, consider a sequence of points y_n of K determined by the conditions

$$y_0=p\ \text{and}\ y_n=\alpha(K,\ 2s(y_{n-1},\ 3x_n)\ \text{for each}\ n\in\mathrm{N},$$

and define $\sigma'(K)=\alpha(K,\ \lim y_n)$.

2) If $p\in T\setminus K$, observe that K is an arc xy in T. Let $|x|\geqq|y|$ and put $\sigma'(K)=x$.

It is easy to see that σ' is a continuous selection for $C(T)$ having the following properties:

$$\text{if}\ p\in K=\overline{px_n}\ \text{for some}\ n\in\mathrm{N},\ \text{then}\ \sigma'(K)=x_n; \tag{3.12}$$

$$\sigma'(T)=p. \tag{3.13}$$

Now let a mapping $\beta:X_3\to T$ be a retraction from X_3 onto T such that for each n, $m\in\mathrm{N}$ we have

$$\beta(a_{n,m})=a_n\ \text{and}\ \beta(b_m)=\beta(p_{n,m})=p$$

and such that β is linear on each straight line segment which is contained in X_3 but not in T. Further, we denote by $\leqq_p$ the partial order on X_3 with respect to the point p, i.e., $x \leqq_p y$ if and only if $px \subset py$. We define a selection $\sigma : C(X_3) \to X_3$ as follows. Let $K \in C(X_3)$ be given. Consider two cases.

(a) If $K \cap T \neq \emptyset$, then we define $\sigma(K) = \sigma'(K \cap T)$.

(b) If $K \cap T = \emptyset$, i.e., $K \subset X_3 \setminus T$, then observe that $K \subset pb_m$ for some $m \in \mathsf{N}$, and therefore the set

$$\beta^{-1}(\sigma'(\beta(K))) \cap K = (\beta \mid K)^{-1}(\sigma'(\beta(K)))$$

is finite; we define $\sigma(K)$ as the first one of them in the ordering $\leqq_p$; i.e. we put

$$\sigma(K) = \min_{\leqq p}\{(\beta \mid K)^{-1}(\sigma'(\beta(K)))\}.$$

Using properties (3.12) and (3.13) of σ', one can verify that σ is continuous. Thus the proof is complete.

The method of construction of a continuous selection σ for the hyperspace of subcontinua shown in Examples 3.1 and 3.10 can also be applied to some other examples, in particular to the fan defined in Example 1.16.

3.14. Example. *There is a countable plane fan which contains a Q-point, is pairwise smooth, and which is selectible (and thus is not of type N).*

Proof. The fan defined in Example 1.16 has the needed properties. The details are left to the reader.

3.15. Remark. Note that the sequence of arcs pb_m which approximates the limit fan T in X_3 of the above Example 3.10 is constructed in such a way that the segment pa_1 is approximated as the last one among all the segments pa_n of T for $n \in \mathsf{N}$. If we change this way of approximation of T into the opposite one in which the segment pa_1 is approximated as the first one, we lost selectibility. To be more precise, consider the following example, in which the same notation as in Example 3.10 is used.

Let T be defined again by formula (3.11), and for each $m \in \mathsf{N}$ let c_m denote the middle point of the straight line segment $b_m a_{1,m}$. Further, note that the union

$$\overline{pa_{m+1}} = \overline{pc_m} \cup \overline{c_m a_{1,m}} \cup \bigcup \{\overline{a_{n,m} p_{n,m}} \cup \overline{p_{n,m} a_{n+1,m}} : n \in \{1, 2 \ldots, m\}\}$$

is an arc from p to a_{m+1} and that the sequence $\{pa_{m+1} : m \in \mathsf{N}\}$ limits on the fan T. Consequently the union

$$X_4 = T \cup \bigcup \{pa_{m+1} : m \in \mathsf{N}\}$$

is a countable fan. It can easily be seen that again p is a Q-point of X_4, but contrary to X_3, the fan X_4 is of type N and thus, according to (2.5), it is not selectible.

As it can be observed from examples considered in other papers, as well as ones constructed above, the only condition we know that implies a fan is nonselectible is (1.13), i.e., that of being of type N. Thus it is natural to ask if the inverse to (2.5) is true.

3.16. Question. Does there exists a nonselectible fan which is not of type N?

Note that a positive answer to Question 0.5 implies a positive answer to 3.16 by Theorem 1.9. Note also that there exists a nonselectible dendroid with exactly two ramification points having the bend intersection property (viz. T. Maćkowiak's Example 1 of [27], p. 548), and therefore it is not of type N by (2.5).

To have a full view of the matter we list below examples of selectible, as well as of nonselectible fans which have or do not have the considered properties (1.14) and (1.15). Because of (2.5) we exclude (1.13) from the discussion below.

A. Selectible fans. 1) The well-known harmonic fan has no Q-point and is pairwise smooth. 2) A fan without any Q-point but which is not pairwise smooth is presented in Fig. 2 of Proposition 4 of the author's [7], p. 111 and 112 (see also Fig. 3 in B. G. Graham's [20], p. 91). 3) A fan containing a Q-point that is pairwise smooth is shown in Example 3.14 above. 4) Finally one with a Q-point that is not pairwise smooth is constructed in Example 3.10.

B. Nonselectible fans. Note that all examples presented in this section are of type N. 1) Fans without any Q-point which are pairwise smooth are presented by B. G. Graham in Fig. 5 and 6 of [20], p. 92. 2) The one point union of the two fans pictured in Fig. 3 and 5 of [20], p. 91 and 92 having their top as the only common point is a fan without any Q-point but which is not pairwise smooth. 3) A fan containing a Q-point that is pairwise smooth is shown by C. A. Eberhart and the author in p. 95 of [10]; see also Fig. 7 of [20], p. 93, and Example 1.2 of L. G. Oversteegen's [31], p. 838. 4) An example of a nonselectible fan containing a Q-point that is not pairwise smooth can be obtained as the point union of two fans, namely of X_3 in Example 3.10 above and of the one in Fig. 5 of [20], p. 92.

The above list shows that conditions (1.14) and (1.15) have no effect on the selectibility of fans.

4. R-continua

Besides (1.13), (1.14) and (1.15) other sets of conditions that imply noncontractibility of dendroids are known from the literature. One such set is formed by conditions considered in [11], [13], [14], [15] and [18], namely, generalizations of the notion of an R-arc and of an R-point defined in [11], p. 230 and 231 and exploited in [7]. The concept of an R-continuum was introduced by S. T. Czuba in [13], Definition 1, p. 300, and renamed an R^1-continuum later in [15]. He also introduced the similar notions of R^2-continuum and R^3-continuum in [15]. Following [15], Definitions 1.1, 1 2 and 1.3, p. 75, a nonempty proper subcontinuum K of a dendroid X is called an R^i-continuum (where $i=1$, 2, or 3) if there exist an open set U containing K and two sequences $\{C_n^1\}$ and $\{C_n^2\}$ of components of U such that

$$K=\begin{cases}\operatorname{Ls} C_n^1 \cap \operatorname{Ls} C_n^2 & \text{for } i=1,\\ \operatorname{Lim} C_n^1 \cap \operatorname{Lim} C_n^2 & \text{for } i=2,\\ \operatorname{Li} C_n^1 & \text{for } i=3.\end{cases}$$

Theorem 9 of [15], p. 78 (see also [13], Theorem 3, p. 300) says that

(4.1) *if a dendroid X contains an R^i-continuum* (*where $i=1$, 2, or* 3), *then X is not contractible.*

Some interrelations between the concept of an R-arc and an R^i-continuum are studied by S. T. Czuba in [15]. For details see [15], Proposition 2 and Examples 3 and 4, p. 75 and 76; Proposition 5 and the paragraph following it, p. 77; and Proposition 10 and Corollary 11, p. 78.

Taking (4.1) into account, one can conjecture that containing an R^i-continuum may imply the nonselectibility of a dendroid or of a fan. However, it is not so. Not only does the existence of a particular R^i-subcontinuum in a fan X not necessarily violate its selectibility, but even the presence of a subcontinuum that is an R^i-continuum for $i=1, 2$ and 3 simultaneously does not imply the fan is selectible. Namely observe that the fan X of the author's Proposition 4 of [7], Fig. 2, p. 111 and 112 contains a singleton which is an R^1-, R^2- and R^3-continuum, so that X is not contractible, while it is known to be selectible.

On the other hand, it can easily be observed that the author's examples of nonselectible fans pictured in p. 95 of [10] (see also B. G. Graham's Fig. 7 in [20], p. 93) and in B. G. Graham's Fig. 5 and 6 of [20], p. 92, do not contain any R^i-continuum. This shows that the property of not being selectible does not imply the existence of an R^i-continuum for fans, and thus for dendroids in general.

Hence, we can conclude that there is no direct relation between the existence of an R^i-subcontinuum in a dendroid and its selectibility.

5. The set function T

Some other conditions implying noncontractibility of dendroids are known which are expressed in terms of the set function T. They were discussed e. g. in [1], [3] and [6]. To formulate them, we need a definition. Given a compact space X and a set $A \subset X$, we define $T(A)$ as the set of all points x of X such that every subcontinuum of X which contains x in its interior must intersect A (see [19], p. 113). D. P. Bellamy and H. S. Davis have shown (see [2], Corollary 1, p. 373) that if X is a continuum and A is a subcontinuum of X, then $T(A)$ is a subcontinuum of X.

One of the conditions mentioned above can be formulated in the following way:

(5.1) the continuum X contains two closed subsets A and B such that $A \cap T(B) = \emptyset = B \cap T(A)$ and $T(A) \cap T(B) \neq \emptyset$.

Then we have the following result (see Corollary 1 in [1], p. 48 and in [6], p. 273).

5.2. Proposition (D. P. Bellamy, J. J. Charatonik). *If a continuum X satisfies condition* (5.1), *then X is not contractible.*

For dendroids condition (5.1) is equivalent (see [14], Lemma 5, p. 304) to the following one, which has been discussed e.g. in [1], p. 47, [3], and [6], p. 271:

(5.3) the dendroid X contains two points a and b having the property that

$$a \in X \setminus T(b),\ b \in X \setminus T(a) \text{ and } T(a) \cap T(b) \neq \emptyset .$$

Moreover, if A and B are subsets of X as in (5.1), then, according to the above quoted result, the points a and b of (5.3) can be chosen so that $a \in A$ and $b \in B$. For a stronger version of Theorem 5.2, namely with (5.3) applied to a subcontinuum of

X, but only in case X is a fan, see L. G. Oversteegen's Theorem 4.2 of [32], p. 394.

Similarly to the previous sections of the paper, we would like to know if condition (5.1) is related in a way to the nonselectibility of dendroids. To this end we discuss four possibilities.

1) The harmonic fan is selectible and does not satisfy (5.1).

2) The one point union of two harmonic fans having the limit point of the set of their end points as the only point in common (see the author's Proposition 3 and Fig. 1 of [7], p. 110) is a selectible countable plane dendroid X with two ramification points a and b satisfying condition (5.3); thus X satisfies (5.1) as well. However, the author does not know any example of a fan with the same properties.

3) There is a countable plane fan which is of type N (thus it is nonselectible), and which does not satisfy condition (5.1). In fact, the fan described and pictured by C. A. Eberhart and the author in [10], p. 95 (see also B. G. Graham's Fig. 7, p. 93 of [20] and L. G. Oversteegen's Example 1.2, p. 838 of [31], and Example 2.3, p. 380 of [33]) meets all the required conditions.

4) There is a countable plane fan which is of type N (thus it is nonselectible), and which satisfies condition (5.1). In fact, let X be the fan mentioned in 3), and let us take as Y the one-point union of two copies of X such that the fans under consideration have the common top as the only point of their intersection. Then Y has all the needed properties.

The examples presented above show that (5.1) neither implies nor is implied by nonselectibility.

Given a compact space X and a set $A \subset X$, we define $K(A)$ as the set of all points x of X such that every subcontinuum of X which contains A in its interior must contain x (see F. B. Jones' [21], p. 404). It is known (see e. g. E. J. Vought's Lemma 1 in [34], p. 374) that if X is a hereditarily unicoherent continuum and if A is connected, then $K(A)$ is a continuum. Further, S. T. Czuba has shown ([16], Lemma 6, p. 197) that for any subcontinuum A of the continuum X we have $K(A) = \{x \in X : T(x) \cap A \neq \emptyset\}$.

Consider the following condition that a dendroid X may satisfy:

(5.4) there is a point p in X with $\{p\} \neq K(p) \subset T(p)$.

5.5. Question. Does condition (5.4) imply nonselectibility of the dendroid X?

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