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Generalization of Ky Fan Inequality

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1. Introduction

The following inequality due to Ky Fan was recorded in [1]:

$$\left[\frac{\Pi_1^n x_i}{\Pi_1^n (1-x_i)}\right]^{1/n} < \frac{\Sigma_1^n x_i}{\Sigma_1^n (1-x_i)}, \quad 0 \leqq x_i \leqq \frac{1}{2}, \tag{1}$$

unless $x_1 = x_2 = \ldots = x_n$.

With the notation

$$M_p(x) = (\frac{1}{n}\Sigma_1^n x_i^p)^{1/p}, \quad x_i > 0;$$

and

$$M_0(x) = \lim_{p \to 0} M_p(x) = (\Pi_1^n x_i)^{1/n},$$

(1) becomes

$$\frac{M_0(x)}{M_0(1-x)} < \frac{M_1(x)}{M_1(1-x)}. \tag{2}$$

D. Segaiman [2] conjectured that

$$\frac{M_p(x)}{M_p(1-x)} < \frac{M_g(x)}{M_g(1-x)}, \quad p < g. \tag{3}$$

F. Chan, D. Goldberg and S. Gonek [2] gave some counterexamples when $0 < 2^p/p < 2^q/q$ or $p+q > 9$. In addition, they proved that (3) is true for $p+q=0>p$ or $0 \leqq p \leqq 1 \leqq q \leqq 2$.

Recently the case $p=-1$ and $q=0$ was proved to be true by Wan-Lan Wang and Peng-Fei Wang [3]. And the case $-1 \leqq p \leqq 0 \leqq q \leqq 1$ was proved by Guang-Xing Li and Ji Chen [4].

In this paper, we determine all the exponents p and q such that (3) is true.

Theorem. *For arbitrary n, $p<q$, the inequality*

$$(4)\qquad \left[\frac{\Sigma_1^n x_i^p}{\Sigma_1^n (1-x_i)^p}\right]^{1/p} \leqq \left[\frac{\Sigma_1^n x_i^q}{\Sigma_1^n (1-x_i)^q}\right]^{1/q} \ (0<x_i\leqq\frac{1}{2})$$

holds if and only if $|p+q|\leqq 3$, $2^p/p \geqq 2^q/q$ *when* $p>0$, $p2^p \leqq q2^q$ *when* $q<0$.

The proof of the sufficiency is contained in Section 3, 4 and 5. In the proof we assume $pq \neq 0$, otherwise by letting p or $q \to 0$, it is easy to see that (4) is also true. In Section 2, we will prove the necessity.

2. Proof of the necessity

In [2], it was proved that (4) and $p<q$ were equivalent when $n=2$, and that if (4) holded, then $2^p/p \geqq 2^q/q$ for $p>0$.

When $q<0$, take $x_1=x_2=\ldots=x_{n-1}=\varepsilon$ $(0<\varepsilon<1/2)$ and $x_n=1/2$, (4) becomes

$$(5)\qquad \left[\frac{(n-1)\varepsilon^p+(\frac{1}{2})^p}{(n-1)(1-\varepsilon)^p+(\frac{1}{2})^p}\right]^{1/p} \leqq \left[\frac{(n-1)\varepsilon^q+(\frac{1}{2})^q}{(n-1)(1-\varepsilon)^q+(\frac{1}{2})^q}\right]^{1/q},$$

or

$$(6)\qquad \frac{[\varepsilon^p+\frac{1}{2^p(n-1)}]^{1/p}}{[\varepsilon^q+\frac{1}{2^q(n-1}]^{1/q}} \leqq \frac{[(1-\varepsilon)^p+\frac{1}{2^p(n-1)}]^{1/p}}{[(1-\varepsilon)^q+\frac{1}{2^q(n-1}]^{1/q}}.$$

Let $\varepsilon \to 0$, (6) yields

$$(7)\qquad 1 \leqq \frac{[1+\frac{1}{2^p(n-1)}]^{1/p}}{[1+\frac{1}{2^q(n-1)}]^{1/q}},$$

hence

$$(8)\qquad \left[1+\frac{1}{2^p(n-1)}\right]^{1/p} \geqq \left[1+\frac{1}{2^q(n-1)}\right]^{1/q}.$$

By using the Maclaurin expansion in $\frac{1}{n}$, we obtain

$$(9)\qquad 1+(p\,2^p n)^{-1}+o(1/n^2) \geqq 1+(q\,2^q n)^{-1}+o(1/n^2).$$

So if $p\,2^p > q\,2^q$, (4) would be faulse for sufficiently large n.

In the equivalent inequality of (4):

$$(10)\qquad \left[\frac{\Sigma_1^n (1-u_i)^p}{\Sigma_1^n (1+u_i)^p}\right]^{1/p} \leqq \left[\frac{\Sigma_1^n (1-u_i)^q}{\Sigma_1^n (1+u_i)^q}\right]^{1/q},\ 0\leqq u_i<1.$$

Let $u_1=u_2=\ldots=u_{n-1}=0$ and $u_n=u$ $(0<u<1)$, then (10) becomes

(11) $$\left[\frac{(n-1)+(1-u)^p}{(n-1)+(1+u)^p}\right]^{1/p}\leqq\left[\frac{(n-1)+(1-u)^q}{(n-1)+(1+u)^q}\right]^{1/q}.$$

Take the Macluarin expansion of (11) in u:

$$1-\frac{2}{n}u+\frac{2}{n^2}u^2-\frac{(n-1)[(n-2)p^2-2np]+2(n^2+2)}{3n^3}u^3+o(u^4)$$

(12) $$\leqq 1-\frac{2}{n}u+\frac{2}{n^2}u^2-\frac{(n-1)[(n-2)q^2-3nq]+2(n^2+2)}{3n^3}u^3+o(u^4).$$

Thus for u sufficiently small, (10) holds only if

(13) $$(n-2)p^2-3np\geqq(n-2)q^2-3nq,$$

or

(14) $$(p-q)[(n-2)(p+q)-3n]\geqq 0.$$

So for $n\geqq 3$, we have

(15) $$p+q\leqq\frac{3n}{n-2}.$$

Let $n\to+\infty$, (15) yields $p+q\leqq 3$.

Similarly, the expansion of (10) with $u_1=u_2=\ldots=u_{n-1}=u$ $(0<u<1)$, $u_n=0$ gives

(16) $$p+q\geqq\frac{-3n}{n-2}.$$

So we obtain $p+q\geqq -3$.

3. An equivalence proposition

In this section, we are to establish an equivalence proposition as follows:

Proposition. *For $p<q$, the following inequalities are equivalent*:

(*i*) $$\left[\frac{\Sigma_1^n \lambda_i x_i^p}{\Sigma_1^n \lambda_i(1-x_i)^p}\right]^{1/p}<\left[\frac{\Sigma_1^n \lambda_i x_i^q}{\Sigma_1^n \lambda_i(1-x_i)^q}\right]^{1/q},$$

where $\lambda_i>0$, $0<x_i\leqq 1/2$, $i=1, 2,\ldots, n$ and $x_1, x_2,\ldots,x_n$ are not all equal:

(*ii*) $$\left[\frac{\lambda x^p+\mu y^p}{\lambda(1-x)^p+\mu(1-y)^p}\right]^{1/p}<\left[\frac{\lambda x^q+\mu y^q}{\lambda(1-x)^q+\mu(1-y)^q}\right]^{1/q},$$

where λ, $\mu>0$, $0<x\neq y\leqq 1/2$;

(*iii*) $$\left[\frac{\lambda+(1-u)^p}{\lambda+(1+u)^p}\right]^{1/p}<\left[\frac{\lambda+(1-u)^q}{\lambda+(1+u)^q}\right]^{1/q},$$

where $\lambda>0$, $0<u<1$.

Proof. (i) obviously implies (iii).

Now suppose (iii) is true, let $x>y$ and $y/x=1-u$, $x/(1-x)=k$, then $0<u<1$, $0<k\leqq 1$ and $(1-y)/(1-x)=1+ku$. So (ii) is equivalent to the following:

$$f(k)=\frac{1}{q}\ln\frac{\lambda+\mu(1-u)^q}{\lambda+\mu(1+ku)^q}-\frac{1}{p}\ln\frac{\lambda+\mu(1-u)^p}{\lambda+\mu(1+ku)^p}>0. \tag{17}$$

Derivate $f(k)$, one can obtain

$$\begin{aligned} f'(k)&=\frac{-\mu(1+ku)^{q-1}u}{\lambda+\mu(1+ku)^q}+\frac{\mu(1+ku)^{p-1}u}{\lambda+\mu(1+ku)^p}\\ &=\frac{u}{1+ku}\left[\frac{\mu(1+ku)^p}{\lambda+\mu(1+ku)^p}-\frac{\mu(1+ku)^q}{\lambda+\mu(1+ku)^q}\right]<0. \end{aligned} \tag{18}$$

Hence

$$f(k)\geqq f(1)=\frac{1}{q}\ln\frac{\lambda+(1-u)^q}{\lambda+(1+u)^q}-\frac{1}{p}\ln\frac{\lambda+(1-u)^p}{\lambda+(1+u)^p}>0. \tag{19}$$

(ii) is established.

We will use induction to show that (i) is true if (ii) holds. At first, (ii) is the case $n=2$ of (i). Now assume that (i) holds for some n ($n\geqq 2$).

Let $1/2\geqq x_1\geqq x_2\geqq\ldots\geqq x_{n+1}$, and x_i are not all equal, then there exist $\mu>0$ and $\nu=\lambda_1\lambda_{n+1}/\mu>0$ such that

$$\begin{aligned} \frac{\Sigma_1^{n+1}\lambda_i x_i^p}{\Sigma_1^{n+1}\lambda_i(1-x_i)^p}&=\frac{\mu x_1^p+\lambda_{n+1}x_{n+1}^p}{\mu(1-x_1)^p+\lambda_{n+1}(1-x_{n+1})^p}\\ &=\frac{\lambda_1 x_1^p+\nu x_{n+1}^p}{\lambda_1(1-x_1)^p+\nu(1-x_{n+1})^p}=R^p. \end{aligned} \tag{20}$$

It is clear that $(\lambda_1-\mu)(\lambda_{n+1}-\nu)\leqq 0$. Without loss of generality, we may assume that $\lambda_1\geqq\mu$. So

$$R^p=\frac{(\lambda_1-\mu)x_1^p+\Sigma_2^n\lambda_i x_i^p}{(\lambda_1-\mu)(1-x_1)^p+\Sigma_2^n\lambda_i(1-x_i)^p}. \tag{21}$$

By the assumption, we have

$$\begin{aligned} R&=\left[\frac{(\lambda_1-\mu)x_1^p+\Sigma_2^n\lambda_i x_i^p}{(\lambda_1-\mu)(1-x_1)^p+\Sigma_2^n\lambda_i(1-x_i)^p}\right]^{1/p}\\ &\leqq\left[\frac{(\lambda_1-\mu)x_1^q+\Sigma_2^n\lambda_i x_i^q}{(\lambda_1-\mu)(1-x_1)^q+\Sigma_2^n\lambda_i(1-x_i)^q}\right]^{1/q}, \end{aligned} \tag{22}$$

and

$$R=\left[\frac{\mu x_1^p+\lambda_{n+1}x_{n+1}^p}{\mu(1-x_1)^p+\lambda_{n+1}(1-x_{n+1})^p}\right]^{1/p}<\left[\frac{\mu x_1^q+\lambda_{n+1}x_{n+1}^q}{\mu(1-x_1)^q+\lambda_{n+1}(1-x_{n+1})^q}\right]^{1/q}. \tag{23}$$

So we have

$$R<\left[\frac{\Sigma_1^{n+1}\lambda_i x_i^q}{\Sigma_1^{n+1}\lambda_i(1-x_i)^q}\right]^{1/q}. \tag{24}$$

Therefore, we get (i) is true for arbitrary n, the proposition is established.

4. Three lemmas

Lemma 1. *If* $\alpha\leqq 0$, $\alpha<\beta\leqq 1-\alpha$, $0\leqq u<1$, *then*

$$(1+u)^\alpha+(1-u)^\alpha\geqq(1+u)^\beta+(1-u)^\beta, \tag{25}$$

the equality is attained if and only if $u=0$ *or* $(\alpha, \beta)=(0, 1)$.

Proof. Let $\varphi(x)=(1+u)^x+(1-u)^x$ $(0<u<1)$, *then*

$$\varphi''(x)=(1+u)^x[\ln(1+u)]^2+(1-u)^x[\ln(1-u)]^2>0. \tag{26}$$

So we have to establish (25) only for $\beta=1-\alpha$, i.e.

$$\Phi(u)=(1+u)^\alpha+(1-u)^\alpha-[(1+u)^{1-\alpha}+(1-u)^{1-\alpha}], \tag{27}$$

where $\alpha<0$, $0<u<1$.

$$\begin{aligned}\Phi(u)&=2\sum_{n=0}^{\infty}[\binom{\alpha}{2n}-\binom{1-\alpha}{2n}]u^{2n}\\ &=2\alpha(\alpha-1)\sum_{n=2}^{\infty}\frac{u^{2n}}{(2n)!}[\prod_{k=2}^{2n-1}(\alpha-k)-\prod_{k=2}^{2n-1}(-\alpha-k+1)]\\ &\geqq 2\alpha(\alpha-1)\sum_{n=2}^{\infty}\frac{u^{2n}}{(2n)!}[\prod_{k=2}^{2n-1}(\alpha-k)-\prod_{k=2}^{2n-1}|-\alpha-k+1|]\\ &>0.\end{aligned} \tag{28}$$

This proofs the lemma

Lemma 2. *If* $0<\alpha<\beta<1-\alpha$ *and* $0<u\leqq 1$. *Let*

$$G(u)=(1+u)^\alpha+(1-u)^\alpha-(1+u)^\beta-(1-u)^\beta, \tag{29}$$

then there exists a unique u_0, *such that*

(i) $G(u)>0$ *for* $0<u<u_0$;

(ii) $G(u)<0$ *for* $u_0<u\leqq 1$.

Proof. We have $\beta(\beta-1)<\alpha(\alpha-1)<0$, hence

$$0<\frac{\alpha(\alpha-1)}{\beta(\beta-1)}<1. \tag{30}$$

Define

$$g(u)=\frac{(1+u)^{\beta-2}+(1-u)^{\beta-2}}{(1+u)^{\alpha-2}+(1-u)^{\alpha-2}},\quad 0<u<1. \tag{31}$$

We have

$$g'(u)=\frac{(\beta-2)[(1+u)^{\beta-3}-(1-u)^{\beta-3}]}{(1+u)^{\alpha-2}+(1-u)^{\alpha-2}}$$

$$-\frac{(\alpha-2)[(1+u)^{\alpha-3}-(1-u)^{\alpha-3}][(1+u)^{\beta-2}+(1-u)^{\beta-2}]}{[(1+u)^{\alpha-2}+(1-u)^{\alpha-2}]^2}$$

$$<\frac{\alpha-2}{[(1+u)^{\alpha-2}+(1-u)^{\alpha-2}]^2}\{[(1+u)^{\beta-3}-(1-u)^{\beta-3}][(1+u)^{\alpha-2}+(1-u)^{\alpha-2}]$$

$$-[(1+u)^{\alpha-3}-(1-u)^{\alpha-3}][(1+u)^{\beta-2}+(1-u)^{\beta-2}]\}$$

$$(32)\qquad =\frac{2(\alpha-2)(1+u)^{\alpha+\beta-6}}{[(1+u)^{\alpha-2}+(1-u)^{\alpha-2}]^2}[(\frac{1-u}{1+u})^{\alpha-3}-(\frac{1-u}{1+u})^{\beta-3}]<0.$$

So $g(u)$ is strictly decreasing with $g(0)=1$ and $g(1)=0$. Hence there exists a unique $u_1\in(0,\ 1)$ such that

$$(33)\qquad g(u_1)=\frac{\alpha(\alpha-1)}{\beta(\beta-1)}.$$

Note that

$$(34)\qquad G'(u)=\alpha[(1+u)^{\alpha-1}-(1-u)^{\alpha-1}]-\beta[(1+u)^{\beta-1}-(1-u)^{\beta-1}],$$

$$(35)\qquad G''(u)=-\beta(\beta-1)[(1+u)^{\alpha-2}+(1-u)^{\alpha-2}][g(u)-\frac{\alpha(\alpha-1)}{\beta(\beta-1)}],$$

and from above we know that $G''(u)>0$ for $u\in(0,\ u_1)$, $G''(u)<0$ for $u\in(u_1,\ 1)$.

Because $G(0)=G'(0)=0$, we have $G'(u)>0$, $G(u)>0$ for $u\in[0,\ u_1]$. But $G'(1)=-\infty$, so there exists a unique $u_2\in(u_1,\ 1)$ such that $G'(u)>0$ when $u\in(u_1, u_2)$, $G'(u)<0$ when $u\in(u_2,\ 1)$.

Then $G(u)$ strictly increases in $(0,\ u_2)$ and strictly decreases in $(u_2,\ 1)$, and since $G(1)=2^\alpha-2^\beta<0$, we can find a unique $u_0\in(u_2,\ 1)$ such that $G(u)>0$ in $(0,\ u_0)$, $G(u)<0$ in $(u_0,\ 1)$.

Lemma 3. *If $p<q$, $p+q\leqq 3$ and $2^p/p\geqq 2^q/q$ for $p>0$, then*

$$(36)\qquad \frac{(1+u)^p-(1-u)^p}{p}\geqq\frac{(1+u)^q-(1-u)^q}{q},\quad 0\leqq u<1,$$

equality occurs if and only if $u=0$ or $(p,\ q)=(1,\ 2)$.

Proof. Let

$$(37)\qquad H(u)=\frac{(1+u)^p-(1-u)^p}{p}-\frac{(1+u)^q-(1-u)^q}{q},\quad 0\leqq u<1.$$

Then

$$(38)\qquad H'(u)=[(1+u)^{p-1}+(1-u)^{p-1}]-[(1+u)^{q-1}\,|\,(1-u)^{q-1}].$$

When $p\leqq 1$, $p-1<q-1\leqq 1-(p-1)$, by Lemma 1 we obtain $H'(u)\geqq 0$. Thus $H(u)\geqq H(0)=0$ with equality if and only if $u=0$ or $(p,\ q)=(1,\ 2)$.

When $p>1$, then $q<3-p$. Otherwise $q-1=1-(p-1)$, then

$$(39)\qquad \frac{(p-1)(p-2)}{(q-1)(q-2)}=1.$$

Repeat the steps in the proof of Lemma 2, it should have $H'(u)<0$. So $H(0)>H(1)$, i.e. $2^p/p<2^q/q$. It is a contradiction. Thus $0<p-1<q-1<1-(p-1)$. By

Lemma 2, $H'(u)$ has its unique zero point u_0 in (0, 1), such that the following is true:

$$H'(u)>0 \text{ for } 0<u<u_0, \text{ hence } H(u)>H(0)=0;$$
$$H'(u)<0 \text{ for } u_0<u<1, \text{ hence } H(u)>H(1)=2^p/p-2^q/q\geqslant 0;$$
$$\text{and } H(u_0)>0.$$

These establish the lemma.

5. Proof of the sufficiency of the theorem

From the equivalence proposition in Section 3, we only need to prove the following inequality:

$$(40)\qquad [\frac{\lambda+(1-u)^p}{\lambda+(1+u)^p}]^{1/p}<[\frac{\lambda+(1-u)^q}{\lambda+(1+u)^q}]^{1/q},$$

where $\lambda>0, 0<u<1, p<q, |p+q|\leqq 3, 2^p/p\geqq 2^q/q$ when $p>0$, $p\,2^p\leqq q\,2^q$ when $q<0$.

The above inequality is equivalent to

$$(41)\qquad F(\lambda)=\frac{1}{q}\ln\frac{\lambda+(1-u)^q}{\lambda+(1+u)^q}-\frac{1}{p}\ln\frac{\lambda+(1-u)^p}{\lambda+(1+u)^p}>0.$$

But

$$F'(\lambda)=\frac{1}{q}[\frac{1}{\lambda+(1-u)^q}-\frac{1}{\lambda+(1+u)^q}]-\frac{1}{p}[\frac{1}{\lambda+(1-u)^p}-\frac{1}{\lambda+(1+u)^p}]$$

$$(42)\qquad =(A\lambda^2+B\lambda+C)/Q(\lambda),$$

where

$$(43)\qquad Q(\lambda)=[\lambda+(1-u)^q][\lambda+(1+u)^q][\lambda+(1-u)^p][\lambda+(1+u)^p],$$

$$(44)\qquad A=\frac{(1+u)^q-(1-u)^q}{q}-\frac{(1+u)^p-(1-u)^p}{p},$$

$$(45)\qquad B=[(1+u)^p+(1-u)^p]\frac{(1+u)^q-(1-u)^q}{q}$$
$$-[(1+u)^q+(1-u)^q]\frac{(1+u)^p-(1-u)^p}{p},$$

$$(46)\qquad C=(1-u)^{p+q}[\frac{(1+u)^{-q}-(1-u)^{-q}}{-q}-\frac{(1+u)^{-p}-(1-u)^{-p}}{-p}].$$

By Lemma 3, when $(p, q)\neq(1, 2)$ and $(p, q)\neq(-2, -1)$, we have $A<0$ and $C>0$. If $(p, q)=(1, 2)$ then $A=0$, $B=-4u^3<0$, $C=2u^3(1-u)^2>0$. If $(p, q)=(-2, -1)$ then $A=-2u^3/(1-u^2)^2<0, B=4u^3/(1-u^2)^3>0$, $C=0$. Thus for all these cases, $F'(\lambda)$ has a unique positive root λ_0 such that $F'(\lambda)>0$ for $0<\lambda<\lambda_0$; $F'(\lambda)<0$ for $\lambda>\lambda_0$. So

$$(47)\qquad F(\lambda)>F(0)=F(+\infty)=0 \text{ for } \lambda>0.$$

Now the theorem is proved.

6. Some remarks and a conjecture

Remark 2. In the case $pq \neq 0$, from the processes of the proof, we can see the equality in (4) is attained if and only if $x_1 = x_2 = \ldots = x_n$. If $pq = 0$, all the results in Section 2 to 5 can be founded without any difference with the case $pq \neq 0$. So the equality in (4) occurs if and only if x_i are all equal.

Remark 2. Inequality (4) is for all natural numbers n, and there is not the best result for each fixed n except $n=2$. We propose a conjecture for this condition as follows:

Conjecture. *If* $p<q$, $|p+q| \leqq 3n/(n-2)$,

$$[1+2^p/(n-1)]^{1/p} \geqq [1+2^q/(n-1)]^{1/q} \text{ when } p>0, \tag{48}$$

and

$$[1+1/2^p(n-1)]^{1/p} \geqq [1+1/2^q(n-1)]^{1/q} \text{ when } q<0, \tag{49}$$

then

$$\left[\frac{\Sigma_1^n x_i^p}{\Sigma_1^n (1-x_i)^p}\right]^{1/p} < \left[\frac{\Sigma_1^n x_i^q}{\Sigma_1^n (1-x_i)^q}\right]^{1/q}, \quad 0<x_i \leqq \frac{1}{2}, \tag{50}$$

unless $x_1 = x_2 = \ldots = x_n$.

References

1. E. F. Beckenbach. Inequalities. Springer-Verlag, Berlin, 1961, 5.
2. F. Chan, D. Goldberg, S. Gonek. On extensions of an inequality among means. *Proc. Amer. Math. Soc.*, **42**, 1974, 202-207.
3. W.-L. Wang, P.-F. Wang. A class of inequalities for the symmetric function (Chinese). *Acta Math. Sinica*, **27**, 1984, 485-497.
4. Guang-Xing Li, Ji Chen. A generalization of the Ky Fan inequality (Chinese). *Hunan Shuxue Tongxun*, 1989, No 4, 37-39.

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