Mathematica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

Mathematica
Balkanica

New Series Vol. 6, 1992, Fasc. 1

Queuing System in Special Transport Situation

Monem Azez Mohammed

Presented by P. Kenderov

This paper deals with queuing processes in transport situations. It is shown that they can be model with Markov chains. The transition matrix is calculated and it is considered numerical examples where the mean queues are calculated.

Introduction

A "Queue" is awaiting line of units demanding service facility, the unit demanding service is called the "Customer" and device at which or the person, by whom it gets served is known as the "Server". For example:
– vehicles demanding service arrive in a garage and depending on the number of employees, one or more vehicles, may be served at each time.
– patients arrive at a doctor's clinic for treatment.
– passangers demanding tickets queue up in front of a ticket counter.

A. K. Erlang (1917) was the first to investigate such simple problem, also F. Benson and D. R. Cox (1951), D. G. Kendall (1951) and with imbedded Markov chain (1953).

In some cases the serves is able to attent to a batch of several customers simultaneously. For example: traffic road, left, and ferry. This problem has been considered by Bailey (1953). The batch service process may be generalize various ways (Newts (1967) and Medhi (1975)).

In this paper we suppose that the maximum batch size, on successive services are independent and identical distributed random variables of finite range.

1. Single Queue with Random Maximum Batch Size of Finite Range

Let X_n be the number of customers in the system immediately after n^{th} service period. Let Y_n be the number of new arrivals during the n^{th} service, and let M_n be the maximum permitted batch size for the n^{th} service. Then

$$X_n=\begin{cases} X_{n-1}-M_n+Y_n & \text{if} \quad X_{n-1}\geqq M_n \\ Y_n & \text{if} \quad X_{n-1}<M_n \end{cases}.$$

Let $\{b_r\}$ be the probability distribution of the number of new arrivals Y_n.

Denote p_r the probability of $M_n = r$ where $r = 1, 2, \ldots, m$ and $\sum_{r=1}^{m} p_r = 1$. Denote $p(X_n = r/X_{n-1} = s)$ the conditional probability of $X_n = r$ when $X_{n-1} = s$. Then we have

$$p(X_n = r/X_{n-1} = 0) = p(Y_n = r) = b_r \qquad (r = 0,\ 1,\ 2, \ldots)$$

$$p(X_n = r/X_{n-1} = 1) = p(Y_n = r) = b_r \qquad (r = 0,\ 1,\ 2, \ldots)$$

$$p(X_n = 0/X_{n-1} = 2) = p(M_n \geqq 2) \times p(Y_n = 0) = (p_2 + p_3 + \ldots + p_m) \times b_0$$

$$p(X_n = r/X_{n-1} = 2) = p(M_n \geqq 2) \times p(Y_n = r) + p(M_n = 1) \times p(Y_n = r-1)$$

$$= (p_2 + p_3 + \ldots + p_m) \times b_r + p_1 \times b_{r-1} \qquad (r = 1,\ 2,\ 3, \ldots)$$

$$p(X_n = 0/X_{n-1} = 3) = p(M_n \geqq 3) \times p(Y_n = 0) = (p_3 + p_4 + \ldots + p_m) \times b_0$$

$$p(X_n = 1/X_{n-1} = 3) = p(M_n \geqq 3) \times p(Y_n = 1) + p(M_n = 2) \times p(Y_n = 0)$$

$$= (p_3 + p_4 + \ldots + p_m) \times b_1 + p_2 \times b_0$$

$\vdots$

$$p(X_n = r/X_{n-1} = 3) = p(M_n \geqq 3) \times p(Y_n = r) + p(M_n = 2) \times p(Y_n = r-1)$$

$$+ p(M_n = 1) \times p(Y_n = r-2) = (p_3 + p_4 + \ldots + p_m) \times b_r + p_2 \times b_{r-1} + p_1 \times b_{r-2}$$

$$(r = 2,\ 3,\ 4, \ldots)$$

$\vdots$

$$p(X_n = 0/X_{n-1} = m) = p(M_n = m) \times p(Y_n = 0) = p_m \times b_0 = A_0$$

$$p(X_n = 1/X_{n-1} = m) = p(M_n = m) \times p(Y_n = 1) + p(M_n = m-1) \times p(Y_n = 0)$$

$$= p_m \times b_1 + p_{m-1} \times b_0$$

$\vdots$

$$p(X_n = r/X_{n-1} = m) = p(M_n = m) \times p(Y_n = r)$$

$$+ p(M_n = m-1) \times p(Y_n = r-1) + \ldots + p(M_n = 1) \times p(Y_n = 0)$$

$$= p_m \times b_r + p_{m-1} \times b_{r-1} + \ldots + p_1 \times b_{r-m+1}.$$

Define $c_{i0} = (p_i + p_{i+1} + \ldots + p_m)$ $(i = 2, 3, \ldots, m-1)$, $c_{ir} = (p_i + p_{i+1} + \ldots + p_m) \times b_r + p_{i-1} \times b_{r-1} + \ldots + p_{i-r} \times b_0$ for $i > r > 0$ and for $2 \leqq i < r$ $c_{ir} = (p_i + p_{i+1} + \ldots + p_m) \times b_r + p_{i-1} \times b_{r-1} + \ldots + p_1 \times b_{r-i+1}$. Also denote $A_r = p_m \times b_r + p_{m-1} \times b_{r-1} + \ldots + p_{m-r} \times b_0$ and if $r \geqq m$ $A_r = p_m \times b_r + p_{m-1} \times b_{r-1} + \ldots + p_1 \times b_{r-m+1}$ $(r = 1, \ldots, m-1)$.

Thus we have Markov chain with transition matrix

$$\underline{P} = \left|\begin{array}{llllll} b_0 & b_1 & b_2 & \ldots & b_r & \ldots \\ b_0 & b_1 & b_2 & \ldots & b_r & \ldots \\ c_{20} & c_{21} & c_{22} & \ldots & c_{2r} & \ldots \\ c_{30} & c_{31} & c_{32} & \ldots & c_{3r} & \ldots \\ \vdots & & & & & \\ c_{(m-1)0} & c_{(m-1)1} & c_{(m-1)2} & \ldots & c_{(m-1)} & \ldots \\ A_0 & A_1 & A_2 & \ldots & A_r & \ldots \\ 0 & A_0 & A_1 & \ldots & A_{r-1} & \ldots \\ \vdots & & & & & \end{array}\right|$$

So we have to solve $\pi = \pi \times \underline{P}$ where $\sum_{j=0}^{\infty} \pi_j = 1$ i.e.

$$\pi_0 = \pi_0 b_0 + \pi_1 b_0 + \pi_2 c_{20} + \ldots + \pi_m A_0$$
$$\pi_1 = \pi_0 b_1 + \pi_1 b_1 + \pi_2 c_{21} + \ldots + \pi_m A_1 + \pi_{m+1} A_0$$
$$\vdots$$
$$\pi_r = \pi_0 b_r + \pi_1 b_r + \pi_2 c_{2r} + \ldots + \pi_m A_r + \pi_{m+1} A_{r-1} + \ldots + \pi_{m+r} A_0$$
$$\vdots$$

Define $c_i(z) = \sum_{j=1}^{i} p_j z^{i+1-j} + \sum_{j=i}^{m} p_j$ $(i = 2, 3, \ldots, (m-1))$, $A(z) = \sum_{j=1}^{m} p_j z^{m-j}$ and $B(z) = \sum_{j=0}^{\infty} b_j z^j$. Then $c_i(z) \times B(z)$ generates the c_{ir} and $A(z) \times B(z)$ generates the A_r. Multiplying the above equations successively by $z^0, z^1, \ldots$ and taking their sum we obtain

$$\pi(z) = B(z) \times (\pi_0 + \pi_1 + \sum_{j=2}^{m-1} \pi_j c_j(z) + A(z) \times \sum_{j=m}^{\infty} \pi_j z^{j-m})$$ i.e.

$$z^m \pi(z) = B(z)(z^m(\pi_0 + \pi_1 + \sum_{j=2}^{m-1} \pi_j c_j(z)) + A(z) \times \sum_{j=m}^{\infty} \pi_j z^j)$$ i.e.

$$z^m \pi(z) = B(z)(z^m (\pi_0 + \pi_1 + \sum_{j=2}^{m-1} \pi_j c_j(z)) + A(z)(\pi(z) - \sum_{j=0}^{m-1} \pi_j z^j)).$$

Therefore

$$\pi(z) = \frac{B(z)(z^m (\pi_0 + \pi_1 + \sum_{j=2}^{m-1} \pi_j c_j(z)) - A(z) \sum_{j=0}^{m-1} \pi_j z^j))}{z^m - B(z) \times A(z)}.$$

In the numerator the terms in brackets of degree higher than z^m cancel out and $\pi(z)$ may be expressed as:

$$\pi(z) = \frac{\sum_{j=0}^{m-1} \pi_j (D_{m-1-j} z^m - z^j E_{m-1-j}(z)) \times B(z)}{z^m - B(z) \times A(z)}, \tag{1}$$

where $D_r = \sum_{j=0}^{r} p_{m-j}$ and $E_r(z) = \sum_{j=0}^{r} p_{m-j} z^j$ $(r = 0, 1, \ldots, (m-1))$. This expression contains m unknown $\pi_0, \pi_1, \ldots, \pi_{m-1}$, they may be determined as follows.

For the system to be ergodic it is clear that the mean number of arrivals in one service period must be less than the mean maximum number which can be served in a batch – i.e. $\rho < E[M]$ where M is the maximum batch size. Now $\rho = B'(1)$ and $E[M] = m - A'(1)$ therefore, the condition is $A'(1) + B'(1) < m$ i.e.

$\frac{d}{dz}(A(z)\times B(z))|_{z=1} < m$. Now it can be shown (see later) that when this condition is satisfied, the denominator, $z^m - A(z)\times B(z)$ in (1) has exactly $(m-1)$ zeros inside the unit circle, $\omega_1, \omega_2, \dots, \omega_{m-1}$. Since $\pi(z)$ is regular $|z|<1$ the numerator must vanish at $z=\omega_i$ $(i=1, 2, \dots, (m-1))$. In addition we must have $\lim_{z\to 1}\pi(z)=1$.

Then we have m equations to determine the m unknowns, $\pi_0, \pi_1, \dots, \pi_{m-1}$. Rather than solve these equations we can take advantage of the fact that the factor multiplying $B(z)$ in the numerator of (1) is a polynomial of degree m. Since $B(\omega_i) \neq 0$, the polynomial must vanish at $z=\omega_1, \omega_2, \dots, \omega_{m-1}$. It also vanishes at $z=1$ and so must be

$$G(z-1)(z-\omega_1)(z-\omega_2)\dots(z-\omega_{m-1})$$

for some G. G is then determined by the condition $\pi(1)=1$. Thus

$$\pi(z)=\frac{G(z-1)\prod_{i=1}^{m-1}(z-\omega_i)B(z)}{z^m - A(z)B(z)}.$$

Evaluating $\pi(1)$ by L'Hospital's rule we find $1=\frac{G\prod_{i=1}^{m-1}(1-\omega_i)}{m-\rho-A'(1)}$. Since $A'(1)=E[m-M]=m-\nu$ say where ν is the mean of maximum batch size, this gives $G=\frac{\nu-\rho}{\prod_{i=1}^{m-1}(1-\omega_i)}$ and so the limiting distribution has probability generating function

$$\pi(z)=(\nu-\rho)\frac{(z-1)\prod_{i=1}^{m-1}\frac{z-\omega_i}{1-\omega_i}B(z)}{z^m - A(z)B(z)}, \tag{2}$$

where ρ is the traffic intensity λ/μ.

2. The Expected Number of Customers in the Queue

Let the expected number of customers in queue $\pi'(1)=\xi$ say. From the limiting distribution we have

$$\pi(z)(z^m - A(z)B(z)) = \frac{\nu-\rho}{\prod_{i=1}^{m-1}(1-\omega_i)}(z-1)\prod_{i=1}^{m-1}(z-\omega_i)\times B(z).$$

By differentiating twice w.r.t. z. and putting $z=1$, we have

$$m(m-1)-B''(1)-2\rho A'(1)-A''(1)+2\xi(m-\rho-A'(1))=2\frac{\nu-\rho}{\prod_{i=1}^{m-1}(1-\omega_i)}.$$

But $A(z)$ be probability generating function of $(m-M)$. So $A'(1)=m-\nu$ where ν is the mean of maximum batch size, and $A''(1)=\mathrm{var}(M)+(m-\nu)^2-(m-\nu)$ and $B''(1)=\rho^2(r^2+1)$ where r is the coefficient of variation of service time. Therefore

$$\xi=\sum_{i=1}^{m-1}(1-\omega_i)^{-1}+\frac{\rho^2(r^2+1)-m(m-1)+(m-\nu)(2\rho+(m-\nu)-1)+\mathrm{var}(M)}{2(\nu-\rho)}.$$

3. Numerical Study

We calculate the expected number of customers in the queue for a number of different traffic intensities and with different probabilities.

Suppose the service time has Gamma distribution with parameter $n=2$, taking the case when the random maximum batch size is 1, 2 or 3 i.e. $m=3$.

Then from limiting distribution in (2) we have

$$\pi(z)=(\nu-\rho)\frac{(z-1)\prod_{i=1}^{2}\frac{z-\omega_i}{1-\omega_i}B(z)}{z^3-A(z)B(z)},$$

where $B(z)=\dfrac{1}{(1-\frac{\rho}{2}(z-1)^2)}$ since $\rho=2\lambda/\mu$ (the traffic intensity) and $A(z)=P_1z^2+P_2z+P_3$. $\pi(z)$ has the denominator $z^3-A(z)B(z)$ whose zeros are the solutions of the equation

$$z^3(1-\frac{\rho}{2}(z-1))^2-A(z)=0 \text{ i.e.}$$

$$z^3(1-\rho(z-1)+\frac{\rho^2}{4}(z-1)^2)-(P_1z^2+P_2z+P_3)=0.$$

The solutions of the polynomial with degree 4 were found by using Bainstows method on a Hewlett Packard micro-computer for various values of ρ and various probability distributions $\{p_i\}$. In each case we have the mean maximum batch size $E[M_n]=2$. We find only two zeros ω_1 and ω_2 inside the unit circle as follows.

Table 1

P	Distribution of maximum batch size			ω_1 and ω_2
	P_1	P_2	P_3	
1/2	1/3	1/3	1/3	$-0.3024 \pm 0.3948i$
	1/4	1/2	1/4	$-0.31454 \pm 0.2897i$
	1/8	3/4	1/8	-0.47814 & -0.187224
	1/16	7/8	1/16	-0.61097 & -0.0725
7/8	1/3	1/3	1/3	$-0.2805 \pm 0.35666i$
	1/4	1/2	1/4	$-0.28376 \pm 0.26859i$
	1/8	3/4	1/8	-0.381512 & -0.19665
	1/16	7/8	1/16	-0.511096 & -0.072757
3/2	1/3	1/3	1/3	$-0.25013 \pm 0.31224i$
	1/4	1/2	1/4	$-0.24676 \pm 0.242625i$
	1/8	3/4	1/8	$-0.2419 \pm 0.033225i$
	1/16	7/8	1/16	-0.073277 & -0.40597
1.9	1/3	1/3	1/3	$-0.23426 \pm 0.291345i$
	1/4	1/2	1/4	$-0.2288 \pm 0.22974i$
	1/8	3/4	1/8	$-0.2207 \pm 0.06425i$
	1/16	7/8	1/16	-0.3595 & -0.07368

To evaluate the mean of queue we substitute ω_1 and ω_2 in the following expression obtained from the paragraph 2

$$\xi = \sum_{i=1}^{2} (1-\omega_i)^{-1} + \frac{\rho^2(r^2+1)-6+(3-\nu)(2\rho+(3-\nu)-1)+\mathrm{var}(M_n)}{2(\nu-\rho)}$$

where $\rho = 1/2,\ 7/8,\ 3/2,\ 1.9$ and $r^2 = 1/2$, $\nu = 2$.

The following table shows the values of mean queue

Table 2

ρ	Distribution of maximum batch size			Mean Queue
	P_1	P_2	P_3	
1/2	1/3	1/3	1/3	0.102
	1/4	1/2	1/4	0.076
	1/8	3/4	1/8	0.062
	1/16	7/8	1/16	0.05
7/8	1/3	1/3	1/3	0.5
	1/4	1/2	1/4	0.34
	1/8	3/4	1/8	0.29
	1/16	7/8	1/16	0.28
3/2	1/3	1/3	1/3	2.58
	1/4	1/2	1/4	2.4
	1/8	3/4	1/8	2.2
	1/16	7/8	1/16	2.1
1.9	1/3	1/3	1/3	21.1
	1/4	1/2	1/4	20.2
	1/8	3/4	1/8	18.9
	1/16	7/8	1/16	18.4

Note that the mean queue gets smaller as the variance of M_m decreases for given ρ. These results may be compared with the mean queue when a fixed maximum batch size is used.

References

1. F. Benson, D. R. Cox. The Productivity of Machines Requiring Attention at Random Intervals. *J. Roy. Statist. Soc., Ser. B*, **13**, 1951, 65-82.
2. N. T. J. Bailey. On Queueing Processes with Bulk Service. *J. Roy. Statist. Soc., Ser. B*, **16**, 1954, 80-87.
3. D. R. Cox, H. D. Miller. The Theory of Stochastic Processes. Methuen, London, 1965.
4. D. R. Cox, W. L. Smith. Queues. Menthuen, London, 1961.
5. A. K. Erlang. Solution of Some Problems in the Theory of Probabilities of Significance in Automatic Telephone Exchanges. *P. O. Elec. Engrs., J.*, **10**, 189, 1918.
6. D. G. Kendall. Some Problems in the Theory of Queues. *J. Roy Statist. Soc., Ser. B*, **13**, 1951.
7. W. Feller. An Introduction to Probability Theory and Its Applications. V. 1, Wiley, New York, 1950.
8. S. Karlin, H. M. Taylor. A First Course in Stochastic Processes. Second Edition. Academic Press, New York-San Francisco-London, 1975.

Institute of Mathematics
Bulgarian Academy of Sciences
P.O.B. 373
1090 Sofia
BULGARIA

Received 31.08.1990