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On First Order Partial Differential-Functional Inequalities

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Suppose that $f: E\times R\times C(E_0\cup E,\ R)\times R^n\to R$ where E_0, $E\subseteq R^{1+n}$ are given sets. Differential-functional inequalities

$$D_x z(x, y)-f(x, y, z(x, y), z, D_y z(x, y))=P[z](x, y)\leqq 0,$$

$(x, y)\in \mathrm{Int}\, E$, are considered where $D_y z=(D_{y_1}z,\dots,D_{y_n}z)$ and z denotes an element of the space of continuous functions from $E_0\cup E$ to R. We prove the following result. Suppose that $P[u](x, y)\leqq 0$ and $P[v](x, y)\geqq 0$ on Int E. If $u(x, y)\leqq v(x, y)$ on the initial set E_0 then under certain conditions concerning f and E we have $u(x, y)\leqq v(x, y)$ on E. Our result is generalization of some results on first order partial differential or differential-functional inequalities considered in [2]–[8]. It is essential fact in our considerations that f satisfies the Volterra condition and it is non-decreasing with respect to the functional argument.

1. Introduction

We denote by $C(X, Y)$ the set of continuous functions defined on X taking values in Y where X, Y are metric spaces. For $y=(y_1,\dots,y_n)\in R^n$ we define the norm $\|y\|_p=(|y_1|+\dots+|y_n|)^{1/p}$ if $1\leqq p<\infty$ and $\|y\|_\infty=\max\{|y_i|: 1\leqq i\leqq n\}$. Let $p^*=\dfrac{1}{p-1}$ for $1<p<\infty$, $p^*=\infty$ for $p=1$ and $p^*=1$ for $p=\infty$.

Let

$$E=\{(x, y)\in R^{1+n}: x\in(0, a],\ \|y\|_p\leqq b-Mx\},$$

$$E_0=\{(x, y)\in R^{1+n}: x\in[-\tau_0, 0],\ \|y\|_p\leqq b\},$$

$$H_x=\{(\xi, \eta)=(\xi, \eta_1,\dots,\eta_n\}\in E_0\cup E: \xi\leqq x\},\ 0\leqq x\leqq a,$$

where a, b, $M>0$, $\tau_0\geqq 0$ and $Ma\leqq b$.

Assume that $f: E\times R\times C(E_0\cup E,\ R)\times R^n\to R$. The paper deals with first order partial differential-functional inequalities

$$(1)\qquad D_x z(x, y)\leqq f(x, y, z(x, y), z, D_y z(x, y)) \text{ for } (x, y)\in \mathrm{Int}\, E.$$

The solution of (1) are supposed to be continuous on $E_0 \cup E$ and to have first order partial derivatives in Int E. We assume that the function f satisfies the following Volterra condition: if $z, \bar{z} \in C(E_0 \cup E, \mathrm{R})$, $z = \bar{z}$ on H_x and $(x, y, s, q) \in E \times \mathrm{R} \times \mathrm{R}^n$ then $f(x, y, s, z, q) = f(x, y, s, \bar{z}, q)$.

First order partial differential or differential-functional inequalities have been studied by many authors under various assumptions. The classical theory of partial differential inequalities is described in [7], [11]. Solutions of differential inequalities are supposed to be of class D in the Haar pyramid, (i. e. if we denote by E the pyramid and by E_0 the initial set then $z: E_0 \cup E \to \mathrm{R}$ is called to be a function of class D if it is continuous, possesses the derivatives $D_x z, D_y z$ in Int E and the total derivative on Fr $E \cap ((0, a) \times \mathrm{R})$, where $(0, a]$ is a projection of E on time axis). Differential-functional inequalities of the Volterra type in the Haar pyramid are discussed in [4], [2], [3], [12]. There are considered solutions of class D in these papers. The following inequality

$$D_x z(x, y) \leqq f(x, y, z(x, y), z(x, \cdot), D_y z(x, y)),$$

where the functional argument depends only on space variable, is analysed in [2]. Another inequality

$$D_x z(x, y) \leqq f(x, y, A(x, y, z), D_y z(x, y)),$$

where A is an operator satisfying Volterra condition and is increasing with regard to functional argument, is considered in [3]. Differential inequalities in the following form

$$D_x z(x, y) \leqq \int_0^\infty f(x, y, z(x-s, y), D_y z(x, y))\, \mathrm{d}R_s(s, x, y) + g(x, y),$$

where R, g are given functions, are considered in [12]. Solutions of differential-functional inequalities cosidered in [2]-[4], [12] are supposed to be of class D in the Haar pyramid.

Now we will not need the assumption that the solutions are of class D. We will assume, however, that they possess derivatives in Int E. Such a class of solutions of differential inequalities (without functional argument) is considered in [10].

The result contained in this paper is a generalization of theorems on differential-functional inequalities given in [2]-[4], [12] and also of some results considered in the monographs [7] (Chapter IX), [11] (Chapter IX). Our result is also motivated by applications of partial differential-functional inequalities considered in [4].

Differential-integral inequalities and differential inequalities with a retarded argument are special cases of (1).

In the sequel we will use the following

Lemma 1. *Suppose that:*

1) $u \in C(E, \mathrm{R})$ *and* $u(0, y) \leqq 0$ *for* $(0, y) \in E$,

2) *there exists a positive number r such that for* $(x, y) \in \operatorname{Int} E$ *satisfying* $0 < u(x, y) < r$ *function u possesses the derivatives* $D_x u(x, y)$, $D_y u(x, y)$ *and* $D_x u(x, y) < M \| D_y u(x, y) \|_{p^*}$.

These assumptions imply the inequality $u(x, y) \leqq 0$ *for* $(x, y) \in E$.

The proof of the lemma you can find in [10].

2. Differential-functional inequalities

Let $S_x = \{y \in R^n : (x, y) \in E_0 \cup E\}$ for $x \in [-\tau_0, a]$. Suppose that $z \in C(E_0 \cup E, R)$ and denote by Tz the function defined in the following way: $Tz(x) = \max \{z(x, y) : y \in S_x\}$, $x \in [-\tau_0, a]$. Since z is continuous, it follows that $Tz \in C([-\tau_0, a], R)$. (See [11], Theorem 34.1.) We thus get $T : C(E_0 \cup E, R) \to C([-\tau_0, a], R)$.

If $X \subseteq R^k$ and $\alpha, \beta : X \to R$ are given functions then we define $\max[\alpha, \beta] : X \to R$ by $\max[\alpha, \beta](\xi) = \max\{\alpha(\xi), \beta(\xi)\}$, $\xi \in X$.

The following assumption will be needed throughout the paper.

Assumption H. *Suppose that the function* $\sigma : [0, a] \times R_+ \times C([-\tau_0, a], R_+) \to R_+$, $R_+ = [0, +\infty)$, *satisfies the following conditions:*

1) *σ is continuous and non-decreasing with respect to the functional argument,*

2) *if* $\eta, \bar{\eta} \in C([-\tau_0, a], R_+)$, $x \in [0, a]$, $s \in R_+$ *and* $\eta(t) = \bar{\eta}(t)$ *for* $t \in [-\tau_0, x]$ *then* $\sigma(x, s, \eta) = \sigma(x, s, \bar{\eta})$ (*Volterra condition*).

We prove the following comparison lemma.

Lemma 2. *Suppose that*

1) *assumption H is satisfied,*

2) $\bar{\theta} \in C([-\tau_0, 0], R_+)$ *and the comparison differential-functional problem:*

$$\eta'(x) = \sigma(x, \eta(x), \eta) \quad \text{for } x \in [0, a],$$
$$\eta'(x) = \bar{\theta}(x) \quad \text{for } x \in [-\tau_0, 0] \tag{2}$$

possesses the right-hand maximum solution θ *defined on* $[-\tau_0, a]$,

3) $\omega \in C(E_0 \cup E, R)$ *and* $\omega(x, y) \leqq \bar{\theta}(x)$ *for* $(x, y) \in E_0$,

4) *there exists a positive number r such that for* $(x, y) \in \operatorname{Int} E$ *satisfying* $\theta(x) < \omega(x, y) < \theta(x) + r$ *function* ω *possesses the derivatives* $D_x \omega(x, y)$, $D_y \omega(x, y)$ *and satisfies*

$$D_x \omega(x, y) \leqq M \| D_y \omega(x, y) \|_{p^*} + \sigma(x, \omega(x, y), \max[T\omega, 0]).$$

Under these assumptions we have the estimation $\omega(x, y) \leqq \theta(x)$ *for* $(x, y) \in E$.

Proof. From the theorem about the continuous dependence maximum solutions on right-hand sides and initial conditions we know that there exists $\varepsilon_0 > 0$ that for arbitrary $0 < \varepsilon < \varepsilon_0$ and arbitrary $m \in N$ (natural number) the initial problem:

$$\eta'(x)=\sigma(x,\ \eta(x),\ \eta)+\frac{\varepsilon}{m}\ \text{for}\ x\in[0,\ a],$$

$$\eta(x)=\bar{\theta}(x)+\frac{\varepsilon}{m}\ \text{for}\ x\in[-\tau_0,\ 0]$$

possesses the right-hand maximum solution θ_m defined on $[-\tau_0, a]$ and satisfying the condition $\lim_{m\to\infty}\theta_m(x)=\theta(x)$ uniformly on $[0, a]$. Since

$$\theta(x)<\theta(x)+\frac{\varepsilon}{m}=\theta_m(x)\ \text{for}\ x\in[-\tau_0,\ 0],$$

$$\theta'(x)=\sigma(x,\ \theta(x),\ \theta)+\frac{\varepsilon}{m}\ \text{for}\ x\in[0,\ a],$$

$$\theta'_m(x)=\sigma(x,\ \theta_m(x),\ \theta_m)+\frac{\varepsilon}{m}>\sigma(x,\ \theta_m(x),\ \theta_m)\ \text{for}\ x\in[0,\ a],$$

it follows from the theorem on strong differential-functional inequalities that $\theta(x)<\theta_m(x)$ for $x\in[0, a]$. We choose ε such as to $\theta_m(x)<\theta(x)+\frac{r}{2}$ for $x\in[0, a]$.

Since $\theta'_m(x)>\sigma(x,\ \theta_m(x),\ \theta_m)$ for $x\in[0, a]$, it follows that there exists $0<r^*<\frac{r}{2}$ such that $\theta'_m(x)>\sigma(x,\ t,\ g)$, $x\in[0, a]$, for an arbitrary function $g\in C([-\tau_0,\ a],\ R_+)$ satisfying $\theta_m(\tau)\leqq g(\tau)\leqq\theta_m(\tau)+r^*$, $\tau\in[-\tau_0,\ a]$ and for each $t\in[\theta_m(\tau),\ \theta_m(\tau)+r^*]$. Suppose the assertion of the lemma is false. Then there exist $x_0\in[0, a]$ and $\delta_0>0$ satisfying the following two conditions:

(i) $\omega(x,\ y)\leqq\theta(x)$ for $(x,\ y)\in H_{x_0}$,

(ii) for each $k\in N$ there exists $(x^{(k)},\ y^{(k)})\in H_{x_0+\delta_0}$ such that

$$\omega(x^{(k)},\ y^{(k)})>\theta(x^{(k)})\ \text{and}\ \lim_{k\to\infty}x^{(k)}=x_0. \tag{3}$$

Thus we have $T\omega(x)\leqq\theta(x)$ for $x\in[-\tau_0,\ x_0]$. Since $T\omega$ and θ are continuous functions, we conclude that there exist $0<r^{**}<r^*$ and $0<\delta<\delta_0$ such that $T\omega(x)\leqq\theta(x)+r^{**}$ for $x\in[x_0,\ x_0+\delta]$. Hence

$$T\omega(x)\leqq\theta_m(x)+r^*\ \text{for}\ x\in[x_0,\ x_0+\delta]. \tag{4}$$

Let $u: H_{x_0+\delta}\to R$ be defined in this way $u(x, y)=\omega(x, y)-\theta_m(x)$ for $(x, y)\in H_{x_0+\delta}$. We can conclude from assumption 3) that $u\in C(H_{x_0+\delta},\ R)$ and $u(0, y)=\omega(0, y)-\theta_m(0)<\omega(0,\ y)-\theta(0)\leqq 0$ for $\|y\|_p\leqq b$. We check the second condition of lemma 1. Let us assume that $(\bar{x},\ \bar{y})\in\text{Int}\,H_{x_0+\delta}$ and $0<u(\bar{x},\ \bar{y})<r^{**}$. We see at once that $\theta(\bar{x})<\theta_m(\bar{x})<\omega(\bar{x},\ \bar{y})<\theta_m(\bar{x})+r^{**}<\theta(\bar{x})+\frac{r}{2}+r^{**}<\theta(\bar{x})+r$. From assumption 4) we derive the following estimations:

$$D_x\omega(\bar{x}, \bar{y}) = D_x u(\bar{x}, \bar{y}) + \theta'_m(\bar{x}) \leqq M \| D_y\omega(\bar{x}, \bar{y}) \|_{p^*} + \sigma(\bar{x}, \omega(\bar{x}, \bar{y}), \max[T\omega, 0])$$
$$= M \| D_y u(\bar{x}, \bar{y}) \|_{p^*} + \sigma(\bar{x}, \omega(\bar{x}, \bar{y}), \max[T\omega, 0]).$$

Since $\theta_m(\bar{x}) < \omega(\bar{x}, \bar{y}) < \theta_m(\bar{x}) + r^*$, it follows that

$$\theta'_m(\bar{x}) > \sigma(\bar{x}, \omega(\bar{x}, \bar{y}), \theta_m). \tag{5}$$

Let us denote by $g: [-\tau_0, a] \to R_+$ a new continuous function such that $g = \max[\theta_m, T\omega]$. Since (4) and for $x \in [-\tau_0, x_0]$ we have the estimations $T\omega(x) \leqq \theta(x) < \theta_m(x) < \theta_m(x) + r^*$, it follows that $\theta_m(x) \leqq g(x) \leqq \theta_m(x) + r^*$ for $x \in [-\tau_0, x_0 + \delta]$. Hence from (5) we obtain $\theta'_m(\bar{x}) > \sigma(\bar{x}, \omega(\bar{x}, \bar{y}), g) \geqq \sigma(\bar{x}, \omega(\bar{x}, \bar{y}), \max[0, T\omega])$. The last inequality is true because δ is non-decreasing with respect to the functional argument and satisfies the Volterra condition. So $D_x u(\bar{x}, \bar{y}) \leqq M \| D_y u(\bar{x}, \bar{y}) \|_{p^*} + \sigma(\bar{x}, \omega(\bar{x}, \bar{y}), g) - \theta'_m(\bar{x}) < M \| D_y u(\bar{x}, \bar{y}) \|_{p^*}$. It is easily seen that function u satisfies the second condition of lemma 1. We can use the theses of this lemma and obtain the inequality $\omega(x, y) \leqq \theta_m(x)$ for $(x, y) \in H_{x_0+\delta}$. Letting $m \to \infty$ we conclude that $\omega(x, y) \leqq \theta(x)$ for $(x, y) \in H_{x_0+\delta}$, which contradicts (3). Thus lemma 2 is proved.

Now we formulate the main result.

Theorem. *Suppose that*

1) *assumption H is satisfied and the initial problem:*

$$\eta'(x) = \sigma(x, \eta(x), \eta) \text{ for } x \in [0, a],$$
$$\eta(x) = 0 \text{ for } x \in [-\tau_0, 0]$$

possesses the only solution $\eta(x) = 0$ *for* $x \in [-\tau_0, a]$,

2) $f: E \times R \times C(E_0 \cup E, R) \times R^n \to R$ *satisfies the Volterra condition and is non-decreasing with respect to the functional argument,*

3) *the estimation*

$$f(x, y, s, z, q) - f(x, y, \bar{s}, \bar{z}, \bar{q}) \leqq \sigma(x, s - \bar{s}, T(z - \bar{z})) + M \| q - \bar{q} \|_{p^*}$$

is true for $(x, y) \in \text{Int } E$, $s, \bar{s} \in R$, $s \geqq \bar{s}$, $z, \bar{z} \in C(E_0 \cup E, R)$, $z \geqq \bar{z}$, $q, \bar{q} \in R^n$.

4) $u, v \in C(E_0 \cup E, R)$ *and the derivatives* $D_x u, D_x v, D_y u, D_y v$ *exist on* Int E,

5) *the initial inequality*

$$u(x, y) \leqq v(x, y) \text{ for } (x, y) \in E_0 \tag{6}$$

and the differential-functional inequalities

$$\begin{aligned} &D_x u(x, y) \leqq f(x, y, u(x, y), u, D_y u(x, y)), \\ &D_x v(x, y) \geqq f(x, y, v(x, y), v, D_y v(x, y)) \text{ for } (x, y) \in \text{Int } E \end{aligned} \tag{7}$$

are satisfied.

Under these assumptions the estimation: $u(x, y) \leqq v(x, y)$ *for* $(x, y) \in E_0 \cup E$ *is true.*

Proof. Let us define function $\omega: E_0 \cup E \to \mathrm{R}$ as follows $\omega(x, y) = u(x, y) - v(x, y)$, $(x, y) \in E_0 \cup E$. From assumption 4) and (6) we obtain $\omega \in C(E_0 \cup E, \mathrm{R})$ and $\omega(x, y) \leqq 0$ for $(x, y) \in E_0$. We use lemma 2 with $\theta(x) = 0$ for $x \in [-\tau_0, a]$. Suppose that $(x, y) \in \mathrm{Int}\, E$ and $0 < \omega(x, y) < r$, $r > 0$. It follows immediately from assumptions 3) and 4) that

$$D_x\omega(x, y) = D_x u(x, y) - D_x v(x, y)$$
$$\leqq f(x, y, u(x, y), u, D_y u(x, y)) - f(x, y, v(x, y), v, D_y v(x, y))$$
$$\leqq f(x, y, u(x, y), \max[u, v], D_y u(x, y)) - f(x, y, v(x, y), v, D_y v(x, y))$$
$$\leqq \sigma(x, \omega(x, y), \max[0, T\omega]) + M \| D_y\omega(x, y) \|_{p^*}.$$

It follows from lemma 2 that $u(x, y) \leqq v(x, y)$ for $(x, y) \in E$. This completes the proof of the theorem.

Remark 1. In Theorem we can assume instead of (7) that

$$D_x u(x, y) - f(x, y, u(x, y), u, D_y u(x, y))$$
$$\leqq D_x v(x, y) - f(x, y, v(x, y), v, D_y v(x, y)) \text{ for } (x, y) \in \mathrm{Int}\, E.$$

Remark 2. Let us consider the following Cauchy problem:

$$\text{(8)} \qquad D_x z(x, y) = f(x, y, z(x, y), z, D_y z(x, y)) \text{ for } (x, y) \in \mathrm{Int}\, E,$$
$$z(x, y) = z_0(x, y) \text{ for } (x, y) \in E_0,$$

where $z_0 : E_0 \to \mathrm{R}$ is given initial function. If f satisfies condition 3) of Theorem with σ satisfying assumption 1) of the same Theorem then problem (8) will not be able to have different solutions.

The uniqueness results of the solutions of the problem (8) in more narrow class of functions are given in [1] and for differential problems in [11], [7]. Solutions of class D in the Haar pyramid are considered in [1], [7], [11].

Remark 3. It is important fact that the problem (2) is differential-functional. If we consider initial value problem as the comparison problem

$$\text{(9)} \qquad \eta'(x) = \sqrt{\eta(x)}, \text{ where } \eta(x) \geqq 0, \qquad \eta(0) = 0$$

then we find out that $s \equiv 0$ on $(-\infty, +\infty)$ is a solution of (9). But the following function $s(x) = \frac{x^2}{4}$ for $x \geqq 0$ and $s(x) = 0$ for $x \leqq 0$ is also a solution. Therefore the problem (9) does not satisfy condition 1) of Theorem. But we can consider differential-functional comparison problem in the following form:

$$\text{(10)} \qquad \eta'(x) = \sqrt{\eta(x^2)}, \text{ for } x \in [0,1], \ \eta(x^2) \geqq 0, \qquad \eta(0) = 0.$$

It is shown in [1] that (10) has the unique solution $s\equiv 0$ and is a comparison problem of the Perron type.

3. Applications

We list below some problems which are solved by using differential-functional inequalities theorems.

1. Suppose that $\bar{z}: E_0 \cup E \to \mathrm{R}$ is a solution of the differential-functional equation

$$(11) \quad D_x z(x, y) = f(x, y, z(x, y), z) + \sum_{i=1}^{n} g_i(x, y), D_{y_i}(x, y), (x, y) \in E$$

and consider the following approximate problem. We are interested in finding a sequence $\{u^{(m)}, v^{(m)}\}_{m=0}^{\infty}$, $u^{(m)}, v^{(m)}: E_0 \cup E \to \mathrm{R}$ (Chaplygin sequence) satisfying the conditions:

(i) for every m functions $u^{(m)}$, $v^{(m)}$ are solutions of linear equations associated with (11),

(ii) $u^{(m)} \leqq u^{(m+1)} \leqq \bar{z} \leqq v^{(m+1)} \leqq v^{(m)}$, $m=0, 1, \ldots$ on $E_0 \cup E$,

(iii) $\lim_{m\to\infty} u^{(m)} = \lim_{m\to\infty} v^{(m)} = \bar{z}$ on $E_0 \cup E$.

This problem is considered in [4] for functions $u^{(m)}$, $v^{(m)}$, $\bar{z}$ of class D. The existence and properties of the Chaplygin sequence are proved in [4] by employing differential-functional inequalities theorem. The main Theorem of this paper enable us to obtain more general theorem on approximation. We can assume in [4] that functions $u^{(m)}$, $v^{(m)}$, $\bar{z}$ are continuous on $E_0 \cup E$ and possess partial derivatives on E.

2. Let us consider an infinite system of first order differential-functional equations

$$(12) \quad D_x z_j(x, y) + \sum_{i=1}^{n} a_{ij}(x, y) D_{y_i} z_j(x, y) = f(x, y, z(x, y), z),$$
$$j = 1, 2, \ldots, (x, y) \in [0, a) \times \mathrm{R}^n$$

with the initial condition

$$(13) \quad z(0, y) = w(y), \ y \in \mathrm{R}^n.$$

The Chaplygin approximate method is used in [9] to prove the existence of the solution of the problem (12), (13). Differential-functional inequalities theorems are the basic tools in [9].

3. Let us consider the following nonlinear differential-functional equation

$$(14) \quad D_x z(x, y) = f(x, y, z(x, y), z, D_y z(x, y)).$$

Sufficient conditions for the stability and asymptotic stability of solutions of (14) by means of Lapunov functions and the theory of differential-functional inequalities are given in [5]. There are studied solutions of class D. Using our Theorem on differential-functional inequalities we can give up the assumption of existence of the total derivative for solutions of (14).

4. Monotone iterative methods are used in theorems on existence of solutions for nonlinear differential equations. Initial and boundary value problems for ordinary or partial equations are considered ([8], [6]). The basic idea is consist of three steps:

(i) constructing a sequence of some kind of approximate solutions,

(ii) showing the convergence of the constructed sequence of approximate solutions, and

(iii) proving that the limit function is actually a solution of the given problem.

Monotone iterative methods are applied in the monograph [6] (see also [8]) for the initial value problem for the equation:

$$D_x z(x, y) = f(x, y, z(x, y)) + \sum_{i=1}^{n} g_i(x, y) D_{y_i} z(x, y). \tag{15}$$

Differential inequalities are the basic tool in these works. Results from [8] can be easily generalized on the differential-functional case by using our Theorem on differential-functional inequalities.

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