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Hilbert's Boundary Value Problem for the Generalized Analytic Functions

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In this paper the author considers applications of theory of Hilbert's problem in theory of momentless tension state of the elastic shells and gives a physical interpretation of the results.

It is known that equations of the state of stress of the elastic shell have the form:

$$\text{(1)} \qquad \frac{1}{\sqrt{f}}\frac{\partial\sqrt{f}T^{ab}}{\partial x^a}+G^b_{ac}T^{ac}=X^b, \qquad (b=1,\ 2)$$

$$p_{ab}T^{ab}=Z$$

where T^{11}, $T^{12}=T^{21}$, T^{22} are the contravariant components of the stress tensor and X^1, X^2, Z are given functions of the point $(x^1,\ x^2)$ of the mean shell surface.

Complex function of the state of stress of the elastic shell is the function having the form

$$\text{(2)} \qquad U=f.\ \sqrt[4]{K}.(T^{11}-iT^{12})$$

where K is the principal curvature of a surface. The value f is determined by expression

$$\text{(3)} \qquad f=f_{11}f_{22}-f^2_{12}>0$$

where $f_{ab}=t_a t_b$ and $p_{ab}=nt_{ab}$ are symmetrical covariant tensors of the second order. Also, here is $t=t(x^1,\ x^2)$ the radius vector of the surface, $t_1=\frac{\partial t}{\partial x^1}$, $t_2=\frac{\partial t}{\partial x^2}$, – basic vectors of the system of coordinates $(x^1,\ x^2)$ and n is the unit vector of the normal to surface to which applies the formula

$$\text{(4)} \qquad n=\frac{t_1 x t_2}{\sqrt{f}}.$$

I. Vekua has demonstrated in his monograph [1] that the equations (1) can be reduced to the complex differential equation

$$\frac{\partial U}{\partial \bar{z}} - A(z, \bar{z})\,\overline{U} = F(z, \bar{z}) \tag{5}$$

where it is

$$A(z, \bar{z}) = G^2_{12} + iG^1_{12} - \frac{1}{4K}\frac{\partial K}{\partial \bar{z}} - \frac{1}{\sqrt{f}}\frac{\partial \sqrt{f}}{\partial \bar{z}} \tag{6}$$

and

$$F(z, \bar{z}) = \frac{1}{2} f \sqrt[4]{K}\left[X^1 - iX^2 - \frac{Z}{D}(G^1_{22} - i\,G^2_{22}) + \frac{i}{\sqrt{f}}\frac{\partial}{\partial y}\frac{\sqrt{f}Z}{D}\right]. \tag{7}$$

Here the quantities

$$f^{11} = \frac{f_{22}}{f},\ f^{21} = f^{12} = -\frac{f_{12}}{f},\ f^{22} = \frac{f_{11}}{f} \tag{8}$$

represent components of the contravariant tensor of the second order, the quantities

$$t^a = f^{ab}\,t_b, \qquad (a = 1, 2) \tag{9}$$

represent conjugate basic vectors to basic vectors t_1, t_2 and quantities

$$G^d_{ab} = t_{ab}\,t^d, \qquad (a, b, d = 1, 2) \tag{10}$$

represent Christoffel's symbols of the second order. The quantity D is determined from relation

$$K = \frac{D^2}{f}. \tag{11}$$

Besides that, it can be shown that

$$T^{22} = -T^{11} + \frac{Z}{D}. \tag{11a}$$

Solving of the equation (5) is of the great theoretical and practical significance. General solution of this equation contains one arbitrary analytical function $\Phi(z)$ and, in the monograph [1] it is given by the following formula

$$U = \Phi(z) + \iint_T \Gamma_1(z, t)\Phi(t)\,dT + \iint_T \Gamma_2(z, t)\overline{\Phi(t)}\,dT \tag{12}$$

$$-\frac{1}{\pi}\iint_T \Omega_1(z, t)F(t)\,dT - \frac{1}{\pi}\iint_T \Omega_2(z, t)\overline{F(t)}\,dT$$

with

$$(13)\quad \Omega_1(z,\ t)=\frac{1}{t-z}+\iint_T \frac{\Gamma_1(z,\ \sigma)}{t-\sigma}d\,T_\sigma\,;\quad \Omega_2(z,\ t)=\iint_T \frac{\Gamma_2(z,\ \sigma)}{\bar t-\bar\sigma}d\,T_\sigma$$

$$(14)\qquad \Gamma_1(z,\ t)=\sum_{j=1}^{\infty} K_{2j}(z,\ t),\quad \Gamma_2(z,\ t)=\sum_{j=1}^{\infty} K_{2j+1}(z,\ t)$$

$$(15)\quad K_1(z,\ t)=-\frac{A(t)}{\pi(t-z)};\quad K_n(z,\ t)=\iint_T K_1(z,\ \sigma)\overline{K_{n-1}(\sigma,\ t)}d\,T_\sigma.$$

As solving of the double singular integrals with Cauchy's kernel (13) in the general case is not possible in the finite form, the formula (12) is practically inapplicable. That is why the author has considered in his papers some of the cases, important for the practice, of integration of the equation (5) without utilization of singular integrals of Cauchy type.

In the paper [2] the equation (5) was solved in the case when function Im $[\int \frac{2F(z,\ \bar z)}{U_0}d\bar z]$ is harmonic, at which U_0 represents one known particular solution of corresponding homogeneous equation

$$(16)\qquad \frac{\partial U}{\partial \bar z}-A(z.\ \bar z)\,\overline{U}=0.$$

The paper [3] considers the case of rotary shells when coefficient A can be presented in the form

$$(17)\qquad A=\frac{\partial[\log(l\ \sqrt[4]{K})]}{\partial \bar z}$$

where l is length of the segment of normal to a surface till the section with axis of rotation.

The equation (5) with analytical coefficients were considered, in the paper [4]. General solution of equation is determined by method of generalized areolar series.

General solution of equation (5) can be presented in the form

$$(18)\qquad U=U_0+V$$

at which U_0 is one particular solution of this equation and V is general solution of the corresponding homogeneous equation (16).

If the mean shell surface is such that at any point we have $A(z,\ \bar z)=0$ i. e.

$$(19)\qquad \Gamma^2_{12}+i\Gamma^1_{12}=\frac{1}{4K}\frac{\partial K}{\partial \bar z}+\frac{1}{\sqrt{f}}\frac{\partial\sqrt{f}}{\partial \bar z}$$

then the equation (16) is transformed into Cauchy-Riemann's differential equation

$$\frac{\partial V}{\partial \bar{z}}=0. \tag{20}$$

For the shells that satisfy condition (19), $V(z)$ will be an analytical function. But then, using (2), (11a) and (18) we can express contravariant components of the stress tensor through the following formulas

$$\begin{aligned} T^{11}&=\frac{1}{f\sqrt[4]{K}}\ \mathrm{Re}\ (U_0)+\frac{1}{f\sqrt[4]{K}}\ \mathrm{Re}\,[\Phi(z)] \\ T^{12}&=-\frac{1}{f\sqrt[4]{K}}.\mathrm{Im}\,(U_0)-\frac{1}{f\sqrt[4]{K}}\ \mathrm{Im}\,[\Phi(z)] \\ T^{22}&=-T^{11}+\frac{Z}{D} \end{aligned} \tag{21}$$

where $\Phi(z)$ is an arbitrary analytical function. There is a series of shells important for the practice, that satisfies the condition (19) (see [5], [6]). The algebraic rotary surfaces of the second order also belong to this group.

For determining of the unknown function U, it is necessary to have some condition given, which that function satisfies on given closed contour L. In practice, we very often encounter Hilbert's contour condition in the form:

$$\mathrm{Re}\,[g(t).U]=\gamma(t), \qquad g=\alpha-i\beta,\ |g|\neq 0 \tag{22}$$

at which $g(t)$ and $\gamma(t)$ are given functions of the point $t\in L$. Such a form also has the following condition

$$N_v \cos\sigma(t)+S_v \sin\sigma(t)=\gamma(t) \tag{23}$$

at which $\sigma(t)$ and $\gamma(t)$ are given real functions of the point $t\in L$, and N_v and S_v are corresponding normal and tangential forces. Mechanical meaning of this condition is the following: at every point t of the contour L, there is associated one force (projection of the vector of force) in direction of $\cos\sigma(t)$ and $\sin\sigma(t)$.

If it is $\sigma(t)=0$ at every point on L, then boundary condition (23) obtains the form

$$N_v=\gamma(t), \tag{24}$$

i. e. at every point of contour there is associated one normal force. However, if it is $\sigma(t)=\frac{\pi}{2}$ at every point on L, then the boundary condition (23) gets transformed into

$$S_v=\gamma(t); \tag{25}$$

i. e. at every point of the contour there is associated one tangential force.

The boundary value problem (24) for the equation (5) and for the shells with positive Gauss's curvature is considered in this paper. The boundary value problem (24) can be presented in the following form (see [1], pp. 80-81)

$$\mathrm{Re}\,[(v_1+iv_2)^2\,U]=\gamma_0(t) \tag{26}$$

at which v_1, v_2 are covariant components of the unit vector $\vec{v}$ which at the point (x^1, x^2) passes through the mean surface and lies in the tangential plane and

$$\gamma_0(t)=(\gamma-N_v^0).f\,\sqrt[4]{K}. \tag{27}$$

However, the condition (26) can further be written in the form

$$pU_1+qU_2=\gamma_0(t) \tag{28}$$

$$(p=v_1^2-v_2^2,\quad q=-2v_1v_2,\quad U=U_1+iU_2)$$

i. e. in the form of the boundary condition of Hilbert.

It is known that in the general case does not exist the procedure for effective finding of the solution of the stated problem. That is why we shall here cite some cases when it is possible to come to the solution in the finite form.

I case: Let it be

$$A(z,\bar z)=\Gamma_{12}^2+i\Gamma_{12}^1-\frac{1}{4K}\frac{\partial K}{\partial\bar z}-\frac{1}{\sqrt{f}}\frac{\partial\sqrt{f}}{\partial\bar z}=0. \tag{29}$$

Then the equation (5) gets transformed into

$$\frac{\partial U}{\partial\bar z}=F(z,\bar z). \tag{30}$$

Its general solution (see [7] pp. 115-120) is

$$U=\int F(z,\bar z)\,d\bar z+Q(z)=U_0(z,\bar z)+Q(z) \tag{31}$$

where $Q(z)$ is an arbitrary analytical function.

If we introduce designations

$$U_0(z,\bar z)=u_0(x,y)+iv_0(x,y),\qquad Q(z)=q_1(x,y)+iq_2(x,y)$$

boundary condition (28) moves into

$$p(u_0+q_1)+q(v_0+q_2)=\gamma_0(t)$$

or, after settlement

$$pq_1+qq_2=\gamma_0(t)-pu_0-qv_0 \tag{32}$$

what represents Hilbert's boundary value problem for determining of analytical function $Q(z)=q_1+iq_2$, whose solving is known (see for example [8], pp. 250-253. The author has given his contribution to the problematics in the paper [9]). By finding unknown analytical function $Q(z)$ function $U=U_1+iU_2$, which represents solution of placed boundary value problem, is also determined.

II case: Let it be that one of two covariant components of vector $\vec{v}$, v_1 or v_2 is equal to zero. Then boundary condition (28) results to

$$U_1=\frac{\gamma_0(t)}{p} \tag{33}$$

i. e., it necessary to find that solution of the equation (5) whose real part U_1 on contour L takes the value γ_0/p .

By means of harmonic extension, we can determine in unique way, harmonic function $U_1(x, y)=\alpha(x, y)$ which on L takes the value (33). In order to determine the unknown imaginary part $U_2(x, y)$, let us write equation (5) in the form

$$\frac{1}{2}\left(\frac{\partial\alpha}{\partial x}-\frac{\partial U_2}{\partial y}\right)+\frac{i}{2}\left(\frac{\partial\alpha}{\partial y}+\frac{\partial U_2}{\partial x}\right)-(a_1+ia_2)(\alpha-iU_2)=f_1+if_2$$

or, by separation of real and imaginary part,

$$\frac{\partial U_2}{\partial x}+A_1U_2=B_1 \tag{34}$$

$$\frac{\partial U_2}{\partial y}+A_2U_2=B_2$$

with

$$A_1=2a_1, \qquad B_1=2f_2-\frac{\partial\alpha}{\partial y}+2a_2\alpha \tag{35}$$

$$A_2=2a_2, \qquad B_2=\frac{\partial\alpha}{\partial x}-2a_1\alpha-2f_1.$$

If the second equation (34) is multiplied by i and *it* we add to the first the previous one, the following complex equation will be obtained

$$2\frac{\partial U_2}{\partial\bar{z}}+(A_1+iA_2)U_2=B_1+iB_2. \tag{36}$$

General solution of the equation (36), on the basis of [7] is

$$U_2=e^{-\frac{1}{2}\int A(z,\bar{z})\,d\bar{z}}\left[Q(z)+\frac{1}{2}\int B(z,\bar{z})\,e^{\frac{1}{2}\int A(z,\bar{z})\,d\bar{z}}\,d\bar{z}\right. \tag{37}$$

where $Q(z)$ is arbitrary analytical function and $A(z,\bar{z})=A_1+iA_2$. $B(z,\bar{z})=B_1+iB_2$.

If we introduce new designations

$$C(z, \bar{z})=C_1+iC_2=e^{-\frac{1}{2}\int A(z, \bar{z})\,dz},$$

$$D(z, \bar{z})=D_1+iD_2=\frac{1}{2}e^{-\frac{1}{2}\int A(z, \bar{z})\,dz}\cdot\int B(z, \bar{z})\, e^{\frac{1}{2}\int A(z, \bar{z})\,dz}\, d\bar{z}$$

we shall have

$$\begin{aligned}U_2&=C(z, \bar{z})\,Q(z)+D(z, \bar{z})=(C_1+iC_2)(Q_1+iQ_2)+D_1+iD_2\\&=C_1Q_1+iC_1Q_2+iC_2Q_1-C_2Q_2+D_1+iD_2=(C_1Q_1-C_2Q_2+D_1)\\&\quad+i(C_1Q_2+C_2Q_1+D_2).\end{aligned}$$

Having on mind, however, that $U_2(x, y)$ is real function, it must be

$$C_1Q_2+C_2Q_1+D_2=0 \tag{38}$$

and by that, the problem has been reduced to known Mitrinović's problem for determining of harmonical-conjugated functions $Q_1(x, y)$ and $Q_2(x, y)$, whose solution is known (see [10], pp. 1-5).

Let us designate with U_{rh}, the class of differential complex functions whose real part is harmonic function. If Mitrinović's problem is solvable (what is the most frequent case), then by substitution of obtained analytical function $Q(z)$ in (20) we find as well required imaginary part of the solution of set boundary value problem. However, if it is impossible, we make conclusion that stated method unables finding the solution of boundary value problem in the class U_{rh}.

III case: $v_1=\pm v_2$. Then boundary condition (28) is reduced to

$$U_2=\frac{\gamma_0(t)}{q} \tag{39}$$

namely it is necessary to determine that solution of the equation (5) whose imaginary part U_2 on the contour L takes the value $\gamma_0(t)/q$. This case is considered in the same way, as previous one.

The importance of the application of theory of Hilbert's boundary value problem for generalized analytical functions can be seen in recent I. Vekua's monography. Considering the character of material, monography can be divided into two parts. In the first part, general methods of reduction of three-dimensional problems of the equilibrium of elastic shells to two-dimensional problems are exposed. Vekua limits himself to the consideration of the case of isotropic homogeneous shells, for which applies generalized Hooke's law. Nevertheless, obtained results are easily generalized to the case of anisotropic shells.

In the second part, special problems of shell equilibrium are studied, and new method is applied on that occasion, which relies upon presentation of required stress field in the form of the sum, so-called tangent and transversal stress field. This method enables reduction of considered problem on coordinate surface

x^3 = const in the system of partial equations of the first order or one partial equation of the second order (Weingarten's non-homogeneous equation). From mathematical point of view, these equations coincide with equations of membrane shells theory. Type of equation is determined by the sign of the main curvature of mean surface, if it is used in the feature of basis of shell parametrization, so that in case of convex shells, we have elliptical systems of equations of the first order or the elliptical equation of the second order. Consequently, for convex shells, considered problems might be reduced to generalized equations of Cauchy-Riemann. This enables wide utilization of the method of theory of generalized analytical functions.

In special case, when mean surface of the shell represents algebraical surface of the second order, the problem is reduced to ordinary Cauchy-Riemann equations. In this way, for very wide class of shells that have great practical application, problem is reduced to Hilbert's boundary value problem for analytical functions.

In this monography, Vekua has shown that stated, statically determined problems of convex shell equilibrium reduce to Hilbert's boundary value problem in the form:

$$\frac{\partial w}{\partial \bar{z}} + B\bar{w} = F, \qquad (\text{in } \hat{G}) \tag{40}$$

$$\operatorname{Re}[\lambda(z)\, w(z)] = f, \qquad (\text{on } \partial \hat{G}) \tag{41}$$

where $\hat{G}$ is topological image of coordinated surface $\hat{S} : x^3 = \text{const}$, which represents convex surface, and functions $B.F.\lambda$ and f depend not only on conjugated-isometric coordinate, $z = x + iy$ of the surface S, but also on problem consists of the fact that conjugated-isometric coordinate $z = x + iy$ of the surface $\hat{S}$, depends also on x^3, and consequently, it is necessary to determine dependence z from x^3 and domain $\hat{G}$, on the other hand, it is necessary to solve infinite number of boundary value problems of the forms (40)-(41) (for each fixed $x^3 \in [-h, h]$ we have got separate boundary value problem).

However, in practical problems, it is sufficient to find approximate solutions only of finite number of such problems. Namely, let us fix m values for $x^3 : x_1^3, x_2^3, \ldots x_m^3$ from segment $[-h, h]$ and then determine conjugated-isometric coordinates for the surfaces $\hat{S}_k : x^3 = x_k^3$ ($k = 1, 2, \ldots m$) and for corresponding values of the coordinate z, we solve m of boundary value problems

$$\frac{\partial w_k}{\partial \bar{z}} + B_k \bar{w}_k = F_k, \qquad (\text{in } \hat{G}_k) \tag{42}$$

$$\operatorname{Re}[\lambda_k w_k] = f_k, \qquad (\text{on } \partial \hat{G}_k) \tag{43}$$

where w_k, B_k, F_k, λ_k, f_k designate values of the corresponding functions on coordinate surfaces $\hat{S}_k : x^3 = x_k^3$ ($k = 1, 2, \ldots m$). When solutions $w_1, w_2, \ldots w_k$ of the problem (42)-(43) have been determined, by application of some interpolation formula, we can determine approximate values of the function w for each value $x^3 \in [-h, h]$.

In general case for the approximate solving of the problem (40)-(41), paper [12] can be used. Also, it is possible to use analogue methods of Hilbert's boundary value problem and particularly method of electromodelling (see [13], pp. 441-446).

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