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Relatively Complete 2-Extensions of Boolean Algebras

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In this paper we give a characterization of relatively complete extensions of Boolean algebras where each ultrafilter on subalgebra has at most two extensions to an ultrafilter of big algebra.

Let C be a subalgebra of A. We say that $q \in \mathrm{Ult}\, C$ splits in A if there are distinct $p, p' \in \mathrm{Ult}\, A$ which extend p i. e. $p \cap C = p' \cap C = q$. Let C and B be Boolean algebras. C is relatively complete (rc) subalgebra of B if for each $b \in B$ there is a greatest element $c \in C$ such that $c \leqq b$. We denote that element by $\mathrm{pr}_C b$. We also denote by $\mathrm{ind}_C b = (\mathrm{pr}(b) + \mathrm{pr}(-b))$. It is a clopen set in $\mathrm{Ult}\, C$ consisting of points that have at least one extension to an ultrafilter of B containg a, and at least one containg $-a$. B is a 2-extension of C if every ultrafilter in $\mathrm{Ult}\, C$ has at most two extensions to an ultrafilter on B. B is a projective extension of C if there exists a free Boolean algebra F and mappings $e\colon B \to C \oplus F$ and $q\colon C \oplus F \to B$ so that $q \circ e = id_B$ and $e \mid C = q \mid C = id \mid C$. If F is 2^2 then we will say that it is a projective extension by 4. Finally, B is a rcs-extension of C if B is a relatively complete simple extension of C, i. e. there is $a \in B$ such that $B = C(a)$ and C is a relatively complete subalgebra of B.

In the following proposition we list some known facts. Proofs could be found in (1).

Proposition. *Let B be a rc extension of C.*
i) *If B is a rcs-extension of C then it is a- 2-extension.*
ii) *Let $U = \{q \in \mathrm{Ult}\, C \mid q$ splits in $B\}$. Then $U = \cup \{s(j) \mid j \in J\}$ where $s: C \to$ $\mathrm{ClopUlt}\, C$ is the Stone isomorphism. In particular U is open in $\mathrm{Ult}\, C$.*
iii) *$J = \{\mathrm{indp}_c(x) \mid x \in B\}$ is an ideal in C, in fact the ideal dual to $U \in \mathrm{Ult}\, C$.*
iv) *Let α, β, γ be pairwise disjoint elements of C such that $\alpha + \beta + \gamma = 1$ and $\alpha \in J$. Assume $x \in A$ and $\mathrm{indp}(x) \leqq \alpha$. Then there is some $z \in A$ such that $\mathrm{indp}(z) = \alpha$, $\mathrm{pr}(z) = \beta$, $\mathrm{pr}(-z) = \gamma$ and $x \in C(z)$.*
v) *Let $\mathrm{indp}(a) = C$. Then $C(a) \cong C \oplus 4$.*
vi) *Let C be a Boolean algebra and $\alpha \in C$. There exists an rcs extension $B = C(a)$ of C such that $\mathrm{ind}\, p(a) = \alpha$.*
vii) *If $b \in C(a)$ then $\mathrm{indp}b \leqq \mathrm{indp}a$ and the equality holds iff $C(b) = C(a)$.*
viii) *If $C(a)$ and $C(b)$ are two rcs extensions of C such that $\mathrm{indp}(a) = \mathrm{indp}(b)$ then*

there is an isomorphism $f: C(a) \to C(b)$ *such that* $f \mid C = id_C$ *and* $f(a) = b$.
ix) *If* B *is an rc2-extension of* C *then* U_C^B *is clopen.*
x) *Canonical mapping* $f: \mathrm{Uit}B \to \mathrm{Ult}C$ *is open.*

Proposition 2. *Let* $C <_{rc2} B$. *Then*
(*) $\forall\, a, b \in B\ (\mathrm{indp}b \leqq \mathrm{indp}a \Rightarrow C(b) \subset C(a))$.

Proof. The implication from right to left is just proposition 1 (vii). Let us prove the other direction. Suppose $\mathrm{indp}(a) \leqq \mathrm{indp}(b)$, and let $\varphi: \mathrm{Ult}\, C(a) \to \mathrm{Ult}\, C$ be the canonical mapping. We claim that $\varphi(\mathrm{indp}_{C(a)}(b)) \cap \mathrm{indp}(a) = \emptyset$. Really, if it was not the case then there would exist $p \in \mathrm{Ult}\, C$ such that $p \in \varphi(\mathrm{indp}_{C(a)}(b)) \cap \mathrm{indp}(a)$. $p \in \mathrm{indp}(a)$ hence p splits in $C(a)$ and since each of these extensions belongs to $\mathrm{ind}_{C(a)}(b)$, they split in $C(a, b)$. Henceforth p would have at least four extensions to $C(a, b)$ and therefore in B, contrary to our assumption. This contradiction proves our claim. Since $\varphi(\mathrm{indp}_{C(a)}(b)) \subset \mathrm{indp}(b) \subset \mathrm{indp}(a)$, we have $\varphi(\mathrm{indp}_{C(a)}(b)) = 0$, hence $\mathrm{indp}_{C(a)}(b) = \emptyset$. That means $b \in C(a)$ i. e. $C(a) \subset C(b)$.

Corollary. *Let* $C <_{rc2} B$. *B is a simple extension of C iff* U_C^B *is clopen.*

Proof. One direction is Proposition 1 (ix). For the other one, if a is an element from B such that $\mathrm{indp}(a) = U_C^B$, then for every $b \in B$, we have $\mathrm{indp}(b) \leqq \mathrm{indp}(a)$ hence $b \in C(a)$.

(*) property of rc2-extensions actually characterizes them among rc extensions:

Proposition 3. *Let B be an rc2-extension of C. Then it is an rc2 extension iff it satisfies* (*).

Proof. Let B satisfy (*). Let $p \in U_C^B$. Then $p \in \mathrm{indp}(a)$ for some $a \in B$. Let p_1, p_2 be the extensions of p to ultrafilters of B so that $a \in p_1$, $-a \in p_2$. We want to prove that they are the only two extensions of p to B. Suppose to the contrary that there is another one q. Wlog we can suppose that $a \in q$ (switch a and $-a$ otherwise). Since $q \neq p_1$ there exist $b \in B$ such that $b \in q$ and $-b \in p_1$. Let $c \in B$ be an element such that $\mathrm{indp}(c) = \mathrm{indp}(a) + \mathrm{indp}(b)$. Then by (*) $a, b \in C(c)$. Then $p_1 \cap C(c)$, $q \cap C(c)$, $p_2 \cap C(c)$ would be three different extensions of p to an ultrafilter of $C(c)$ contrary to Proposition 1 (i).

There is another characterization of rcs extensions:

Proposition 4. *B is an rcs extension of C iff B is a projective extension of C by* 4.

Proof. ($\leftarrow$) Let $e: B \to C+4$ and $q: C+4 \to B$ so that $q \circ e = id_B$ and $e \mid C = q \mid C = id \mid C$. Then B is a simple extension as a homomorphic image of $C \oplus 4$. Also, since C is rc in $C \oplus 4$, $e(C) = C$ is rc in $e(B)$, hence C is rc in B (e is monomorphism).

($\to$) Let $B = C(a)$. Let u be a generator of 4 and let $q \mid C = id_C$ and $q(u) = a$. Then q has a unique extension to a homomorphism $q: C \oplus 4 \to B$. Let $\alpha = pr_C a$ and $\gamma = \mathrm{indp}_C(a)$. If we define $e \mid C = id_C$ and $e(a) = \alpha + \gamma u$ then by Sikorsky extension criterion it could be extended to a homomorphism $e: B \to C \oplus 4$. e, q obviously satisfy conditions for projective extenson.

In (2) there were given examples of 2-extensions which are not simple. The following example shows that situation remains the same even for projective extensions.

Example. Let $C = F_\omega$ free Boolean algebra on ω generators $a_1, a_2, \ldots$, and $A = F_\omega(a) \cong F_\omega \oplus 4$. Let $A_n = \langle F_\omega \cup \{aa_1, \ldots, aa_n\}\rangle$ be a subalgebra of A. Then $B = \bigcup \{A_n \mid n \in \omega\}$ is a subalgebra of A and a projective extension of C ($\{A_n \mid n \in \omega\}$ is its skeleton cf. [1]). As a subalgebra of A it is an rc2-extension of C. It is not simple since $U^B_C = \bigcup \{aa_n \mid n \in \omega\}$ is not clopen (it does not have finite subcover since aa_n' s are independent).

It is interesting that being simple extension is not a hereditary property among the extensions of a Boolean algebra, and being 2-extension is hereditary. The following theorem explains the situation.

Theorem 1. *B is an rc2-extension of C iff there exists an embedding $c: B \to C \oplus 4$ such that $e \mid C = id_C$.*

Proof. Let $b \in B$ and $e_b: C(b) \to C \oplus 4$ an embedding from Proposition 4. We claim that for $C(b) \subset C(c)$ we have $e_b = e_c \mid C(b)$. Since they both agree on C, it is enough to check that they have the same value at b. Wlog we can suppose that $b \leqq c$ (b partitions into a part below c and a part below c') Let $b = \delta c$ for $\delta \in C$. $e_c(b) = \delta(\alpha_c + \gamma_c u) = \delta\alpha_c + \delta\gamma_c u$, and $e_b(b) = \alpha_b + \gamma_b u$. But $\alpha_b = \delta\alpha_c$, and also $\beta_b = \delta\beta_c$ (projections are homomorphisms), hence because of $\alpha + \beta + \gamma = 1$, we also have $\gamma_b = \delta\gamma_c$. So we have $e_c(b) = e_b(b)$. We also have that for $b_1, \ldots, b_n \in B$, and $b = \bigvee \{B_k \mid k \leqq n\}$ $C(b_k) \subset C(b)$, $k \leqq n$. Now we have actually proved that $\{(C(b), f_b) \mid b \in B\}$ is a directed system. Let (D, f) be its limit. Since for every $b \in B$ $b \in C(b)$ we have $D = B$. f is the desired monomorphism from B into $C(a)$. This monomorphism actually maps B into a subalgebra $C(d)$ of $C(a)$ for any $d \in C(a)$ such that $\mathrm{ind}p(d) \supset U^B_C$.

Corollary. *Let $C <_{rc} B$. B is an rc2-extension of C iff it is a subalgebra of an rcs-extension of C.*

Using this theorem we could give an easier proof of Theorem 3 in (2):

Theorem 2. *Let B be a complete rc2-extension of C. Then B is an rcs-extension of C.*

Proof. Let $\alpha = \mathrm{cl}U^B_C$. Since C is complete (as an rc subalgebra of a complete one) α is clopen in C. Then we can suppose that $B \subset C(a)$ for $\mathrm{ind}p(a) = \alpha$. We will prove that $B = C(a)$. Since B is complete it is rc in $C(a)$. If $a \in B$ then $\mathrm{ind}p_B(a) \neq 0$. Now we have for canonical (open) projection $f: \mathrm{Ult}B \to \mathrm{Ult}C$ that $f(\mathrm{ind}p_B(a))$ is a nonempty open set in $\mathrm{Ult}C$. On the other hand it is a subset of $\mathrm{ind}p_C(a) - U^B_C$ i. e. $\mathrm{cl}\,U^B_C \setminus U^B_C$. Contradiction. Hence $a \in B$, i. e. $B = C(a)$.

It seems that these are the minimal conditions that reduce rc2-extensions to rcs-extensions. It is fairly easy to make necessary counterexamples for weaker conditions.

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